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# Robust Control of Linear Systems Under Structured Nonlinear Time-Varying Perturbations I : Analysis

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Abstract

In this paper robust stability conditions are obtained for linear dynamical systems under structured nonlinear time-varying perturbations, using absolute stability theory and the concept of dissipative systems. The conditions are expressed in terms of solutions to linear matrix inequality(LMI). Based on this result, a synthesis methodology is developed for robust feedback controllers with worst-case $H_2$ performance via convex optimization and LMI formulation.

## 1 Introduction

In the analysis and synthesis of feedback control systems, it is important not only to determine the stability and performance properties of the nominal closed-loop system, but also to guarantee that such properties are achieved for an entire set of neighboring plants that arise from inevitable presence of modeling errors and plant uncertainties [29]. Thus, robust stability and performance has attracted considerable interest, and depending on how the set of plant perturbations are defined, various results have been proposed to check such properties. Much research in robust analysis has been conducted for problems with respect to norm bounded perturbation which are either unstructured or structured and modeled as otherwise unknown transfer functions. In particular, $\mathcal{H}_\infty$ theory has been recognized as an important tool for guaranteeing robust stability with respect to unstructured uncertainties. However, $\mathcal{H}_\infty$ design may lead to conservatism when uncertainty is known to be structured. Structured singular value($\mu$) analysis developed by Doyle is frequently applied to reduce significantly the conservatism [10, 11, 12, 29].

Recently, robust analysis and synthesis tools that are less conservative for real parameter uncertainties, are developed by Safonov [12, 13] using absolute stability theory and by How [7, 8, 15] using the combination of absolute stability theory and the concepts of dissipative dynamical systems. The stability condition in [7, 8, 15] is established for system under nonlinear uncertainty model with linear uncertainty as special case of this much broader class, and is formulated in state space in terms of Riccati and Lyapunov equations, whereas robust stability condition in [12, 13] involves complex diagonal multipliers acting on a positive-real, bilinearly-transformed system. The corresponding frequency domain stability test in [7, 8, 15] gives an interesting connection to an upperbound of $\mu$ with mixed real and complex linear time-invariant perturbations [32]. The analysis tool developed in [7, 8, 15], however, is restricted to structured time-invariant nonlinear perturbations.

In this paper, robust stability conditions are developed for structured nonlinear time-varying perturbations. This paper heavily relies on the recent results reported in [7, 8, 15]. It is found that stability conditions established in [7, 8, 15] is a special case of those developed in this paper. Furthermore, our result goes beyond those of [7, 8, 15] by considering nonmonotonic nonlinear uncertainty with improved stability condition and nonlinearity with saturation. Instead of employing Riccati equations as done in [7, 8, 15], the robust stability conditions derived in this paper will be stated in terms of LMI, which is in our opinion more natural and could provide a valuable alternative to analysis and synthesis of robust control [27, ?, 29, 30, 31, 36, 37]. Our paper also differs from [7, 8, 15] in that convex optimization and LMI approaches are used in the synthesis of feedback control with robust $\mathcal{H}_2$ performance under structured nonlinear time-varying perturbations, extending the previous result of ours on mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust control design [27]. A closely related analysis problem is considered recently by Rantzer [38] in the frequency domain.

Notation used in this paper is fairly standard. For a given matrix $A$, $A'$ and $tr(A)$ denotes its transpose and trace, respectively. If $A$ and $B$ are hermitian matrices, $A \geq B$ (resp. $A > B$) denotes $A - B$ positive definite(resp., definite). The Hardy space $\mathcal{H}_2$(resp., $\mathcal{H}_\infty$) consists of matrix-valued functions that are square integrable(essentially bounded) on the imaginary axis with analytic continuation into the right-half plane. The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms are defined as

$$\|G(s)\|_2 := \{(\frac{1}{2\pi})\int_{-\infty}^{\infty} tr[G^*(j\omega)G(j\omega)]d\omega\}^{1/2}$$
$$\|G(s)\|_\infty := \sup_{\omega\in\Re} \sigma_{max}\{G(j\omega)\}$$

Let

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

denote a state space realization of a transfer function $G(s) = C(sI - A)^{-1}B + D$. A square transfer function $G(s)$ is said to be *positive real* if: (1) All poles of $G(s)$ in the closed left half plane, and (2) $G(s) + G^*(s) \geq 0$ for $Re[s] > 0$. A square transfer function $G(s)$ is said to be *strictly positive real* if: (1) $G(s)$ is asymptotically stable (2) $G(j\omega) + G^*(j\omega) \geq 0$ for all real $\omega$. A square transfer function $G(s)$ is *strongly positive real* if it is strictly positive real and $D + D' > 0$ where $D = G(\infty)$. A minimal realization of a positive real transfer function is known to be stable in the sense of Lyapunov, and a strictly positive real transfer function is asymptotically stable.

## 2 Formulation of Robust Stability Problem

We consider robust control analysis problem with setup shown in Figure 1. In this figure $\Delta = diag(\Delta_1, \Delta_2, ..., \Delta_m)$ consists of $m$

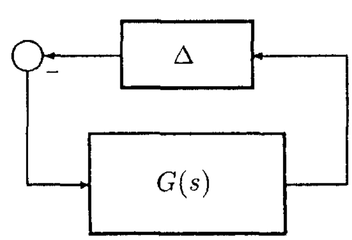


Figure 1: Robust control analysis framework

diagonal memoryless nonlinear time-varying (NLTV) elements, while $G(s)$ is an LTI feedback interconnection matrix with state space representation $G(s) \sim \left[\begin{array}{c|c} A & B_0 \\ \hline C_0 & 0 \end{array}\right]$. It is well known that nonlinear elements arising at different locations in the feedback loop can be put in this framework [29].

Robust stability problem considered in this paper is to determine conditions for stability of the feedback interconnection shown in Figure 1 for all time-varying memoriless nonlinearities $\Delta_i$ within certain classes defined precisely in the sequel. The nonlinearities of interest in this paper are real, continous, single-valued scalar functions, and are denoted by $f_i(.,t)$. We will impose restriction on the rate of time-variation of the nonlinearities. In general this can be described in several ways. In this paper, we use

$$\int_0^{\sigma_i} \frac{\partial f_i(\sigma_i,t)}{\partial t} d\sigma_i \leq \epsilon_i \sigma_i f_i(\sigma_i,t), \ \forall i, \ \epsilon_i \geq 0 \quad (2.1)$$

We shall consider several classes of nonlinearities as follows:

1. **Time-Varying Monotonic Nonlinearities.**
This class of nonlinearities satisfies the following relations

$$f_i(0,t) = 0, \ \forall t \geq 0, \forall i \quad (2.2)$$
$$m_{li} \leq \frac{df_i(t,\sigma)}{d\sigma} < m_{ui}, \ \forall \sigma, \forall t \geq 0, \forall i \quad (2.3)$$

where $m_{li}$ and $m_{ui}$ represents upper and lower bound on the slope restrictions(see Figure 2). Such a class of nonlinearities will be denoted by $\mathcal{N}_m$.

2. **Time-Varying Odd Monotonic Nonlinearities.**
In addition to (2.1)- (2.3), this class of nonlinearities also satisfies

$$f_i(-\sigma,t) = -f_i(\sigma,t), \ \forall t \geq 0, \forall i \quad (2.4)$$

Such a class of nonlinearities will be denoted by $\mathcal{N}_{om}$.

3. **Time-Varying Odd Monotonic Nonlinearities with Saturations.**
In addition to (2.1)-(2.4), this class of nonlinearities also satisfies

$$\sigma \frac{d^2 f_i(\sigma,t)}{d\sigma^2} \leq 0, \ \forall \sigma, \forall t \geq 0, \forall i \quad (2.5)$$

It is shown in [21, 25], that such nonlinearities satisfy

$$\begin{aligned} &\theta_i \sigma_1 f_i(\sigma_1,t) + [1-\theta_i]\sigma_2 f_i(\sigma_2,t) \pm [\sigma_2 f_i(\sigma_1) \\ &-\sigma_1 f_i(\sigma_2,t)] \geq 0, \ \forall \sigma_1,\sigma_2,\forall t, 0 \leq \theta_i \leq 1, \forall i \end{aligned} \quad (2.6)$$

Such a class of nonlinearities will be denoted by $\mathcal{N}_{oms}$.

4. **Time-Varying Nonmonotone Nonlinearities.**
In addition to (2.1) and (2.2), the functions $f_i(\sigma,t)$ are assumed to satisfy

$$\begin{aligned} &(\sigma_1 - \sigma_2)[f_i(\sigma_1,t) - f_i(\sigma_2,t)] \geq -(\lambda_i - 1)[\sigma_1 f_i(\sigma_1,t) \\ &+\sigma_2 f_i(\sigma_2,t)], \ \forall \sigma_1,\sigma_2,\forall t,\forall \lambda_i \geq 1, \forall i \end{aligned} \quad (2.7)$$

These functions reduce to monotonic nonlinearities if $\lambda_i = 1$ [20]. Such a class of nonlinearities will be denoted by $\mathcal{N}_{nm}$.

The reason for considering more restricted classes of nonlinearities, e.g. the classes $\mathcal{N}_{om}$ and $\mathcal{N}_{oms}$ compared to the class $\mathcal{N}_m$, is that we would like to reduce the conservatism of the robust stability test by enlarging the class of the multipliers. Thus, for a more restricted class of nonlinearity, the condition imposed on the LTI system is expected to be loosened. In some cases, however, one should be able to handle a more general class of nonlinearities. This is the reason why the general class of time-varying nonmonotone nonlinearities, i.e. the class $\mathcal{N}_{nm}$, is also addressed in this paper.

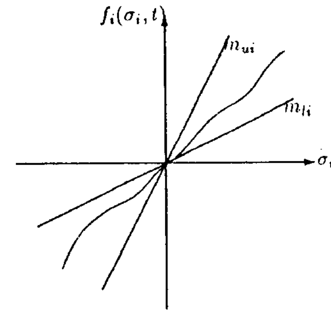


Figure 2: Nonlinearities with bounded sector $[m_{li}, m_{lu}]$.

## 3 Dissipative Dynamical Systems

Let us consider a dynamic system $\Sigma$ of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (3.1)$$
$$y(t) = f(x(t),t) + Du(t) \quad (3.2)$$

where $u(t) \in \Re^m, y(t) \in \Re^l$, and $x(t) \in \Re^n$. Along with the dynamical system $\Sigma$, suppose that there is given a function $\mathcal{V} : \Re^n \times \Re^+ \rightarrow \Re$, called *storage function*, and a function $\mathcal{W} : \Re^l \times \Re^m \times \Re^+ \rightarrow \Re$, called the *supply rate*. Assume that for any $u(.)$ and $x(.)$

$$\int_{t_0}^{t_1} |\mathcal{W}(t)| dt < \infty \quad \text{for all } t_0, t_1 > 0,$$

i.e. $\mathcal{W}$ is locally integrable.

**Definition 3.1** *(Willems[2, 5]) The triple $\{\Sigma, \mathcal{W}, \mathcal{V}\}$ defines a* dissipative *dynamical system if*

*1. $\mathcal{V}(x,t) \geq 0$ for all $t \geq 0$ and for all $x(.)$ satisfying (3.1),*

*2. the following* dissipation inequality*(DI) is satisfied,*

$$\mathcal{V}(x(t_1),t_1) - \mathcal{V}(x(t_0),t_0) \leq \int_{t_0}^{t_1} \mathcal{W}(y(t),u(t),t)dt, \quad (3.3)$$

*for all $t_0, t_1$, and for all $x(.)$, $u(.)$, and $y(.)$ satisfying (3.1) and (3.2).*

*When $\mathcal{V}(x(t),t)$ is differentiable, then DI reduces to*

$$\dot{\mathcal{V}}(x(t),t) \leq \mathcal{W}(y(t),u(t),t), \quad t \geq 0 \quad (3.4)$$

*with $\dot{\mathcal{V}}(x(t),t)$ a total derivative of $\mathcal{V}(x(t),t)$ along the state trajectory $x(t)$.*

As shown in [2, 3, 4, 5], an appropriate supply rate for testing passivity of a system $y = G(s)u$ is $\mathcal{W}(y,u) = u'y$, while for small gain theorem is $\mathcal{W}(y,u) = u'u - \gamma^{-2}y'y$, for the latter results in $\|G(s)\|_\infty < \gamma$.

The concepts of supply rate and storage function are very useful in checking the stability of interconnected systems. If, for each subsystem it can be found that there exists a storage function which is dissipative with respect to an appropriate supply rate, then these functions can be combined to form a Lyapunov function for the interconnected systems. Let us illustrate this concept via a special case of interconnected systems that is relevant in robust stability problem stated in previous section. Let $\Sigma_1$ and $\Sigma_2$ be two dynamical systems interconnected as in Figure 3. Suppose $v_1 = v_2 = 0$. Let $\Sigma_{12}$ be the resulting interconnected system. Let $x_1$ and $x_2$ be the states associated with $\Sigma_1$ and $\Sigma_2$, respectively and assume that $\{\Sigma_1, \mathcal{V}_1, \mathcal{W}_1\}$ and

$\{\Sigma_2, \mathcal{V}_2, \mathcal{W}_2\}$ are both dissipative. Then as shown in [2, 5, 7, 8, 15] $\{\Sigma_{12}, \mathcal{V}_1 + \mathcal{V}_2, \mathcal{W}_1 + \mathcal{W}_2\}$ is also dissipative. In many cases, the supply rates satisfy $\mathcal{W}_1(y_1, u_1) + \mathcal{W}_2(y_2, u_2) = 0$, for all $u_1 = -y_2$ and $u_2 = y_1$, which implies that the solution $(x_1, x_2) = 0$ of the feedback interconnection $\Sigma_{12}$ is Lyapunov stable with Lyapunov function $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2$ [2, 5, 7, 8, 15]. We will apply this result in deriving conditions for stability of the setup shown in Figure 1. Extension of the above result to the case where a single

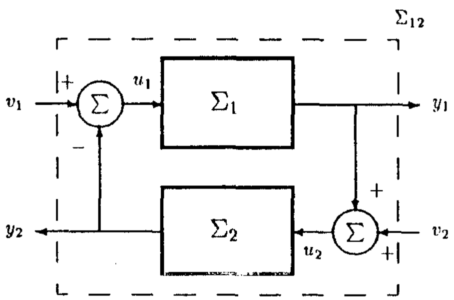


Figure 3: Interconnected systems under consideration

LTI system $G(s)$ is interconnected to $m$ independent systems, is discussed in [7, 8, 15].

# 4 Robust Stability Conditions For Nonlinear Time-Varying Uncertainties

In this section we develop robust stability condition for LTI system with scalar nonlinear time-varying functions, as illustrated in Figure 1. As discussed in Section 2, $G(s)$ is an LTI system and $\Delta := f(.,t) = diag[f_1(y_1,t), ..., f_m(y_m,t)]$ consists of $m$ decoupled nonlinear functions. Robust stability for such system will be developed by combining results from absolute stability theory and the concepts of supply rates and storage functions, a technique initially introduced in [7, 8, 15]. Absolute stability theory, which is discussed in Popov [24], among others, concerns with determining stability of an LTI system coupled with a nonlinear feedback which is either time-invariant or time-varying. The well known results in this theory is the frequency domain criterion which states that if there exists a multiplier $Z(s)$ belonging to a particular class of function $\mathcal{Z}_c$ associated with a class of non-linearities $\mathcal{F}_c$ such that $Re[Z(j\omega)G(j\omega)] \geq 0, \forall \omega \in [0, \infty)$, then the system containing the linear part transfer function $G(s)$ having all the poles in the open left half plane and a nonlinearity belonging to $\mathcal{F}_c$ in cascade, in a negative feedback loop, is absolutely stable. These criteria, however, suffer from a major drawback in that they are graphical in nature, and thus difficult to apply. Many works have been done with the aim at broadening the class of multiplier functions by restricting the nonlinearity to be monotonic, odd monotonic etc. [21, 22, 25].

## 4.1 Time-Varying Monotonic Nonlinearities

In this subsection robust stability is developed for linear system $G(s)$ coupled with time-varying monotonic nonlinearities, i.e. nonlinearities in the class $\mathcal{N}_m$. Since decoupled nonlinearities are considered in this paper, the corresponding multipliers will take the form $W(s) = diag(W_1(s), ..., W_m(s))$. In view of the results presented in [19, 22, 7, 8, 15], the appropriate frequency dependent multiplier associated with nonlinearities in the class $\mathcal{N}_m$ would be of the form

$$W_i(s) = \alpha_{i0} + \epsilon_i\beta_{i0} + \beta_{i0}s + \sum_{j=1}^{m_{i1}} \alpha_{ij}\left(1 - \frac{\alpha_{ij}}{\beta_{ij}(s + \eta_{ij} + \epsilon_i)}\right) \quad (4.1)$$

$$\alpha_{ij}, \beta_{ij}, \eta_{ij} \geq 0, \quad \eta_{ij}\beta_{ij} - \alpha_{ij} \geq 0 \quad (4.2)$$

The class of multipliers that has the form (4.1) and that satisfies (4.2) will be denoted by $\mathcal{M}_m$. For convenience in stating the main result of this section, let us represent the bounds on the nonlinearity sector in terms of diagonal matrices as follows:

$$M_U = M = diag(M_{11}, ..., M_{mm}) = diag(m_{u1}, ..., m_{um}) \in \Re^{m \times m}$$
$$M_L = diag(m_{l1}, ..., m_{lm}) \in \Re^{m \times m}$$

Prior to presenting the main theorem in this section, let us define

$$\hat{C}_i := \begin{bmatrix} \frac{\alpha_{i1}}{\beta_{i1}} \\ \frac{\alpha_{i2}}{\beta_{i2}} \\ \vdots \\ \frac{\alpha_{im_{i1}}}{\beta_{im_{i1}}} \end{bmatrix} (C_0)_i, \quad \hat{M}_i := \begin{bmatrix} \frac{\alpha_{i1}}{\beta_{i1}} \\ \frac{\alpha_{i2}}{\beta_{i2}} \\ \vdots \\ \frac{\alpha_{im_{i1}}}{\beta_{im_{i1}}} \end{bmatrix} (M^{-1})_i$$

$$A_i = diag(-(\eta_{ij} + \epsilon_i)), \quad j = 1, ..., m_{i1}$$

where $(.)_i$ denotes the $i^{th}$ row of $(.)$.

Following [7, 8, 15] let us further define matrices which corresponds to the $(A, B, C)$ matrices for the system dynamics resulting from the augmentation of the multiplier to a shifted system,

$$A_a = \begin{bmatrix} A & 0 & 0 & \dots & 0 \\ \hat{C}_1 & A_1 & 0 & & 0 \\ \hat{C}_2 & 0 & A_2 & & 0 \\ \vdots & & & \ddots & \\ \hat{C}_m & 0 & 0 & & A_m \end{bmatrix}, \quad B_a = \begin{bmatrix} B_0 \\ \hat{M}_1 \\ \hat{M}_2 \\ \vdots \\ \hat{M}_m \end{bmatrix}$$

$$C_a = [\; C_0 \;\; 0 \;\; 0 \;\; \dots 0 \;\;]$$

Let $m_1 = max_i(m_{i1})$. For each $i = 1, ..., m$ and $j = 1, ..., m_1$, let $\alpha_{ij} = 0, \beta_{ij} = 0, \eta_{ij} = 0$, and $R_{ij} = 0$ if $j > m_{i1}$. Next, define

$$\begin{aligned} H_j &= diag(\alpha_{1j}, ..., \alpha_{mj}), \quad j = 0, 1, ..., m_1 \\ N_j &= diag(\beta_{1j}, ..., \beta_{mj}), \quad j = 0, 1, ..., m_1 \\ S_j &= diag(\eta_{j1}, ..., \eta_{jm}), \quad j = 0, 1, ..., m_1 \\ Q_0 &= diag(\epsilon_1, ..., \epsilon_m) \\ R_j &= [\; R'_{1j}, R'_{2j}, ..., R'_{mj} \;]' \end{aligned}$$

where $R_{ij}$ are matrices with zero elements except the $(\sum_{l=1}^{i-1} m_{l1} + j)^{th}$-term, which is 1. Now, we are ready to present the main results of this section, which can be viewed as generalization of Theorem 4.1 in [7, 8, 15].

**Theorem 4.1** *Sufficient condition for asymptotic stability of interconnection of the system $G(s)$ and the nonlinearities $f_i(.,t) \in \mathcal{N}_m, i = 1, ..., m$, is that there exist multipliers $W_i(s) \in \mathcal{M}_m$ such that the following LMI*

$$\begin{bmatrix} A'_a P + P A_a & P B_a - \hat{C}'_a \\ (P B_a - \hat{C}'_a)' & -(\hat{R} + \hat{R}') \end{bmatrix} < 0 \quad (4.3)$$

*has a symmetric positive definite solution $P > 0$, where $\hat{C}_a$ and $\hat{R}$ are defined by*

$$\hat{C}_a := (H_0 C_a + N_0 Q_0 C_a + N_0 C_a A_a + \sum_{j=1}^{m_1} H_j (C_a - R_j)) \quad (4.4)$$

$$\hat{R} := (N_0 C_a B_a + N_0 Q_0 M^{-1} + \sum_{j=0}^{m_1} H_j M^{-1}) \quad (4.5)$$

Before proving this theorem, we present an intermediate result concerning with the storage functions and supply rates for the system dynamics $\mathcal{G}_i$ with state space representation

$$\dot{x}_i(t) = A_i x_i(t) + B_i y_i(t) \quad (4.6)$$
$$-u_i(t) = g_i(x_i(t), t), \quad (4.7)$$

that are constructed by combining $f_i(.,t)$ and $W_i^{-1}(s)$ in case $M_L = 0$(refer to Figure 4). The proof is based on the idea and procedures proposed in [7, 8, 15].

**Proposition 4.1** *Consider a differentiable monotonic time-varying nonlinear functions $f_i(.,t) \in \mathcal{N}_m$, along with multipliers $W_i(s) \in \mathcal{M}_m$. Define a shifted nonlinearity $\tilde{f}_i(\tilde{y}_i,t)$ as*

$$\frac{\tilde{f}_i(\tilde{y}_i,t)}{\tilde{y}_i} = \frac{f_i(y_i,t)/y_i}{1 - M_{ii}^{-1} f_i(y_i,t)/y_i} \qquad (4.8)$$

*Then, the triple $\{\mathcal{G}_i, \mathcal{V}_i, \mathcal{W}_i\}$ is dissipative, with storage function $\mathcal{V}_i$ and supply rate $\mathcal{W}_i$ given by*

$$\begin{aligned}\mathcal{V}_i(\tilde{y}_i, z_{i1}, \dots, z_{im_{i1}}) &= \beta_{i0}\left(\textstyle\int_0^{\tilde{y}_i} \tilde{f}_i(\sigma,t)d\sigma + \frac{1}{2}M_{ii}^{-1}u_i^2\right) \\ &\quad + \textstyle\sum_{j=1}^{m_{i1}} \beta_{ij} \int_0^{z_{ij}} \tilde{f}_i(\sigma,t)d\sigma \qquad (4.9)\\ \mathcal{W}_i(\bar{y}_i,u_i) &= [\textstyle\sum_{j=1}^{m_{i1}} \alpha_{ij}(\tilde{y}_i - z_{ij}) + \beta_{i0}\dot{y}_i + (\alpha_{i0} + \beta_{i0}\epsilon_i)\tilde{y}_i]u_i, \qquad (4.10)\end{aligned}$$

*where the signals $\tilde{y}_i$, $z_{ij}$ and $\bar{y}_i$ are given by*

$$\begin{aligned}\tilde{y}_i &= y_i - M^{-1}\tilde{f}_i(\tilde{y}_i,t) &(4.11)\\ z_{ij} &= \frac{\alpha_{ij}\tilde{y}_i}{\beta_{ij}(s + \eta_{ij} + \epsilon_i)} &(4.12)\\ \bar{y}_i &= W_i(s)\tilde{y}_i + \beta_{i0}M_{ii}^{-1}su_i, &(4.13)\end{aligned}$$

*respectively.*

*Proof*: In the following we apply three major steps from [7, 8, 15]: loop transformation, multiplier augmentation and checking dissipativeness.

*Step 1 : Loop Transformation.* To apply absolute stability theory, a loop-transformation is needed to convert the sector restriction $[0, M_U]$ to an infinite sector $[0, \infty)$ [23]. Such a loop transformation is depicted in Figure 4.

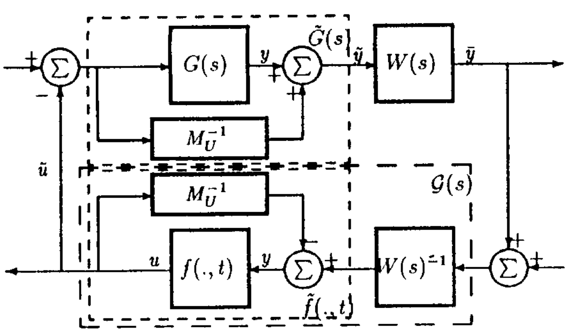

Figure 4: Loop transformation which converts bounded sector $[0, M_U]$ to to infinite one $[0, \infty)$, followed by multiplier augmentation ;

For each nonlinearity $f_i(y_i,t)$, with $y_i \neq 0$, let us define $\tilde{f}_i(\tilde{y}_i,t)$ as in equation (4.8). It has been shown in [7, 8, 15] that $\tilde{f}_i(.,t)$ is indeed differentiable and satisfies $0 \leq \frac{d\tilde{f}_i(\sigma,t)}{d\sigma}$, and hence is monotonic. Thus, the following conditions hold for this shifted system [25, 21]

$$\sigma\tilde{f}_i(\sigma,t) \geq 0, \;\; \forall\sigma \in \Re, \;\; \forall t, \;\; \forall i \qquad (4.14)$$

$$(\tilde{f}_i(\sigma_1,t) - \tilde{f}_i(\sigma_2,t))(\sigma_1 - \sigma_2) \geq 0, \forall\sigma_1,\sigma_2 \in \Re, \forall t, \forall i \qquad (4.15)$$

while restriction on the time-variation (2.1) reduces to

$$\int_0^{\sigma_i} \frac{\partial\tilde{f}_i(\sigma_i,t)}{\partial t} d\sigma_i \leq \epsilon_i\sigma_i\tilde{f}(\sigma_i,t), \;\; \epsilon_i \geq 0, \;\; \forall i \qquad (4.16)$$

Note that under the transformation the shifted LTI system $\tilde{G}(s)$ is given by

$$\tilde{G}(s) = G(s) + M^{-1} \qquad (4.17)$$

*Step2: Multiplier Augmentation.* It is pointed out in [7, 8, 15] that the expression $W_i(s)\tilde{y}_i$ contains terms of the form described in (4.12). Denoting $z_{ij}$ as the system states of $\mathcal{G}_i$, the augmentation of the dynamic of each term in the multiplier to the system can be done by representing (4.12) as [7, 8, 15]

$$\dot{z}_{ij} + (\eta_{ij} + \epsilon_i)z_{ij} = \frac{\alpha_{ij}}{\beta_{ij}}\tilde{y}_i = \frac{\alpha_{ij}}{\beta_{ij}}(C_0 x - M^{-1}u)_i \qquad (4.18)$$

where $(.)_i$ denotes the $i^{th}$ row of $(.)$.

*Step 3: Checking dissipativity.* It has been shown in [2, 3, 4] that the appropriate supply rate for checking system's passivity is the product of its inputs and outputs. Following [7, 8, 15], we consider output signal $\bar{y}_i$ of the form (4.13), which is actually the signal $\tilde{y}_i$ in Figure 4 added with signal of the form $\beta_{i0}M_{ii}^{-1}su_i$. This signal $\bar{y}_i$ can be represented as follows:

$$\begin{aligned}\bar{y}_i &= W_i(s)\tilde{y}_i + \beta_{i0}M_{ii}^{-1}su_i\\ &= \sum_{j=1}^{m_{i1}} \alpha_{ij}\left(1 - \frac{\alpha_{ij}}{\beta_{ij}(s + \eta_{ij} + \epsilon_i)}\right)\tilde{y}_i + (\alpha_{i0} + \epsilon_i\beta_{i0} + \beta_{i0}s)\\ &\quad \times(y - M^{-1}u)_i + \beta_{i0}M_{ii}^{-1}su_i\\ &= \sum_{j=1}^{m_{i1}} \alpha_{ij}(\tilde{y}_i - z_{ij}) + (\alpha_{i0} + \epsilon_i\beta_{i0} + \beta_{i0}s)y_i\\ &\quad -(\alpha_{i0} + \epsilon_i\beta_{i0})M_{ii}^{-1}u_i\\ &= \sum_{j=1}^{m_{i1}} \alpha_{ij}(\tilde{y}_i - z_{ij}) + \beta_{i0}sy_i + \alpha_{i0}\tilde{y}_i + \epsilon_i\beta_{i0}\tilde{y}_i \qquad (4.19)\end{aligned}$$

In the above calculation, we have used the definition of $W_i(s)$ in (4.1), $\tilde{y}_i$ in (4.11), and $z_{ij}$ in (4.12).

Now, the supply rate $\mathcal{W}_i(\bar{y}_i, u_i)$ can be chosen as follows [7, 8, 15],

$$\mathcal{W}_i(\bar{y}_i,u_i) = [\sum_{j=1}^{m_{i1}} \alpha_{ij}(\tilde{y}_i - z_{ij}) + \beta_{i0}\dot{y}_i + \alpha_{i0}\tilde{y}_i + \epsilon_i\beta_{i0}\tilde{y}_i]u_i \qquad (4.20)$$

Note that $\mathcal{V}_i(0,t) = 0$ and $\mathcal{V}_i$ is nonnegative. By defining $u_{ij} = \tilde{f}_i(z_{ij},t)$ and $u_i = \tilde{f}_i(\tilde{y}_i,t)$, we obtain

$$\begin{aligned}\dot{\mathcal{V}}_i &= \sum_{j=1}^{m_{i1}} \beta_{ij}\tilde{f}_i(z_{ij},t)\dot{z}_{ij} + \sum_{j=1}^{m_{i1}} \beta_{ij}\int_0^{z_{ij}} \frac{\partial\tilde{f}_i(\sigma,t)}{\partial t}d\sigma + \beta_{i0}[\tilde{f}_i(\tilde{y}_i,t)\\ &\quad + M_{ii}^{-1}\tilde{f}_i(\tilde{y}_i,t)\frac{d\tilde{f}_i(\tilde{y}_i,t)}{dt} + \int_0^{\tilde{y}_i} \frac{\partial\tilde{f}_i(\sigma,t)}{\partial t}d\sigma]\\ &= \sum_{j=1}^{m_{i1}} \beta_{ij}u_{ij}\dot{z}_{ij} + \sum_{j=1}^{m_{i1}} \beta_{ij}\int_0^{z_{ij}} \frac{\partial\tilde{f}_i(\sigma,t)}{\partial t}d\sigma + \beta_{i0}[\dot{\tilde{y}}_i u_i\\ &\quad + M_{ii}^{-1}\frac{du_i}{dt}u_i + \int_0^{\tilde{y}_i} \frac{\partial\tilde{f}_i(\sigma,t)}{\partial t}d\sigma]\\ &= \sum_{j=1}^{m_{i1}} \beta_{ij}u_{ij}\dot{z}_{ij} + \sum_{j=1}^{m_{i1}} \beta_{ij}\int_0^{z_{ij}} \frac{\partial\tilde{f}_i(\sigma,t)}{\partial t}d\sigma + \beta_{i0}[u_i\dot{y}_i\\ &\quad + \int_0^{\tilde{y}_i} \frac{\partial\tilde{f}_i(\sigma,t)}{\partial t}d\sigma] \qquad (4.21)\end{aligned}$$

where in the last equation we have used the definition of $\tilde{y}_i$ described in (4.11). From equations (4.2), and (4.14)-(4.16), the following inequalities can be infered for $i = 1, \dots, m$ and $j = 1, \dots, m_{i1}$,

$$0 \;\leq\; \alpha_{i0}\tilde{y}_i u_i \qquad (4.22)$$

$$0 \leq \alpha_{ij}(\tilde{y}_i - z_{ij})(u_i - u_{ij}) \quad (4.23)$$
$$0 \leq (\eta_{ij}\beta_{ij} - \alpha_{ij})z_{ij}u_{ij} \quad (4.24)$$
$$0 \leq [-\beta_{i0}\int_0^{\tilde{y}_i}\frac{\partial \tilde{f}_i(\sigma,t)}{\partial t}d\sigma + \beta_{i0}\epsilon_i\tilde{y}_i u_i] \quad (4.25)$$
$$0 \leq [-\sum_{j=1}^{m_{i1}}\beta_{ij}\int_0^{z_{ij}}\frac{\partial \tilde{f}_i(\sigma,t)}{\partial t}d\sigma + \sum_{j=1}^{m_{i1}}\beta_{ij}\epsilon_i z_{ij}u_{ij}] \quad (4.26)$$

Adding equations (4.22)-(4.26) to equation (4.21), yields

$$\dot{\mathcal{V}}_i \leq \sum_{j=1}^{m_{i1}} u_{ij}(\beta_{ij}\dot{z}_{ij} + (\eta_{ij}\beta_{ij} - \alpha_{ij})z_{ij}) + \sum_{j=1}^{m_{i1}}\alpha_{ij}(\tilde{y}_i - z_{ij})(u_i - u_{ij}) + (\beta_{i0}\dot{y}_i + \alpha_{i0}\tilde{y}_i)u_i + \beta_{i0}\epsilon_i\tilde{y}_i u_i + \sum_{j=1}^{m_{i1}}\beta_{ij}\epsilon_i z_{ij}u_{ij}$$

By collecting terms and using relation (4.18), we finally have

$$\dot{\mathcal{V}}_i \leq [\sum_{j=1}^{m_{i1}}\alpha_{ij}(\tilde{y}_i - z_{ij}) + \beta_{i0}\dot{y}_i + \alpha_{i0}\tilde{y}_i + \beta_{i0}\epsilon_i\tilde{y}_i]u_i = \mathcal{W}_i(\tilde{y}_i, u_i)$$

Now, it follows from Definition 3.1, that the triple $\{\mathcal{G}_i, \mathcal{V}_i, \mathcal{W}_i\}$ corresponding to $f_i(.,t) \in \mathcal{N}_m$ and $W_i(s) \in \mathcal{M}_n$, is dissipative with storage function $\mathcal{V}_i$ given in (4.9) and supply rate $\mathcal{W}_i$ given in (4.10). □

*Proof of Theorem 4.1*: The proof of this theorem follows from the techniques introduced in [7, 8, 15] by employing the result presented in Proposition 4.1, the discussion concerning stability of interconnected systems at the end of Section 3, and the Schur complement formula for matrix inequalities. □

### 4.2 Time-Varying Odd Monotonic Nonlinearities

In this subsection, robust stability is considered for a LTI coupled to time-varying odd monotonic nonlinearities, i.e. time-varying nonlinearities in the class $\mathcal{N}_{om}$. Multipliers associated with such nonlinearities have the form [19, 17, 7, 8, 15]

$$W_i(s) = \alpha_{i0} + \epsilon_i\beta_{i0} + \beta_{i0}s + \sum_{j=1}^{m_{i1}}\alpha_{ij}(1 - \frac{\alpha_{ij}}{\beta_{ij}(s+\eta_{ij}+\epsilon_i)}) + \sum_{j=m_{i1}+1}^{m_{i2}}\alpha_{ij}(1 + \frac{\alpha_{ij}}{\beta_{ij}(s+\eta_{ij}+\epsilon_i)}) \quad (4.27)$$
$$\alpha_{ij}, \beta_{ij}, \eta_{ij} \geq 0, \;\; \eta_{ij}\beta_{ij} - \alpha_{ij} \geq 0 \quad (4.28)$$

with $m_{i2} \neq m_{i1}$. The class of multipliers that has the form (4.27) with $m_{i2} \neq m_{i1}$ and that satisfies (4.28) will be denoted by $\mathcal{M}_{om}$. As expected the class of multipliers $\mathcal{M}_{om}$ is more general than the class $\mathcal{M}_m$. The following holds for nonlinearities in the class $\mathcal{N}_{om}$, [21, 25],

$$0 \leq \sigma_1 f_i(\sigma_1,t) + \sigma_2 f_i(\sigma_2,t) + \sigma_1 f_i(\sigma_2,t) - \sigma_2 f_i(\sigma_1,t) \quad \forall\sigma_1,\sigma_2 \in \Re, \forall t \geq 0, \forall i \quad (4.29)$$

We then have the following theorem concerning the stability of the interconnected system, stated by extending the definition of matrices $H_j, N_j, S_j$ and $R_j$ in Section 4.1 to also include terms having indices $m_1+1, ..., m_2$ with $m_2 = \max_i(m_{i2})$.

**Theorem 4.2** *Sufficient condition for asymptotic stability of the negative feedback interconnection of the system $G(s)$ and the nonlinearities $f_i(.,t) \in \mathcal{N}_{om}, i = 1,...,m$, is that there exist multipliers $W_i(s) \in \mathcal{M}_{om}$ such that the following LMI*

$$\begin{bmatrix} A_a'P + PA_a & PB_a - \hat{C}_a' \\ (PB_a - \hat{C}_a')' & -(\hat{R} + \hat{R}') \end{bmatrix} < 0 \quad (4.30)$$

*has a symmetric positive definite solution $P > 0$, where $\hat{C}_a$ and $\hat{R}$ are defined by*

$$\hat{C}_a := (H_0C_a + N_0Q_0C_a + N_0C_aA_a + \sum_{j=1}^{m_1}H_j(C_a - R_j) + \sum_{j=m_1+1}^{m_2}H_j(C_a + R_j)) \quad (4.31)$$
$$\hat{R} := (N_0C_aB_a + N_0Q_0M^{-1} + \sum_{j=0}^{m_2}H_jM^{-1}) \quad (4.32)$$

The proof of Theorem 4.2 will be based on the following proposition (see [7, 8, 15] for the case of time-invariant nonlinearities).

**Proposition 4.2** *Consider a differentiable odd monotonic time-varying nonlinear functions $f_i(.,t) \in \mathcal{N}_{om}$, along with multipliers $W_i(s) \in \mathcal{M}_{om}$. Define a shifted nonlinearity $\tilde{f}_i(\tilde{y}_i,t)$ as in equation (4.8). Then, the triple $\{\mathcal{G}_i, \mathcal{V}_i, \mathcal{W}_i\}$ is dissipative, with storage function $\mathcal{V}_i$ and supply rate $\mathcal{W}_i$ given by*

$$\mathcal{V}_i(\tilde{y}_i, z_{i1}, ..., z_{im_{i1}}) = \beta_{i0}(\int_0^{\tilde{y}_i}\tilde{f}_i(\sigma,t)d\sigma + \tfrac{1}{2}M_{ii}^{-1}u_i^2) + \sum_{j=1}^{m_{i2}}\beta_{ij}\int_0^{z_{ij}}\tilde{f}_i(\sigma,t)d\sigma \quad (4.33)$$
$$\mathcal{W}_i(\tilde{y}_i, u_i) = [\sum_{j=1}^{m_{i1}}\alpha_{ij}(\tilde{y}_i - z_{ij}) + \sum_{j=m_{i1}+1}^{m_{i2}}\alpha_{ij}(\tilde{y}_i + z_{ij}) + \beta_{i0}\dot{y}_i + (\alpha_{i0} + \beta_{i0}\epsilon_i)\tilde{y}_i]u_i \quad (4.34)$$

*where $\tilde{y}_i$, $z_{ij}$ and $\bar{y}_i$ are given by equations (4.11), 4.12) and (4.13), respectively.*

*Proof:* The loop transformation is done in the same way as those of Proposition 4.1. The establishment of the supply rate (4.34) follows from similar technique adopted in the proof of Proposition 4.1, while noting that now $m_{i1} \neq m_{i2}$.

Using the result of Proposition 4.1, in particular and expression for $\dot{\mathcal{V}}_i$ and inequalities (4.26), we have

$$\dot{\mathcal{V}}_i \leq [\sum_{j=1}^{m_{i1}}\alpha_{ij}(\tilde{y}_i - z_{ij}) + \beta_{i0}\dot{y}_i + \alpha_{i0}\tilde{y}_i + \beta_{i0}\epsilon_i\tilde{y}_i]u_i + \sum_{j=m_{i1}+1}^{m_{i2}}\beta_{ij}u_{ij}\dot{z}_{ij} + \sum_{j=m_{i1}+1}^{m_{i2}}\beta_{ij}\epsilon_i z_{ij}u_{ij} \quad (4.35)$$

Note that inequality (4.29) yields

$$0 \leq \alpha_{ij}[(z_{ij} + \tilde{y}_i)u_i + (z_{ij} - \tilde{y}_i)u_{ij}] \quad (4.36)$$

Adding equations (4.24) and (4.36) to (4.35), yields

$$\begin{aligned}\dot{\mathcal{V}}_i \leq\; & [\sum_{j=1}^{m_{i1}}\alpha_{ij}(\tilde{y}_i - z_{ij}) + \beta_{i0}\dot{y}_i + \alpha_{i0}\tilde{y}_i + \beta_{i0}\epsilon_i\tilde{y}_i]u_i \\ & + \sum_{j=m_{i1}+1}^{m_{i2}}[\beta_{ij}u_{ij}\dot{z}_{ij} + \beta_{ij}\epsilon_i z_{ij}u_{ij} + \alpha_{ij}((z_{ij} + \tilde{y}_i)u_i \\ & + z_{ij} - \tilde{y}_i)u_{ij} + (\eta_{ij}\beta_{ij} - \alpha_{ij})z_{ij}u_{ij}]\end{aligned} \quad (4.37)$$

Now, replacing $\dot{z}_{ij}$ using equation (4.20), and canceling terms, equation (4.37) becomes

$$\dot{\mathcal{V}}_i \leq [\sum_{j=1}^{m_{i1}}\alpha_{ij}(\tilde{y}_i - z_{ij}) + \beta_{i0}\dot{y}_i + \alpha_{i0}\tilde{y}_i + \beta_{i0}\epsilon_i\tilde{y}_i + \sum_{j=m_{i1}+1}^{m_{i2}}\alpha_{ij}(\tilde{y}_i + z_{ij})]u_i = \mathcal{W}_i(\tilde{y}_i, u_i)$$

which shows that the the triple $\{\mathcal{G}_i, \mathcal{V}_i, \mathcal{W}_i\}$ corresponding to $f_i(.,t) \in \mathcal{N}_{om}$ and $W_i(s) \in \mathcal{M}_{om}$, is dissipative with storage function $\mathcal{V}_i$ given in (4.33) and supply rate $\mathcal{W}_i$ given in (4.34). □

*Proof of Theorem 4.2*: The proof can be verified in the same way as those of Theorem 4.1 using the result presented in Proposition 4.2. □

## 4.3 Time-Varying Odd Monotonic Nonlinearities with Saturations

In this subsection, we consider stability of a LTI system interconnected to time-varying monotonic nonlinearities with saturations, i.e. nonlinearities in the class $\mathcal{N}_{oms}$.

Being the subclass of odd monotonic nonlinearities, the stability criterion derived for the interconnected systems under time-varying monotonic nonlinearities with saturations, will further reduce conservatism of those derived for monotonic and odd monotonic nonlinearities. Following [21, 25], the appropriate multipliers for such nonlinearities are given by

$$W_i(s) = \alpha_{i0} + \epsilon_i\beta_{i0} + \beta_{i0}s + \sum_{j=1}^{m_{i1}} \alpha_{ij}(1 - \frac{\alpha_{ij}}{\beta_{ij}(s+\eta_{ij}+\epsilon_i)}) + \sum_{j=m_{i1}+1}^{m_{i2}} \alpha_{ij}(1 + (1-\tau_{ij}) + \frac{\alpha_{ij}}{\beta_{ij}(s+\tau_{ij}\eta_{ij}+\epsilon_i)}) \quad (4.38)$$

$$0 \le \tau_{ij} \le 1 \quad (4.39)$$

with the other parameters as defined in previous subsections. The class of multipliers that has the form (4.38) and that satisfies (4.28) and (4.39) will be denoted by $\mathcal{M}_{oms}$. As expected, the class of multipliers $\mathcal{M}_{oms}$ is more general than the classes $\mathcal{M}_m$ and $\mathcal{M}_{om}$.

**Theorem 4.3** *Sufficient condition for asymptotic stability of the negative feedback interconnection of the system $G(s)$ and the nonlinearities $f_i(.,t) \in \mathcal{N}_{oms}, i = 1,...,m$, is that there exist multipliers $W_i(s) \in \mathcal{M}_{oms}$ such that the following LMI*

$$\begin{bmatrix} A_a'P + PA_a & PB_a - \hat{C}_a' \\ (PB_a - \hat{C}_a')' & -(\hat{R} + \hat{R}') \end{bmatrix} < 0 \quad (4.40)$$

*has a symmetric positive definite solution $P > 0$, where $\hat{C}_a$ and $\hat{R}$ are defined by*

$$\hat{C}_a := (H_0C_a + N_0Q_0C_a + N_0C_aA_a + \sum_{j=1}^{m_1} H_j(C_a - R_j) + \sum_{j=m_1+1}^{m_2} H_j(C_a + (I - T_j)C_a + R_j)) \quad (4.41)$$

$$\hat{R} := (N_0C_aB_a + N_0Q_0M^{-1} + \sum_{j=0}^{m_1} H_jM^{-1} + \sum_{j=m_1+1}^{m_2} H_j(I + (I - T_j))M^{-1}) \quad (4.42)$$

*and where $T_j := diag(\tau_{1j}, ..., \tau_{mj})$.*

The proof of Theorem 4.3 relies on the following intermediate result.

**Proposition 4.3** *Consider a differentiable odd monotonic time-varying nonlinear functions $f_i(.,t) \in \mathcal{N}_{oms}$, along with multipliers $W_i(s) \in \mathcal{M}_{oms}$. Define a shifted nonlinearity $\tilde{f}_i(\tilde{y}_i, t)$ as in equation (4.8). Then, $\{\mathcal{G}_i, \mathcal{V}_i, \mathcal{W}_i\}$ is dissipative, with storage function $\mathcal{V}_i$ and supply rate $\mathcal{W}_i$ given by*

$$\mathcal{V}_i(\tilde{y}_i, z_{i1}, ..., z_{im_{i1}}) = \beta_{i0}(\int_0^{\tilde{y}_i} \tilde{f}_i(\sigma,t)d\sigma + \tfrac{1}{2}M_{ii}^{-1}u_i^2) + \sum_{j=1}^{m_{i2}} \beta_{ij}\int_0^{z_{ij}} \tilde{f}_i(\sigma,t)d\sigma \quad (4.43)$$

$$\mathcal{W}_i(\bar{y}_i, u_i) = [\sum_{j=1}^{m_{i1}} \alpha_{ij}(\tilde{y}_i - z_{ij}) + \sum_{j=m_{i1}+1}^{m_{i2}} \alpha_{ij}(1 - \tau_{ij}\tilde{y}_i + z_{ij}) + \beta_{i0}\dot{y}_i + (\alpha_{i0} + \beta_{i0}\epsilon_i)\tilde{y}_i]u_i \quad (4.44)$$

*where $\tilde{y}_i$ and $\bar{y}_i$ are given by equations (4.11) and (4.13), respectively, while $z_{ij}$ is now given by*

$$z_{ij} = \begin{cases} \frac{\alpha_{ij}\tilde{y}_i}{\beta_{ij}(s+\eta_{ij}+\epsilon_i)} & \text{if } j = 1,...,m_{i1} \\ \frac{\alpha_{ij}\tilde{y}_i}{\beta_{ij}(s+\tau_{ij}\eta_{ij}+\epsilon_i)} & \text{if } j = m_{i1+1},...,m_{i2} \end{cases} \quad (4.45)$$

*Proof:* The loop transformation is done in the same way as those of Proposition 4.1. The establishment of the supply rate (4.43) follows from similar technique shown in the proof of Proposition 4.1, while noting that now the multipliers belong to the class $\mathcal{M}_{oms}$.

In view of (2.6) and (4.39), the following inequalities hold,

$$(\eta_{ij}\beta_{ij} - \alpha_{ij})\tau_{ij}z_{ij}u_{ij} \ge 0 \quad (4.46)$$

$$\alpha_{ij}[(\tau_{ij}z_{ij} - \tilde{y}_i)u_{ij} + (1 - \tau_{ij})\tilde{y}_iu_i + z_{ij}u_i] \ge 0, \quad \forall 0 \le \tau_{ij} \le 1 \quad (4.47)$$

Next, using the result of Proposition 4.1 and inequalities (4.25) and (4.26), we have

$$\dot{\mathcal{V}}_i \le [\sum_{j=1}^{m_{i1}} \alpha_{ij}(\tilde{y}_i - z_{ij}) + \beta_{i0}\dot{y}_i + \alpha_{i0}\tilde{y}_i + \beta_{i0}\epsilon_i\tilde{y}_i]u_i + \sum_{j=m_{i1}+1}^{m_{i2}} \beta_{ij}u_{ij}\dot{z}_{ij} + \sum_{j=m_{i1}+1}^{m_{i2}} \beta_{ij}\epsilon_iz_{ij}u_{ij} \quad (4.48)$$

Adding equations (4.46) and (4.47) to (4.48) yields

$$\begin{aligned}\dot{\mathcal{V}}_i \le{} & \sum_{j=1}^{m_{i1}} \alpha_{ij}(\tilde{y}_i - z_{ij}) + \beta_{i0}\dot{y}_i + \alpha_{i0}\tilde{y}_i + \beta_{i0}\epsilon_i\tilde{y}_i]u_i \\ & + \sum_{j=m_{i1}+1}^{m_{i2}} [\beta_{ij}u_{ij}\dot{z}_{ij} + \beta_{ij}\epsilon_iz_{ij}u_{ij} + \alpha_{ij}[(\tau_{ij}z_{ij} - \tilde{y}_i)u_{ij} \\ & + (1-\tau_{ij})\tilde{y}_iu_i + z_{ij}u_i] + (\eta_{ij}\beta_{ij} - \alpha_{ij})\tau_{ij}z_{ij}u_{ij}]\end{aligned} \quad (4.49)$$

Now, replacing $\dot{z}_{ij}$ using equation (4.45) for $j = m_{i1+1},...,m_{i2}$, and canceling terms, equation (4.49) results in

$$\dot{\mathcal{V}}_i \le [\sum_{j=1}^{m_{i1}} \alpha_{ij}(\tilde{y}_i - z_{ij}) + \beta_{i0}\dot{y}_i + \alpha_{i0}\tilde{y}_i + \beta_{i0}\epsilon_i\tilde{y}_i + \sum_{j=m_{i1+1}}^{m_{i2}} \alpha_{ij}((1-\tau_{ij})\tilde{y}_i + z_{ij}]u_i = \mathcal{W}_i(\bar{y}_i, u_i) \quad (4.50)$$

which shows that the triple $\{\mathcal{G}_i, \mathcal{V}_i, \mathcal{W}_i\}$ corresponding to $f_i(.,t) \in \mathcal{N}_{oms}$ and $W_i(s) \in \mathcal{M}_{oms}$, is dissipative with storage function $\mathcal{V}_i$ given in (4.43) and supply rate $\mathcal{W}_i$ given in (4.44). □

*Proof of Theorem 4.3*: The proof can be verified in the same way as those of Theorem 4.1 in using the result presented in Proposition 4.3. □

## 4.4 Time-Varying Nonmonotone Nonlinearities

In this subsection we consider stability of a LTI system interconnected to time-varying nonmonotone nonlinearities belonging to the class $\mathcal{N}_{nm}$. The nonlinearities are assumed to be bounded in sector $[0, M]$, but not necessarily monotonic. This class of nonlinearities is more general than the class of monotone nonlinearities and all its subclasses, but more restricted than the class of all functions in sector $[0, M]$. For further properties of the class $\mathcal{N}_{nm}$ see [20]. In view of the results presented in [20], the multipliers for this class of nonlinearities are taken of the form

$$W_i(s) = \alpha_{i0} + \beta_{i0}\epsilon_is + \sum_{j=1}^{m_{i1}} \lambda_i\alpha_{ij}(1 - \frac{\alpha_{ij}}{\beta_{ij}(s + \eta_{ij} + \epsilon_i)}) \quad (4.51)$$

where $\lambda_i \ge 1$ and restriction on parameters (4.2) is replaced by

$$(\frac{\eta_{ij}\beta_{ij}}{\lambda_i^2 - \alpha_{ij}}) \ge 0 \quad (4.52)$$

The class of multipliers that has the form (4.51) and that satisfies (4.52) will be denoted by $\mathcal{M}_{nm}$. Let us define

$$L_0 := diag(\lambda_1, \ldots, \lambda_m) \qquad (4.53)$$

We have the following results concerning the stability of the interconnected system.

**Theorem 4.4** *Sufficient condition for asymptotic stability of the negative feedback interconnection of the system $G(s)$ and the nonlinearities $f_i(.,t) \in \mathcal{N}_{nm}, i = 1, \ldots, m$, is that there exist multipliers $W_i(s) \in \mathcal{M}_{nm}$ such that the following LMI*

$$\begin{bmatrix} A_a'P + PA_a & PB_a - \hat{C}_a' \\ (PB_a - \hat{C}_a')' & -(\hat{R} + \hat{R}') \end{bmatrix} < 0 \qquad (4.54)$$

*has a symmetric positive definite solution $P > 0$, where $\hat{C}_a$ and $\hat{R}$ are defined by*

$$\hat{C}_a := (H_0C_a + N_0Q_0C_a + N_0C_aA_a + \sum_{j=1}^{m_1} H_j(L_0C_a - R_j)) \quad (4.55)$$

$$\hat{R} := (N_0C_aB_a + N_0Q_0M^{-1} + H_0M^{-1} + \sum_{j=1}^{m_1} H_jL_0M^{-1}) \quad (4.56)$$

The proof of Theorem 4.2 relies on the following intermediate result, the proof of which is omitted for lack of space.

**Proposition 4.4** *Consider a differentiable nonmonotonic time-varying nonlinear functions $f_i(.,t) \in \mathcal{N}_{nm}$, along with multipliers $W_i(s) \in \mathcal{M}_{nm}$. Define a shifted nonlinearity $\tilde{f}_i(\tilde{y}_i, t)$ as in (4.8). Then, $\{\mathcal{G}_i, \mathcal{V}_i, \mathcal{W}_i\}$ is dissipative, with storage function $\mathcal{V}_i$ and supply rate $\mathcal{W}_i$ given by*

$$\mathcal{V}_i(\tilde{y}_i, z_{i1}, \ldots, z_{im_{i1}}) = \beta_{i0}(\int_0^{\tilde{y}_i} \tilde{f}_i(\sigma,t)d\sigma + \tfrac{1}{2}M_{ii}^{-1}u_i^2) + \sum_{j=1}^{m_{i1}} \beta_{ij} \int_0^{z_{ij}} \tilde{f}_i(\sigma,t)d\sigma \qquad (4.57)$$

$$\mathcal{W}_i(\tilde{y}_i, u_i) = [\sum_{j=1}^{m_{i1}} \alpha_{ij}(\lambda_i\tilde{y}_i - z_{ij}) + \beta_{i0}\dot{y}_i + (\alpha_{i0} + \beta_{i0}c_i)\tilde{y}_i]u_i \qquad (4.58)$$

*where the signals $\tilde{y}_i$, $z_{ij}$ and $\bar{y}_i$ are given by (4.11), (4.12) and (4.13), respectively.*

*Remark 4.1*
Stability conditions stated in the above theorems can be equivalently expressed in terms of solutions to Riccati equations, instead of LMI. LMI in Theorem 4.1, for example, can be equivalently expressed as

$$\begin{aligned} &A_a'P + PA_a + \hat{R} + [H_0C_a + N_0Q_0C_a + N_0C_aA_a \\ &+ \textstyle\sum_{j=1}^{m_1} H_j(C_a - R_j) - B_a'P]'\hat{R}_0^{-1} \\ &\times[H_0C_a + N_0Q_0C_a + N_0C_aA_a + \textstyle\sum_{j=1}^{m_1} H_j(C_a - R_j) - B_a'P] \\ &= 0 \end{aligned} \qquad (4.59)$$

with $\hat{R}$ positive definite, provided that $\hat{R}_0 := (\hat{R} + \hat{R}')$ is positive definite. Since the LMI defines a convex set [3], and since efficient convex optimization algorithm exists to find its solutions [30], it could provide a valuable alternative to analysis and synthesis of robust control, as has been shown recently in [27, 28, 29, 30, 36, 37].

*Remark 4.2*
When the nonlinearities are restricted to be time-invariant in Theorems 4.1 and 4.2, the results of How [7, 8, 15] are recovered. It shown in [7, 8, 15], that equivalent frequency domain for Lyapunov stability of $G(s)$ interconnected to $m$ nonlinearities in certain cases is connected to $\mu$ upperbounds with mixed real and complex perturbations. In view of the above observation in [7, 8, 15], the results of the present paper could also be considered as an extension of mixed $\mu$ upperbounds with nonlinear time-varying perturbations.

*Remark 4.3*
Using loop transformation described in [7, 8, 15] and by appropriate modifications to the shifted system $\tilde{G}(s)$ and nonlinearity $\tilde{f}(.,t)$, the previous results can be extended to the case where the sector has both upper and lower bounds.

# 5 Numerical Example

In this section, a numerical example is presented to demonstrate the effectiveness of the robust stability analysis presented in Section 4. Let us consider a closed-loop system, under nonlinearities acting on the sensor as depicted in Figure 5, where the transfer functions of the plant and the controller are given by

$$P(s) = \frac{1}{75s+1}\begin{bmatrix} 0.878 & 0.864 \\ 1.082 & 1.096 \end{bmatrix}$$

$$C(s) = \frac{0.1}{s}\begin{bmatrix} 75s+1 & 0 \\ 0 & 75s+1 \end{bmatrix}$$

respectively. The multipliers are taken of the form (4.1), with parameters chosen as follows,

$$\alpha_{10} = \alpha_{20} = \alpha_{11} = \alpha_{12} = \alpha_{21} = \alpha_{22} = 1,\ \beta_{10} = \beta_{20} = 1$$
$$\beta_{11} = \beta_{12} = \beta_{21} = \beta_{22} = 2$$
$$\eta_{11} = \eta_{12} = \eta_{21} = \eta_{22} = 1,\ \epsilon_1 = \epsilon_2 = 1$$
$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

For the above choice of parameters, we found that there exists positive definite solution $P$ to the LMI (4.3). This indicates that the closed-loop system is asymptotically stable under nonlinear time-varying perturbations with sector bounds $m_{u1} = m_{u2} = 1$ and rate of variation restricted by $\int_0^{\sigma_i} \frac{\partial f_i(\sigma_i,t)}{\partial t} d\sigma_i \le \epsilon_i\sigma_i f_i(\sigma_i, t)$, $\epsilon_i = 1$, for $i = 1, 2$. To confirm this result, time-responses of outputs are shown in Figure 6, with nonlinearities restricted to be time-invariant.

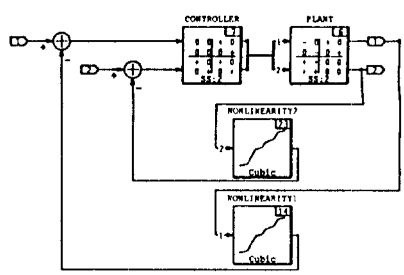


Figure 5: Feedback control systems under structured nonlinear perturbations

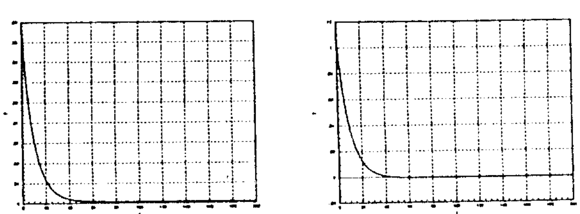

Figure 6: Time responses of outputs $y_1$ and $y_2$ when an impulse is applied at $u_1$