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다몸체 시스템의 운동방정식 형성방법

노 태 수
한국항공우주연구소

A Method of Formulating the Equations of Motion of Multibody Systems

Tae Soo No
Satellite Systems Department, Korea Aerospace Research Institute

Abstract

An efficient method of formulating the equations of motion of multibody systems is presented. The equations of motion for each body are formulated by using Newton-Eulerian approach in their generic form. And then a transformation matrix which relates the global coordinates and relative coordinates is introduced to rewrite the equations of motion in terms of relative cooridnates. When appropriate set of kinematic constraints equations in terms of relative coordinates is provided, the resulting differential and algebraic equations are obtained in a suitable form for computer implementation. The system geometry or topology is effectively described by using the path matrix and reference body operator.

Introduction

Not only a complex and large space system, but also most earth-based mechanical systems in use are essentially multibody systems. Typical multibody systems are shown in Fig. 1. Obviously, the analysis of multibody systems will be more complicated and costly than that of comparatively simple systems such as single or two body systems. This area has seen steady and undimishing progress because of a continuing demand for the analysis of multibody systems and the explosive growth of computing power.

One feature pertaining to multibody systems is that certain elementary subsystem shapes are repeated. This often allows automatic and recursive formulation of fairly general equations of motion for a broad class of multibody systems. There are several general computer program available which has the capability described in the above [1-3].

Formulation of dynamical equations of motion

While Newton-Eulerian vectorial mechanics has been successfully for decades used to obtain equations of motion of a single or multibody system[4-6], the Lagrangian approach has been favored by many who seek to use computer aided automatic generation of generic equations of motion. In this context, the Lagrangian approach includes: classical Lagrange's equations, the principal of virtual work, Lagrange's equations in terms of quasi-coordinate, and Lagrange's form of D'Alembert's principle [7]. All of this formalism are, of course, based on and derivable from D'Alembert's principle. Another variation of D'Alembert's principle is Kane's method [8]. Advantages of using Kane's method are that non-working constraint forces and moments are automatically eliminated and equations of motion in terms of a minimal set of generalized speeds are obtained. In the Newton-Eulerian approach, the non-working constraint forces must be solved for or eliminated.

No matter what formalism is used, its final outcome is equations of motion in the form of

$$M\ddot{x} = f(x, \dot{x}, t) \tag{1}$$

where M is a generalized mass matrix, x is a column matrix of generalized coordinates, and f is a generalized force vector. More often Eq. (1) should be augmented by a set of system constraint equations

$$A\ddot{x} = g(x,t) \tag{2}$$

where A is a Jacobian constraint matrix and g is a constraint vector. In most previous works, attention has focused on how effectively (analytical formulation) and how efficiently (numerical implementation) one can get Eq. (1).

However, when the number of components of a multibody system is large, the procedure for solving Eqs. (1) and (2) for x becomes prohibitedly time consuming. Therefore, an efficient algorithm has been sought since early 1980's. [9-14] Nikravesh, et al., used solely Euler parameters to derive equations of motion of constrained multibody systems and equatios of kinematic constraints for various joints[15]. Kim and Haug's formulation is considered to be very generic and suitable for parallel or concurrent processing, which may reduce computing time drastically. A formulation of equations of motion using velocity transformation was proposed by Kim [16] and Keat [17]. In this paper, Kim's method is further developed by introducing path matrix and reference body matrix so that it may be easily applicable to more complicated systems and suitable for general purpose multibody computer simulation program.

Kinematics

Path matrix and reference body operator

Referring to the generic multibody system shown in Fig. 1 an arbitrary body is chosen as a base body and is numbered "1". If the system contains any closed loop, then one of the joints in each loop may be cut so that the entire system consists of "chain" or "open" tree configurations only. The inboard body of the body j is the one leading to the base body and outboard body is the one leading away from the base body. Note that there is only one inboard body unless it has a closed loop. An end body is one without any outboard bodies. The component bodies may be numbered in an arbitrary manner in the current formulation.

Let's define the element of the path matrix such that

$$\pi_{ij} = \begin{cases} 1, & \text{if body j is located between bodies I and i} \\ 0, & \text{if not} \end{cases}$$

For example, we may construct the follwoing path matrix π for the multibody system shown in Fig. 2.

$$\pi = \begin{bmatrix} 1&0&0&0&0&0&0 \\ 1&1&0&0&0&0&0 \\ 1&0&1&0&0&0&0 \\ 1&1&0&1&0&0&0 \\ 1&1&0&0&1&0&0 \\ 1&0&1&0&0&1&0 \\ 1&0&1&0&0&0&1 \end{bmatrix}$$

If the attitude of body j is defined with respect to its inboard body i, we define an operator such that

$$L(j) = i \tag{3}$$

For the example shown in Fig.3 , we may readily note that

$$L(2) = L(3) = 1$$
$$L(4) = L(5) = 2$$
$$L(6) = L(7) = 3.$$

By definition, we let $L(1) = 0$. That is, the reference body of the base body is the inertial coordiante system. To illustrate the use of the reference body operator L, we try to relate ω_j to $\omega_{L(j)}$. Again referring to Fig. 2, we can set

$$\omega_{L(1)} = 0, \quad \omega_{L(2)} = \omega_1,$$
$$\omega_{L(3)} = \omega_1, \quad \omega_{L(4)} = \omega_2,$$
$$\omega_{L(5)} = \omega_2, \quad \omega_{L(6)} = \omega_3,$$
$$\omega_{L(7)} = \omega_3.$$

In matrix form, we may write

$$\begin{bmatrix} \omega_{L(1)} \\ \omega_{L(2)} \\ \omega_{L(3)} \\ \omega_{L(4)} \\ \omega_{L(5)} \\ \omega_{L(6)} \\ \omega_{L(7)} \end{bmatrix} = \begin{bmatrix} 0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \\ \omega_6 \\ \omega_7 \end{bmatrix}$$

For simplicity, we write

$$\omega_L = \rho\, \omega \tag{4}$$

where each element of the matrix ρ represents the 3x3 null or identity matrix.

Velocity transformation

Referring to Fig. 3, one may write the absolute position vector of P_j, the attachment point of body j to its inboard body, as

$$R_j = \sum_{k=1}^{j} \pi_{jk}(d_k + r_k),\ j=1,2,\ldots,n \tag{5}$$

where the use of notations d_k and r_k may be clear upon examining Fig. 4. By definition, we let $d_1 = 0$. The total time derivative of Eq. (5) expressed in the inertial coordinate system is

$$\dot{R}_j = V_j = \sum_{k=1}^{j} \pi_{jk}\, C_{L(k)}{}^T [\, \dot{r}_k + \omega_{L(k)} \times (d_k + r_k)] \tag{6}$$

where $C_{L(k)}$ denote a direction cosine matrix which transforms a vector in the inertial coordinate system into a vector in the $L(k)^{th}$ body-fixed coordinate system. We may rewite Eq. (6) using matrix notation as follows:

$$\dot{R} = \pi[\pi_{ij}\, C_{L(j)}{}^T]\dot{r} + \pi[\pi_{ij}\, C_{L(j)}{}^T(\tilde{d}_j + \tilde{r}_j)]\, \omega_L \tag{7}$$

where $\dot{r} = (\dot{r}_1{}^T, \dot{r}_1{}^T, \ldots, \dot{r}_n{}^T)^T$, and $\omega_L = (\omega_{L(1)}{}^T, \omega_{L(2)}{}^T, \ldots, \omega_{L(1)}{}^T)^T$. The various path matricies $\pi[\pi_{ij}\, C_{L(j)}{}^T]$ and $[\pi_{ij}\, C_{L(j)}{}^T(\tilde{d}_j + \tilde{r}_j)]$ may constructed from the topology of the system of interest. Tilde "~" denotes the skew symmetric matrix of a vector.

Using similar procedure, we write the absolute angular velocity of the body j as

$$\omega_j = \sum_{k=1}^{j} \pi_{jk} C_{jk}\, \Omega_k \tag{8}$$

In Eq. (8) Ω_k denotes the angular velocity of body k with respect to its inboard body and

$C_{jk} = C_j C_k^T$ is a transformation matrix. Again, rewriting Eq. (8) in matrix form yields

$$\omega = \pi(\pi_{ij}C_{ij})\Omega \qquad (9)$$

Combining Eqs. (4) and (9) results in

$$\omega_L = \rho\pi(\pi_{ij}C_{ij})\Omega \qquad (10)$$

Using Eqs. (7) and (9), we form a velocity transformation matrix such that

$$\begin{bmatrix} \dot{R} \\ \omega \end{bmatrix} = \begin{bmatrix} \pi(\pi_{ij} C_{L(j)}^T) & \pi[\pi_{ij} C_{L(j)}^T(\tilde{d}_j^T + \tilde{r}_j^T)]\rho\pi(\pi_{ij}C_{ij})) \\ 0 & \pi(\pi_{ij}C_{ij}) \end{bmatrix} \begin{bmatrix} \dot{r} \\ \Omega \end{bmatrix}$$

$$= B\begin{bmatrix} \dot{r} \\ \Omega \end{bmatrix}$$

Or,

$$\dot{X} = B\dot{x} \qquad (11)$$

Equation (11) defines the kinematic relationship between the absolute coordinates and relative coordinates. Note that the velocity transformation matrix B is not necessarily a square matrix. In other words, the number of relative cooridantes can be less that that of the absolute coordinates if due consideration on the system geometry and constraints has been made in getting the matrix B.

Kinetics

Now we derive the equations of motion of each component body treating it as an independent one.

<u>Translation</u>

Referring to Fig. 3, we apply Newton's second law to a generic mass dm_j to get

$$[\ddot{R}_j + \dot{\omega}_j x\, \rho_j + \omega_j x(\omega_j x\, \rho_j)]dm_j = d f_j \qquad (12)$$

In the above ρ_j locates m_j in the body j. Integrating Eq. (12) over the volume of body j yields

$$m_j \ddot{R}_j + \dot{\omega}_j x \int \rho_j dm_j + \omega_j x(\omega_j x \int \rho_j dm_j)$$

$$= \int d f_j \qquad (13)$$

Let us define

$$r_{jc}' = \int \rho_j dm_j \qquad (14)$$

= the position vector of the center of mass in the body-fixed coordinate system.

Then we rewrite Eq. (13) using matrix notation and considering coordinate transformation as

$$m_j \ddot{R}_j + C_j^T \tilde{r}_{jc}^T \dot{\omega}_j = E_j^{ext} + E_j^c + E_j^{inertial}$$

$$= E_j + E_j^c \qquad (15)$$

where

E_j^{ext} = external force on body j,

E_j^c = constraint force due to adjacent bodies,

$E_j^{inertial} = -C_j^T \tilde{\omega}_j \tilde{\omega}_j r_{jc}$.

<u>Rotation</u>

To obtain the rotational equations of body j about its attach point P_j, we take the vector cross product of Eq. (12) with ρ_j and integrate over body j. The result in matrix form is,

$$\int(\tilde{\rho}_j dm_j) C_j \ddot{R}_j + \int(\tilde{\rho}_j^T \tilde{\rho}_j dm_j)\dot{\omega}_j$$

$$= -\tilde{\omega}_j \int(\tilde{\rho}_j^T \tilde{\rho}_j dm_j)\omega_j + \int \tilde{\rho}_j d f_j \qquad (16)$$

We define $I_j = \int \tilde{\rho}_j^T \tilde{\rho}_j dm_j$, the moment of inertia matrix of body j about the point P_j. Then Eq. (16) is rewritten as

$$\tilde{r}_{jc} C_j \ddot{R}_j + I_j \dot{\omega}_j = M_j^{ext} + M_j^c + M_j^{inertial}$$

$$= M_j + M_j^c \qquad (17)$$

where

M_j^{ext} = external moment on body j about P_j,

M_j^c = constraint moment due to adjacent bodies,

$M_j^{inertial} = -\tilde{\omega}_j I_j \omega_j$.

<u>Equations of motion of body j</u>

Combining Eqs. (15) and (17) we get the equations of motion of body j in terms of the absolute coordinates as follows:

$$\begin{bmatrix} m_j & (\tilde{r}_j C_j)^T \\ \tilde{r}_{jc} C_j & I_j \end{bmatrix} \begin{bmatrix} \ddot{R}_j \\ \dot{\omega}_j \end{bmatrix} = \begin{bmatrix} E_j + E_j^c \\ M_j + M_j^c \end{bmatrix} \qquad (18)$$

<u>Equations of motion of the system of n bodies</u>

We have n equatins similar to Eq. (18) for each component bodies of the n-body system. If we stack up Eq. (18) in appropriate manner, we get

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \ddot{R} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} E + E^c \\ M + M^c \end{bmatrix}$$

Or

$$M\dot{X} = Q + Q^c \qquad (19)$$

where

$R = (R_1^T, R_2^T, \ldots, R_n^T)^T$,

$\dot{\omega} = (\dot{\omega}_1^T, \dot{\omega}_2^T, \ldots, \dot{\omega}_n^T)^T$,

$$M_{11} = \begin{bmatrix} m_1 & 0 & \ldots & 0 \\ 0 & m_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & m_n \end{bmatrix}, \text{ 3nx3n matrix}$$

$$M_{12} = \begin{bmatrix} (\tilde{r}_{1c}C_1)^T & 0 & \ldots & 0 \\ 0 & (\tilde{r}_{2c}C_2)^T & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & (\tilde{r}_{nc}C_n)^T \end{bmatrix}, \text{ 3nx3n matrix}$$

$$M_{21}' = M_{12}^T$$

$$M_{22} = \begin{bmatrix} I_1 & 0 & \ldots & 0 \\ 0 & I_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & I_n \end{bmatrix}, \text{ 3nx3n matrix}$$

Constraints

In the previous section, we have used two types of coordinates. One is the absolute ones which define the absolute position and rotation of body j. The other ones defines the positon and rotation of body j with respect to its inboard body L(j). If the velocity transformation matrix B is chosen such that the result is a minimal set of coordinates to describe the system state, one may substitute Eq. (11) into Eq. (19) and multiply the resulting equations by B^T to get

$$B^T M B \ddot{x} = B^T Q + B^T Q^c - B^T M \dot{B} \dot{x} \quad (20)$$

The two orthogonal and complimentary subspaces of matrix B which results in a minimal set of relative coordinates, column and null spaces, has very important properties. The column space of matrix satisfies the system constraints and its null space defines the possible direction of motion of the system. Due to this, we have always

$$B^T Q^c = 0 \quad (21)$$

When the transformation matrix B does not result in a minimal set of relative coordinates (it could be a nxn square matrix for a change of variables) or there are additional constraints due to such as a closed loop in the system geometry, $B^T Q^c$ is not always a zero vector.

In most cases, the constraint conditions may be expressed in the form of equations,

$$A(x,t)\dot{x} = g(x,t) \quad (22)$$

Equation (22) may represent holonomic, nonholonomic or mixed constraint equations. If that is the case, the still unknown constraint force $B^T Q^c$ can be rearranged and expressed [18] as

$$B^T Q^c = -A^T \lambda \quad (23)$$

where λ represents the Lagrange's multiplier vector which is related to the constraint forces and moments.

Now if we combine Eqs. (20) and (22), we finally get

$$\begin{bmatrix} B^T M B & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \lambda \end{bmatrix} = \begin{bmatrix} B^T Q - B^T M \dot{B} \dot{x} \\ \dot{g}(x,t) + \frac{\partial g}{\partial x} \dot{x} - \dot{A} \dot{x} \end{bmatrix} \quad (24)$$

Numerical implementation

The procedure addressed in this paper is especially suitable for computer numerical implementation. In the following a step-by-step procedure of obtaining the equations of motion of a multibody system is presented.

Step 1) Define system geometry
- Construct the path matrix
- Define reference body operator

Step 2) Form the transformation matrix [Eq. (11)]
Step 3) Build up the system mass matrix[Eq. (19)]
Step 4) Indentify external force and moment [Eqs. (15) and (17)]
Step 5) Define the constraint matrix[Eq. (22)]
Step 6) Set up augmented equations of motion [See Eq. (24)]
Step 7) Solve Eq. (24) for $\ddot{x}$ and λ and integrate

For a general purpose multibody simulation program, one may automate Steps 2), 3), 6), and 7). The system geometry, mass properties of each component bodies, and special constraints may be provided as input to the main program.

Conclusions

An efficient method of formulating the equations of motion of multibody systems has been presented. This method is especially suitable for computer numerical implementation. The use of path matrix and reference matrix was shown to expedite the development of a general purpose multibody simulation program. Also a step-by-step solution procedure was presented. The current method is belived to be applicable to the flexible multibody systems.

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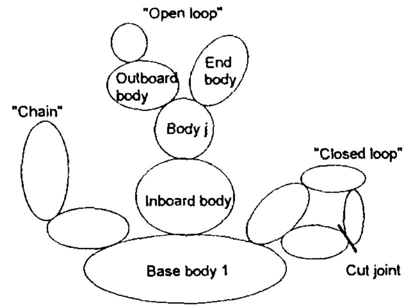

Fig. 1 Representative multibody systems

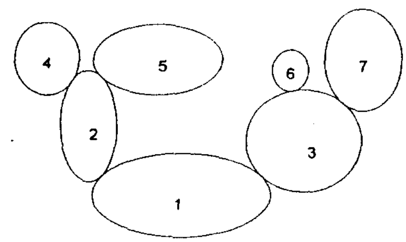

Fig. 2 Example of multibody systems

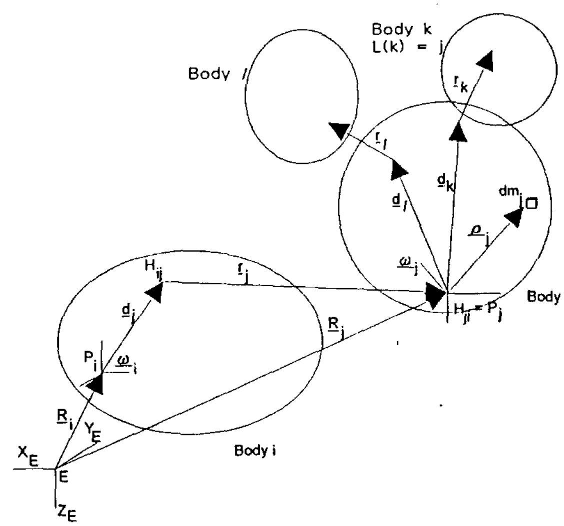

Fig. 3 Mathematical model of component body