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# STABILITY ANALYSIS OF LINGUISTIC FUZZY MODEL SYSTEMS IN STATE SPACE

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**Abstract**

In this paper we propose a new stability theorem and a robust stability condition for linguistic fuzzy model systems in state space. First we define a stability in linear sense. After representing the fuzzy model by a system with disturbances, A necessary and sufficient condition for the stability is derived. This condition is proved to be a sufficient condition of the fuzzy model. The Q in the Lyapunov equation is iteratively adjusted by an gradient-based algorithm to improve its stability test. Finally, stability robustness bounds of a system having modeling error is derived. An example is also included to show that the stability test is powerful.

A nonlinear system can be modeled by a fuzzy function,F, described by *if − then* rules The parameters of the rules are identified well using a least square method from its input output data[1].

$$x(k+1) = F(x(k), u(k)) \quad (1)$$

The function F is represented briefly by

$$x(k+1) = \sum_{i=1}^{m} \alpha_i(x)(A_i x(k) + B(k)u(k)) \quad (2)$$

where $A_i$ is a system matrix of i-th rule and $\alpha_i = \omega_i / \Sigma_j \omega_j$,and $\omega_i$ is a membership value of the rule i. Thus the range of $\alpha_i(x)$ is $[0,1]$ and the summation of all $\alpha_i$'s is 1.

First, we analyze unforced systems, i.e. $u(k) = 0$.

**DEFINITION 1** *An $n-$ th order system is stable in linear sense if there exists a matrix $P \in R^{n \times n}$ such that $V = x^T P x$ is a Lyapunov function. That is,*

*1) $V \geq 0$*

*2) $\Delta V(k) = V(k+1) - V(k) \leq 0$*

*for all $x$, where the equality holds when $x \equiv 0$.*

It is certain that P should be a positive definite matrix. Let $x \in Supp(L_i)$ mean that all states of $x$ are in the supports of the fuzzy sets in the i-th rule, i.e. $x_j \in Support(L_{ij}), j = 1, 2, ..., m$.

If a system (1) has a Lyapunov function, then it is stable. This is a sufficient condition for the system to be stable. When the system is linear, the condition is a necessary and sufficient condition for stability. It is the reason that the name contains *linear sense*. Now we derive a condition under which the system has a Lyapunov function and after then find the bounds in which the robust stability is guaranteed. The bound is significant since the fuzzy functions only approximate nonlinear functions.

The system matrices in the consequent part of rules may be divided into common matrix $A_0$ and rule dependent matrices $\delta A_i$'s. If then, the system matrix $A_i$ of i-th rule is denoted by $A_0 + \delta A_i$. In (2), $\alpha_i(x)$'s are functions of $x$. Thus for a given state $x$, $\alpha_i(x)$'s are deterministic values. Consider uncertain variables $\alpha_i$'s. For a given x, assume that it can be any values in [0,1] if $\alpha_i(x)$ is greater than zero and $\alpha_i = 0$ if $x \notin Supp(L_i)$. But the summation is 1 as in (2). With the $\alpha_i$'s, we define another system which is similar to Eq.(2).

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{m} \alpha_i (A_0 + \delta A_i) x(k) \\ &= (A_0 + \sum_{i=1}^{m} \alpha_i \delta A_i) x(k) \end{aligned} \quad (3)$$

The system (3) is a special form of systems having structured disturbances. Let

$$D_i = (A_0 + \delta A_i)^T P (A_0 + \delta A_i) - P \quad (4)$$

**THEOREM 1** *The system (3) is stable in linear sense if and only if there exists a positive definite matrix P, such that*

*For any state $x$, $x^T D_i x$ is not positive for all $i \in \{j | x \in Supp(L_j)\}$ the equality holds only when $x \equiv 0$.*

Using the theorem, the stability in linear sense can be determined from the equation of each rule. The theorem is a necessary and sufficient condition for stability in linear sense of the system (3). In the system, the $\alpha_i$'s are considered as disturbances when $x \in Supp(L_j)$. Thus if (3) is stable, then (1) is also stable even though the reverse is not true.

To use this theorem, we have to find an appropriate matrix $P$ for all rules, which is very difficult. And since we want to analyze the fuzzy models, not the models with disturbances, we can make it easier by assuming $A_0$ as a value when the system (1) is at a certain state such as equilibrium state. Let $P$ be the solution of the following Lyapunov equation.

$$A_0^T P A_0 - P = -Q \qquad (5)$$

Then $D_i$ becomes

$$D_i = \delta A_i^T P \delta A_i + \delta A_i^T P A_0 + A_0^T P \delta A_i - Q \qquad (6)$$

**COROLLARY 1** *The system (1) is stable in linear sense if there exists a stable $A_0$ and a positive definite matrix $Q$ such that*

*For any rule i, $x^T D_i x \leq 0$, for all $x \in \{j | x \in Supp(L_j)\}$, and the equality holds when $x \equiv 0$.*

To use the corollary, we need to find $Q$ instead of $P$ in the theorem 1. It is not easy to find the appropriate $P$. But the $Q$ can be found systematically which will be shown later. And it is easy to extend the results to robust stability. To use the corollary, we have to decide $A_0$. It is recommended to use $\Sigma\alpha_i(x)A_i$, when $x \equiv 0$, since we are interested in the stability at the origin.

If a system is proved to be stable by T-S(Tanaka and Sugeno)'s Method[2], using the common $P$ we can make a Lyapunov function, $V = x^T P x$. Thus, if we find a $P$ which satisfies T-S's condition, then it satisfies the condition of corollary 1, too. On the other hand, even if a system is stable in linear sense, or it satisfies the conditions in corollary 1, there may be no common solution P in T-S method. We show such a case by an example at the end of this section. To use the corollary we have to know the maximum of $x^T D_i x$ for all $x \in Supp(L_i)$. It is not difficult to find the bound, since the matrix $D_i$ are symmetric, The lower and upper bounds of $x^T D_i x$ are calculated easily by diagonalizing $D_i$ using eigen vectors, even though they are not a tight bounds.

For some $Q$, the matrix $D_i$ may be negative definite, and for the other $Q$'s, not negative definite. If we adjust a $Q$ to the direction which makes the maximum eigen value of the matrix $D_i$ getting smaller, we improve the possibility of determining the stability of systems correctly. Using a grandient-based algorithm, we can decrease the maximum eigen value of the $D_i$ systematically.

Let a function $J_i$ be

$$J_i = \lambda_M(D_i) \qquad (7)$$

Then the $J_i$ is a function of $Q$.

where $\lambda_M(\cdot)$ means the maximum eigen value. And let $Q = L^T L$ where $L$ is a full rank matrix. Then $J_i$ can be minimized by iteratively adjusting $Q$ using the following theorem. Note that from now, we omit subscript i for convenience.

**THEOREM 2** *Let $J$ be defined as in Eq.(7). Then*

$$\frac{\partial J}{\partial L} = 2L(W - p_M p_M^T) \qquad (8)$$

*where $W$ satisfies*

$$A_0 W A_0^T - W = -(A_0 + \delta A) p_M p_M^T (A_0 + \delta A)^T + A_0 p_M p_M^T A_0^T \qquad (9)$$

*$p_M$ is eigen vector corresponding to $\lambda_{max}(D)$*

Adjusting $Q$ to the direction which minimizes the J, i.e. the maximum eigen values of $D_i$, for all $i = 1, ..., m$, or which minimize the eigen values of $D_i$ of positive $xD_ix$'s, we can increase the possibility checking the stability. Using the theorem, we can decrease any eigen values. Thus applying this theorem to all positive eigen values and adjusting them to the common direction to which all the positive eigen values are decreased, we can find more appropriate $Q$.

The fuzzy model system can not exactly model the nonlinear systems. Thus to determine the stability of a nonlinear, we need to know whether the fuzzy model system is stable even if it has modeling errors. The errors are in membership functions and in system matrices of rules. The corollary 1 can be extended to the systems which have modeling errors. Since it does not use the membership functions, the errors in membership functions do not affect determining the robust stability.

Consider a model which has system matrices errors, i.e. the consequent part of rule i is .

$$x(k+1) = (A_1 + \sum_{j=0}^{l_1} e_{1j} E_{1j}) x(k) = (A_0 + \delta A_1 + \sum_{j=0}^{l_1} e_{1j} E_{1j}) x(k)$$

**THEOREM 3** *Let the system described above is stable in linear sense when there is no modeling error, i.e. all $E_{ij} = 0$. Then for rule i robust stability bound, $e_i$, of $e_{ij}$'s, $j = 1, ... l_i$ is given by*

$$e_i = \min_{x \in Supp(L_i)} \frac{-b_i + \sqrt{b_i^2 - 4a_i c_i}}{2a_i} \qquad (10)$$

*where*

$$\begin{aligned} a_i &= l_i \sum_{j=1}^{l_i} x^T E_{ij}^T P E_{ij} x \qquad (11) \\ b_i &= \sum_{j=1}^{l_i} |x^T (E_{ij}^T P \delta A + \delta A^T P E_{ij} + E_{ij} P A + A P E_{ij}) x| \\ c_i &= x^T D x \end{aligned}$$

Since $a_i > 0$ and $c_i < 0$, there always exists an $e_i > 0$. And all the matrices in $a_i$, $b_i$ and $c_i$ are symmetric, we can easilly find their bounds when $x \in Supp(L_i)$.

Using the control $u(k) = kx(k)$ the above results can be extended to forced systems. Substituting $A_0$ and $\delta A_i$ by $A_0 + B_0 k$ and by $A_i + B_i k$ respectively, we can use the corollary 1 to determine the control stabilizing the system. If it is not stable, we adjust the $k$, and use the theorem again. To assist the adjustment of $k$, the following theorem is proposed.

**THEOREM 4** *Let $J$ be defined as in Eq.(7). Then*

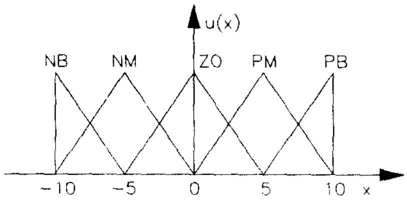
$$\frac{\partial J}{\partial k} = 2((B_0 + B_i)^T P(\delta A_i + \delta B_i k) p_M p_M^T + B_0^T P(A_0 + B_0 k) V \quad (12)$$

*where $V$ satisfies*

$$(A_0 + B_0 k) V (A_0 + B_0 k)^T - V = -(A_i + B_i k) p_M p_M^T (A_i + B_i k)^T + (A_0 + B_0 k) p_M p_M^T (A_0 + B_0 k)^T$$

From the theorem, we can adjust the control gain k effectively using the gradient mathod.

*Example:* Let fuzzy membership functions, NB,NM,ZO,PM and PB are as followings




Let system matrices be

$$A_1 = \begin{bmatrix} 0.95 & -0.1 \\ 0.13 & -0.9 \end{bmatrix} A_2 = \begin{bmatrix} 1.1 & -0.1 \\ 0.11 & -0.6 \end{bmatrix} \quad (13)$$
$$A_3 = \begin{bmatrix} 0.6 & -0.12 \\ 0.15 & -1.2 \end{bmatrix}$$

Consider a system modelled by the following rules.
Rule 1: If $x_1$ is ZO,$x_2$ is ZO, then $x(k+1) = A_1 x(k)$
Rule 2: If $x_1$ is NM, then $x(k+1) = A_1 x(k)$
Rule 3: If $x_1$ is PM, then $x(k+1) = A_1 x(k)$
Rule 4: If $x_1$ is ZO,$x_2$ is NB, then $x(k+1) = A_2 x(k)$
Rule 5: If $x_1$ is ZO,$x_2$ is PB, then $x(k+1) = A_2 x(k)$
Rule 6: If $x_1$ is NB,$x_2$ is ZO, then $x(k+1) = A_3 x(k)$
Rule 7: If $x_1$ is PB,$x_2$ is ZO, then $x(k+1) = A_3 x(k)$

Since $A_2$ and $A_3$ is unstable, there is not a common Lyapunov solution. Thus T-S method does not tell this system as stable. We use corollary 1 for this system. Let $A_0 = A_1$ ,then for $Q = I$, the solution of Eq.(5) is

$$P = \begin{bmatrix} 9.1509 & -0.7790 \\ -0.7790 & 5.0068 \end{bmatrix} \quad (14)$$

And $D_i$ in THM 1 is

$$D_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} D_2 = \begin{bmatrix} 1.7937 & -0.0354 \\ -0.0354 & -3.2063 \end{bmatrix} \quad (15)$$
$$D_3 = \begin{bmatrix} -5.8841 & -0.2062 \\ -0.2062 & 2.1104 \end{bmatrix}$$

After diagonalizing with eigen vectors, we have the following $\Delta V(k)$ bound for each rule.

Rule 1,2,3 : [-200, 0],

Rule 4,5 : [ -322.9137, -32.9044]

Rule 6,7 : [ -603.8211, -81.2002]

Thus, the system is stable.

## References


[1] M.Sugeno and G.T.Kang,"Structure identification of fuzzy model," *Fuzzy set and systems* **28**,pp.15-33,1988.

[2] K.Tanaka and M.Sugeno,"Stability analysis and design of fuzzy control systems," *Fuzzy Set and Systems*, **45**, 1992.