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On Chaotic Behavior of Fuzzy Inference Rule Based Nonlinear Functions

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Abstract

This research provides the results of a trial to generate the chaos by using nonlinear function constructed by fuzzy inference rules. The chaos generating function or chaotic behavior can be obtained by using Takagi-Sugeno fuzzy model with some constraint of the relationship of its parameters. Two examples are shown in this research. The first is simple example that construct of logistic image by fuzzy model. The second is more complicated one that provide the chaotic time series by non-linear autoregression based on fuzzy model. Simulated results are shown in these examples.
Key Words : *Fuzzy model, Chaos, Nonlinear regression*

1 Introduction

Generally, fuzzy inference and its rules have the property which matches our intuition, such that the near conditions will derive the almost same conclusions. For the contrast, by the chaotic system, we obtain pretty different results although by starting from the near initial conditions. Usually, it seems to be almost completely different result of these two systems. And so, it is considered that the fusioning of fuzzy system and chaotic system is difficult. But in the case of repeatedly using the nonlinear function which is constructed by fuzzy inference rules, we may encounter the phenomena like chaos. This paper reports the chaotic behavior of fuzzy inference based nonlinear functions. This might bring us the new look of fuzzy inference rule based function with the results of chaos research.

This research provides the results of a trial to generate the chaos by using nonlinear function constructed by fuzzy inference rules. This is motivated by followings. We assume the situation that we have to use the fuzzy inference system repeatedly. In this case, the stability or another features become important. Recently, the new research field named chaos[] is developed. It treats the behavior under the situation of repeatedly applied nonlinear function. This could serve us the new approach to analyze the fuzzy inference system. In order to accomplish this, for the first step, we construct the nonlinear function, which has the chaotic property, by fuzzy inference rules. And apply it repeatedly to obtain the series of its output. Then we discuss about its chaotic behavior.

This paper contains two topics as following. For the first example, we summarize the fuzzy model proposed by Takagi and Sugeno [4] and we also summarize the chaos, by referring the early research of chaos [1, 2] for simple example. The construction of logistic image by fuzzy inference rules is shown and obtain the same result of chaos of logistic image. The second example treats the chaotic time series generation by fuzzy model based non-linear autoregressive model. Computer simulation will show us its results.

2 Summary of Fuzzy Model

For simplify, we only treat fuzzy model, which was proposed by Takagi and Sugeno [4], as fuzzy inference rule. Fuzzy model has great merits to analyze its behavior, because its conclusion part is described by the linear equation. In this section, we summarize the fuzzy inference rule notation for the preparation of after discussion. We can write down following Rules, which are simplest one to be used for single input x and single output y fuzzy inference,

$$\text{IF } x \text{ IS } X_i \text{ THEN} y = f_i(x), \quad i = 1, 2, ..., R \tag{1}$$

as image $f() \; : \; X \ni x \mapsto y \in Y$

$$f(x) = \sum_{i=1}^{R} \mu_i \cdot f_i(x) / \sum_{i=1}^{R} \mu_i(x). \tag{2}$$

Where, $f_i(\cdot)$ is linear equation

$$f_i(x) = a_0^i + a_1^i \cdot x \tag{3}$$

and $\mu_i(x)$ is a membership function which corresponds to the fuzzy label X_i. Each membership function has the shape of trapezoid and it is described as follows

$$\mu_i(x) = \begin{cases} 0.0 & x \le L_i,\ x \ge R_i \\ 1.0 & l_i \le x \le r_i \\ (x-l_i)/(r_i-l_i) & L_i < x < l_i \\ (x-r_i)/(l_i-r_i) & r_i < x < R_i \end{cases} \tag{4}$$

This formula show us that i-th membership function can be denoted by four parameters L_i, l_i, r_i, and R_i. Here, we assume that the each label is restricted by the followings

$$L_i = \begin{cases} r_{i-1} & i > 1 \\ \inf_{x\in X}\{x\} & otherwise \end{cases} \tag{5}$$

and

$$R_i = \begin{cases} l_{i-1} & i > 1 \\ \sup_{x\in X}\{x\} & otherwise \end{cases} \tag{6}$$

Under these conditions, the number of parameters which describe one membership function is going to be only two. Then we can obtain the more simple description of nonlinear function constructed by fuzzy inference rules, which is partially defined, as follows

$$f(x) = \begin{cases} \mu_i(x)\cdot f_i(x) + [1-\mu_i(x)]\cdot f_{i-1}(x) \\ \quad , L_i < x < l_i \text{ and } i > 0 \\ \mu_i(x)\cdot f_i(x) + [1-\mu_i(x)]\cdot f_{i+1}(x) \\ \quad , r_i < x < R_i \text{ and } i < N \\ 1.0 \quad , otherwise \end{cases} \tag{7}$$

3 Examples of Chaos

It is known that *Chaos* is named by T.Y.Li and J.A.Yorke [1]. But the definition of chaos is slightly different between the researchers. Then we are not concerning about the definition of chaos, only looking the behavior of that system. Simple examples will give us the direct understanding of chaos. For the preparation, we define the situation by the formulation of first order difference equation denoted in the general form as

$$x_{t+1} = F(x_t). \tag{8}$$

In the various images $F(\cdot)$, one image will generate chaos or another will bring no chaotic one. So the choice of image is important.

One of the simplest and most famous example is introduced here. It is the logistic image application shown in M.May [2], in which the logistic image

$$F(x) = a \cdot x(1-x) \tag{9}$$

is used. Where a is the control parameter to govern the behavior of series x_t, if we choose $a \ge 3.57\ldots$, the series will be chaos.

4 Construction of Logistic Image by Fuzzy Model

Chaos by using logistic image is summarized in the previous section. This type of image can be easily constructed by fuzzy model as follows. For the simple discussion, let the range of input variable x normalized on $[0,1]$. In the case we need the only two fuzzy inference rules. These two rules are show as below.

$$\text{IF } x \text{ IS } X_1 \text{ THEN } y = f_1(x) \tag{10}$$

$$\text{IF } x \text{ IS } X_2 \text{ THEN } y = f_2(x) \tag{11}$$

$$a_0^1 = \frac{a}{2},\ a_1^1 = 0, \quad a_0^2 = -\frac{a}{2},\ a_1^2 = \frac{a}{2}, \tag{12}$$

$$l_1 = r_1 = 0,\ l_2 = r_2 = 1. \tag{13}$$

In these rules, a is the same parameter mentioned at logistic image.

5 Nonlinear Autoregressive Model based on Fuzzy Model

Let's consider the time series denoted by

$$x_1, x_2, x_3, \ldots, x_{N-1}, x_N, \tag{14}$$

and lag p autoregression model constructed by fuzzy model

$$x_t = \sum_{i=1}^{R} f_i(\mathbf{y}_t)\mu_i(\mathbf{y}_t) / \sum_{i=1}^{R} \mu_i(\mathbf{y}_t), \tag{15}$$

where R is the number of rules as mentioned before and

$$\mathbf{y}_t = [x_{t-1}, x_{t-2}, \ldots, x_{t-p}]^T. \tag{16}$$

According to the above general notation, the nonlinear function based on fuzzy model is as $(\mathbf{y} \in X^p) \to (x \in X)$. When p increases, the dimension of input space become so high and we will encounter the situation so-called *curse of dimension* . To avoid this, we divide the rule set for each time lag as follows

$$x_t = \frac{\sum_{i=1}^{p}\sum_{j=1}^{N_i} f_{ij}(x_{t-i})\mu_{ij}(x_{t-i})}{\sum_{i=1}^{p}\sum_{j=1}^{N_i} \mu_{ij}(x_{t-i})}, \tag{17}$$

where N_i denotes the number of rules for lag i, f_{ij} and μ_{ij} are corresponding to j-th rule of lag i each other. This regressive model is considered as the one of the linear autoregressive model but coefficients varies according to the input variables.

6 Example of Chaotic Time Series

Non-linear autoregressive model base on fuzzy model is shown in previous section. Here, the generation of chaotic time series are discussed by varying the parameter of that rules. As mentioned above, some combination of parameter values will generate the chaotic time series and another will not. So at first, we impose the more restriction for the model. The pair of membership function $\mu(\cdot)$ and linear function $f(\cdot)$ are only three for each lag. So the membership functions are only described by two parameters such that and they are described as follows

$$l_i \equiv l_{i2}, r_i \equiv r_{i2}, \tag{18}$$

and the linear functions are constant as follows

$$f_{i1} = f_{i3} = a_i \quad f_{i2} = b_i, i = 1, 2, \ldots, p. \tag{19}$$

In this case, the fuzzy model can be written as following simple one

$$x_t = \frac{\sum_{i=1}^{p} \sum_{j=1}^{3} f_{ij}(x_{t-i}) \mu_{ij}(x_{t-i})}{\sum_{i=1}^{p} \sum_{j=1}^{3} \mu_{ij}(x_{t-i})} \tag{20}$$

$$\begin{aligned} x_t &= \frac{1}{p} \sum_{i=1}^{p} \{ a_i[\mu_{i1}(x_{t-i}) + \mu_{i3}(x_{t-i})] \\ &\quad + b_i \mu_{i2}(x_{t-i}) \}. \end{aligned} \tag{21}$$

By using the restricted model with some values of parameters, some examples of time series are generated. In this paper, two kinds of fuzzy model are used. The parameters of each rule are shown in table 1. In each fuzzy model, two different initial values for autoregression are used to generate time series. These initial values are shown in table 2. The results of generated time series of each fuzzy model are shown in figure 1 and 2 respectively. Figure 1 shows the lag 2 autoregressive model and figure 2 shows the lag 4 case. In each figure, (a) and (b) show the results of different initial values.

By looking each figure, figure 1 seems to be the periodic behavior but its amplitude and frequency have correlation. This is the same effect of the model in [3] called as the *amplitude-dependent frequency* . Figure 2 has the typically non-linear behavior of time series. It seems not to be a simple periodic.

Table 1: Conditions of example of time series

Model No.	lag i	l_i	r_i	a_i	b_i
1	1	-0.5	0.5	1.5	0.2
	2	-0.5	0.5	-1.4	-0.4
1	1	-0.5	0.5	3.8	0.2
	2	-0.5	0.5	-1.8	-1.2
	3	-0.5	0.5	1.8	-0.2
	4	-0.5	0.5	-2.8	-0.8

7 Concluding Remark

Some example of chaotic non-linear function constructed by fuzzy inference rules are shown. The simplest example shows that logistic image can be constructed by fuzzy inference rules easily. Generation of chaotic time series are also shown by imposing the restriction of rule parameters to fuzzy model. Some examples of chaotic time series are shown by computer simulation. These results show us that chaos or chaotic series can easily obtained by fuzzy inference rule system cause of its non-linearlity.

In the case of fuzzy inference rule base function, analytical method cannot applied directly. So the criterion what obtained function generates chaos or not cannot calculate by analytic way. The example shown in this parer is only example of chaotic case. By considering this situation, there still remain for the future researches, that how chaos or not depends on the parameters of fuzzy inference rules. It is an interesting topic of fuzzy inference and chaos and will be researched near future.

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Table 2: Conditions of example of time series

Model No.		x_1	x_2	Model No.		x_1	x_2	x_3	x_4
1	(a)	0.25	0.25	2	(a)	0.5	0.2	0.1	0.3
	(b)	0.00	0.00		(b)	0.3	0.2	0.1	0.3

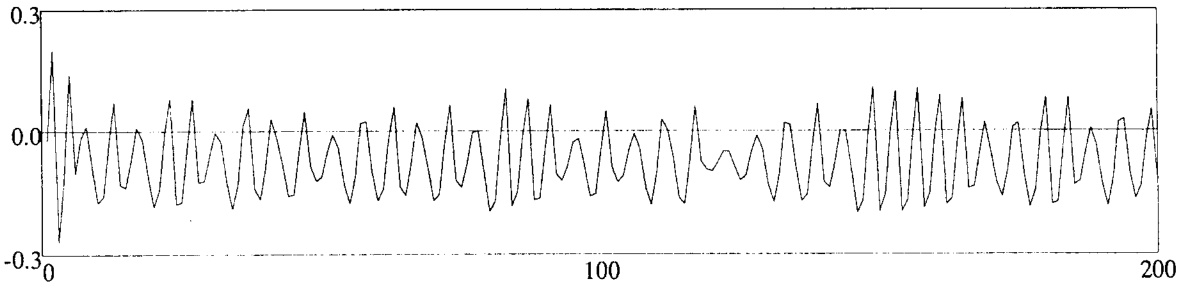

Figure-1:(a) chaotic time series obtained by fuzzy model, with rule 1, initial values (a) of rule 1

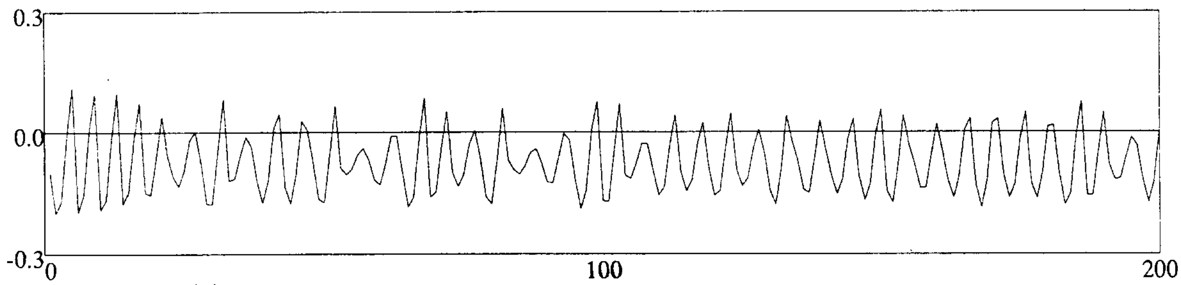

Figure-1:(b) chaotic time series obtained by fuzzy model, with rule 1, initial values (b) of rule 1

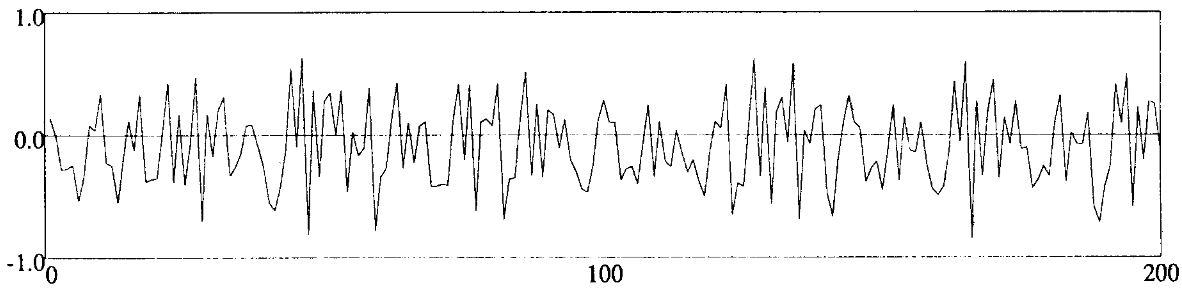

Figure-2:(a) chaotic time series obtained by fuzzy model, with rule 2, initial values (a) of rule 2

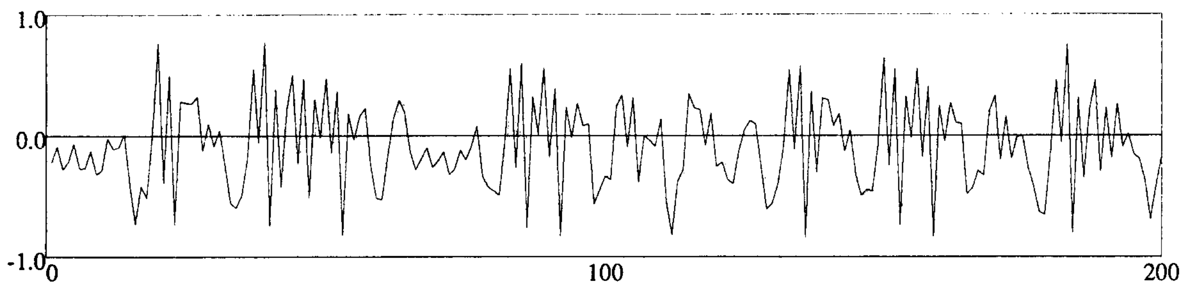

Figure-2:(b) chaotic time series obtained by fuzzy model, with rule 2, initial values (b) of rule 2