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LIMIT ANALYSIS OF CONTINUOUS STRUCTURES USING MATHEMATICAL PROGRAMMING

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ABSTRACT

An efficient approach to limit analysis is presented whereby a continuous perfectly plastic structure is replaced by a discrete mathematical model. It is formulated as a mathematical programming problem using the static theorem of plasticity. The discretization is accomplished by writing the governing equilibrium equations in finite difference form, and is combined with piecewise linearization of the nonlinear yield curve, thus converting the formulation into a linear programming exercise. Examples of reported cases involving plates and shells are solved to illustrate the ease of application of the present method, its flexibility and accuracy - features which it make attractive to practising engineers.

1. INTRODUCTION

The limit analysis of continuous structures such as cylindrical shells and plates has received considerable attention since the early 1950s [e.g. 1-4] and the works of notable early researchers in the area are referenced in the classical texts [5,6]. The methods used in these works were based on analytical solutions which, despite its obvious advantage of being expressed in closed forms, have two major shortcomings. First, it requires considerable skill in choosing the appropriate stress field with an assumed plastic regime or mechanism before proceeding to solve the governing nonlinear algebraic equations. Secondly, it can only be applied in general to relatively simple problems, which even then would require major reformulation for different boundary conditions, loads or yield conditions. In view of these disadvantages, there is evident need for a general, systematic and efficient approaches to the problem, especially if they are to be used by practising engineers.

In this paper, a general, systematic and efficient approach to limit analysis is introduced whereby a continuous structure made of perfectly plastic material is replaced

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by discrete mathematical model. This idea originates from the early works by Ceradini and Gavarini [7] and Koopman and Lance [8]. It involves expressing the lower bound theorem of plasticity as a mathematical programming problem (MP), and with linearized equilibrium and yield conditions as constraints the problem is then converted into a linear programming (LP) problem. Numerous key references outlining the development of the application of MP techniques to plastic limit analysis are given in the comprehensive reviews of Maier and Munro [9] and Maier and Lloyd Smith [10]. These reviews clearly show that MP methods, because of its simplicity and power, are ideally suited for finding the limit loads of a wide range of suitably discretized structural models.

2. METHOD OF ANALYSIS

The basis of the method presented herein is the lower bound (static) theorem of limit analysis. This theorem states that the load on a perfectly plastic structure, which corresponds to any arbitrary stress field that is in equilibrium with the load and nowhere violates the yield condition, is a lower bound to the actual plastic collapse load. The difficulty of analytical methods mentioned earlier is the determination of appropriate stress fields which satisfy equilibrium equations, the force boundary conditions, and yield conformity, whilst at the same time maximizing the load parameter. This difficulty is overcome by replacing the continuous stress fields of the structure by a finite number of stress parameters. This is done quite conveniently by writing the basic differential equations of equilibrium and force boundary conditions in finite difference form. These stress parameters are used together with a linear or specially linearized nonlinear, yield condition and an additional variable, the load parameter. Thus, the aim of the analysis is to maximize the proportionally applied loads, defined by a common load multiplier, whilst satisfying both the equilibrium and yield conditions. At this point, two points are worthy of note. Firstly, as with all discretized analyses, the solution obtained is only an approximation to the actual one. It is therefore necessary to perform convergence studies through mesh refinement of the model in order to obtain indication of the accuracy of the computed collapse load. In fact, a lower bound solution is not guaranteed due to possible yield violations at locations between chosen check points. Secondly, the flexibiltiy of the approach is largely due to a formulation which allows the discretized equilibrium equations to be systematically written and independently of the linearized yield conditions. This feature is particularly useful when performing studies involving changing parameters such as mesh size, types of loadings and boundary conditions.

The essential ingredients of the proposed method will be described in more detail in the following sections through the analysis of axisymmetrically loaded (a) circular orthotropic plates and (b) circular cylindrical shells.

3. CIRCULAR ORTHOTROPIC PLATES

3.1 Equilibrium and Boundary Conditions

Consider a perfectly plastic axisymmetrically loaded circular plate of radius R and thickness 2H. The essential variables of the problem are the circumferential and radial bending moments M_0 and M_r, respectively, with positive directions as shown in Fig. 1 for an element of the plate located r from the centre. These stress resultants are also active in that they contribute to the yield condition [5]. Also given in Fig. 1 are the shear force Q and pressure p. The equilibrium equations then take the form,

$$d(rM_r)/dr - M_\theta = rQ \qquad (1a)$$

$$Q = -(1/r)\int pr\, dr \qquad (1b)$$

In dimensionless form, Eq. 1 becomes

$$d(xm)/dx - n/(1-\alpha) = (QRx)/[M_{\theta 0}(1-\alpha)] \qquad (2)$$

in which $x=r/R$, $m=M_r/M_{r0}$, $n=M_\theta/M_{\theta 0}$, $\alpha=[1-(M_{r0}/M_{\theta 0})]$, $M_{r0}=\sigma_{r0}H^2$, $M_{\theta 0}=\sigma_{\theta 0}H^2$ and σ_{r0}, $\sigma_{\theta 0}$ = the radial and tangential yield strengths, respectively, of the material. It is also assumed that the tensile and compressive yield strengths are equal.

For the structure, equilibrium is ensured if Eq. 2 is supplemented by force or static boundary conditions. The condition given in Eq. 2 will now be linearized.

The plate is divided into equally spaced nodes at intervals of h, as is typical of conventional finite difference schemes. For a circular plate which is subjected to a uniformly distributed load p over a circular area of radius a, the central difference $O(h^2)$ approximation of Eq. 1 at the i-th node is,

$$x_i(m_{i+1}-m_{i-1}) + 2hm_i - [(2h)/(1-\alpha)](n_i) = [1/\{2(1-\alpha)\}](\mu p*)[\{(x_i/y)^2-1\}\delta+1] \qquad (3)$$

in which $y=a/R$, $p*=P/(\pi a^2)$, P is the total load and μ is a monotonically increasing load parameter for a known p_i, and $\delta=0$ (for $x_i>y$) or 1 (for $x_i<y$).

The boundary conditions are specified as follows: (a) simply supported: m=0; (b) fixed: no constraints; and (c) free: m=0, $m_{i+1}-m_{i-1}=0$.

3.2 Yield Conditions

As in reference [4], the circular orthotropic plate is assumed to obey the modified Tresca yield condition shown in dimensionless stress space in Fig. 2. A compact and efficient mathematical scheme for representing the piecewise linear (PWL) yield condition is in terms of vertices or corners of the convex yield polygon, an idea first introduced by Zavelani-Rossi [11]. In essence, the stress vector Q^i with terms $\{n_i\ m_i\}$ at any node i can be expressed by the linear combination of non-negative coefficients or convex multipliers ξ^i of the corner vectors as follow:

$$\{Q^i\} = [V^i]\{\xi^i\}\ ; \quad \{\xi^i\} \geq 0 \qquad (4)$$

where V^i is a matrix which collects all the extreme point vertices of PWL polygon. For example, the relevant vectors and matrices appropriate to the PWL polygon shown in Fig. 2 are

$$\{Q^i\} = \{n_i \quad m_i\} \tag{5a}$$

$$[V^i] = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 & -1 & 0 \end{bmatrix} \tag{5b}$$

$$\{\xi^i\} = \{\xi_{i1} \cdots\cdots \xi_{i6}\} \tag{5c}$$

For node i, plastic conformity is ensured if

$$\xi_{i1}+\xi_{i2}+\xi_{i3}+\xi_{i4}+\xi_{i5}+\xi_{i6} \le 1 \quad \text{or} \quad [U^i]\{\xi^i\} \le 1 \tag{6}$$

where U^i is a row matrix of size [1 x 6] for six vertices. Finally, Eqs. 5 and 6 can be extended over all i = 1 .... s check nodes of the structure to give,

$$\{Q\} = [V]\{\xi\} \ ; \quad \{\xi\} \ge 0 \ ; \quad \text{and} \quad [U]\{\xi\} \le 1 \tag{7}$$

in which for s nodes and six yield vertices per node (Fig. 2), Q is a vector of length 2s of active stress resultants, V is a [2s x 6s] matrix of yield corners, ξ is a vector of length 6s of corner multipliers, U is a [6s x s] boolean matrix (i.e., with only 0 and 1 entries), 0 is a null vector of length 6s and 1 is a vector of length s with only unit entries.

3.3 Linear Programming Formulation

The limit analysis can be carried out through application of the static theorem of plastic theory by maximizing the proportionally applied load multiplier μ while satisfying both the equilibrium and yield conditions. The fundamental relations developed in the previous sections can now be used for the mathematical formulation of the limit analysis problem.

For the discretized plate model under uniformly distributed pressure μp acting over a circular area of varying radius, the relevant MP formulation for finding the limit load factor μ_c can be written compactly as,

$$\mu_c = \max \mu \mid \text{Eq. 3, b.c. and Eq. 7} \tag{8}$$

where "b.c." refers to boundary conditions and the symbol | is read as "such that". The formulation given by Eq. 8 is a linear programming problem in variables μ, Q and ξ since the objective function and all constraints are linear. Note also that μ and ξ are non-negative variables whereas elements of Q are "free" or sign-unconstrained.

Some remarks about the linear programming problem described by Eq. 8 are appropriate at this stage. Firstly, the efficiency of using the vertex formulation, instead of the alternative and commonly used hyperplane description (e.g. Maier [12]), for checking the yield condition can now be explained. Far fewer constraints are typically required to check yield conformity through Eq. 7. In the vertex formulation, only three constraints per node (the non-negativity requirements for ξ are not counted in LP as they automatically taken care of in the simplex solution algorithm) are necessary as compared to six for the hexagon shown in Fig. 2. This is important since the computational effort required to solve LP problems increases rapidly with the number of constraints, whilst the number of variables is not equally crucial. Obviously, the savings increase with finer linearizations of the yield surface since the vertex formulation would still require three constraints per node, while in a hyperplane formulation the number of constraints per node is equal to the number of hyperplanes. Secondly, the dual LP formulation of Eq. 8 represents the kinematic approach and can be neatly obtained from Kuhn-Tucker's conditions of mathematical programming theory. Thirdly, a feasible, not necessarily the actual collapse mechanism for the discretized model can be constructed by inspection of the optimal dual values of the static LP problem. Finally, with LP formulation, the stress field needs not be explicitly generated; the only requirements are that it should satisfy equilibrium and yield. Thus, the finite difference scheme of discretizing the differential equations is indeed very attractive since the resulting finite difference equations need not be explicitly solved, as is the case with an elastic analysis. In addition, the present approach lends itself ideally to computer solution since the problem can be efficiently and systematically set up.

3.4 Examples

Basically, the same problem as analysed by Markowitz and Hu [4] was solved using the LP approach presented above. It involves the calculations of the collapse pressures of simply supported and clamped plates subjected to uniform pressure acting over a circular area of varying radius and different degrees of anistropy as defined by the parameters a/R and α, respectively. To determine the number of mesh points to be used in obtaining the various solutions, the analyses of fully loaded simply supported and clamped plates were carried out for mesh divisions ranging from 10 to 98. The results are shown in Fig. 3, and for mesh discretization of 90-division mesh size, the difference between exact and computed value is about 1%. Various cases (i.e., for a/R = 0.1, 0.2, 0.4, 0.6 0.8 and 1.0 and α = -1, -3/7, 0 (isotropic), 0.3 and 0.5) were run using 90 mesh divisions. The solutions are shown in Fig. 4 which were superimposed on the curves obtained by Markowitz and Hu [4]. Agreement is clearly excellent in all cases.

4. CYLINDRICAL SHELLS

4.1 Equilibrium and Boundary Conditions

Consider a perfectly plastic cylindrical shell of length L, radius of midsurface R, and constant thickness t (Fig. 5a), and is subjected to an internal axially symmetric hydrostatic pressure distribution in the outward radial direction. Without loss of generality, no loads are applied in the axial direction. A typical shell differential element under applied pressure P and stress resultants is shown in Fig.5b. The well-known equilibrium condition for this differential element in a cylindrical coordinate system (X, θ,r) can be easily derived (e.g.[13]) by elimination of shears S from the radial and moment equilibrium equations. In dimensionless form, it is given as,

$$m'' + 2\alpha^2(n - p) = 0 \qquad (9)$$

in which $x=X/L$, $p=PR/\sigma_o t$, $n=N_\theta/N_o$, $m=M_x/M_o$, $N_o=\sigma_o t$, $M_o=\sigma_o t^2/4$, $\alpha=2L^2/Rt$; M_x is the bending moment perpendicular to the shell axis, N_θ is the circumferential membrane force, σ_o is the material yield stress, and prime denotes differentiation with respect to x.

For the structure, equilibrium is ensured if Eq. 9 is supplemented by force bopundary conditions. Both of these will now be linearized.

The shell is divided into s equally spaced nodes at intervals of h, as is typical of conventional finite difference scheme (Fig. 6), and dummy stations are labelled as node 0 and (s+1). The central difference $O(h^2)$ approximation for Eq. 9 is used for all nodes, and can be written as,

$$(m_{i-1} - 2m_i + m_{i+1}) + 2\alpha^2h^2(n_i - \mu p_i) = 0 \qquad (i = 1 \ldots s) \qquad (10)$$

in which μ is a monotonically increasing load parameter for a known p_i. A compact matrix representation of Eq. 10 is

$$[A]\{m\} + \beta\{n\} - \beta\mu\{p\} = \{0\} \qquad (11)$$

in which [A] is a [s x (s+2)] block diagonal equilibrium matrix; {m}, {n}, and {p} are vectors of length (s+2), s and s, respectively; $\beta=2\alpha^2h^2$; and {0} is a null vector.

The force boundary conditions for any boundary node r (nodes 1 and s in this case) can now be specified according to one of three typical edge conditions as follows: (a) free: $m_r=0$, $m_{r-1}-m_{r+1}=0$; (b) simply supported: $m_r=0$; (c) clamped or fixed: no constraints. Note that a central difference approximation has also been used to represent the zero shear condition for case (a).

4.2 Yield Conditions and LP Formulation

It has been shown elsewhere [13,14] that for the case where there is no axial load, only the stress resultants m and n are "active" or enter into the yield condition; M_θ and S (Fig. 5b) are "reactions" in Prager's sense since they do not a priori vanish for reasons of symmetry or equilibrium, and nevertheless do not appear in the dissipation function. The exact nonlinear yield curve for a circular cylindrical shell of uniform section made of Tresca material has been given by Hodge [15], and is shown in Fig. 7. Also shown in Fig. 7 is a piecewise linear (PWL) approximation to the nonlinear curve, and is used in the present analysis. In effect, the hexagon corresponds to the exact yield condition of a Tresca sandwich shell. It is also a lower bound approximation to sandwich and uiniforn shells obeying von Mises' yield criterion as given elsewhere [5].

A compact mathematical expression of the PWL yield condition is again achieved by means of a vertex representation similar to those described by Eq. 7. Also, the present limit analysis problem follows the same LP formulation described in Sec. 3.3 with the appropopriate constraints.

4.3 Examples

The same structure as analyzed by Cinquini et al. [16] using analytical method, was solved using the LP approach. It involves the calculation of collapse loads for a hydrostatically loaded circular cylindrical shell with its lower edge built-in.

A free upper edge case with shell parameter α = 2.958 is first considered; a complete solution for this case is given by Cinquini et al. [16]. The finite difference discretization adopted, as shown in Fig. 8a, has nine equally spaced nodes with node 1 free and node 9 fixed, and the dummy stations at nodes 0 and 10. Thus, h = 0.125 and the parameter β = 0.27343. The LP problem involved 38 constraints (9 equilibrium, 2 boundary, 27 yield) and 75 variables. A collapse load of μ_c = 2.6352 was obtained; this is about 0.3% greater than the exact solution of 2.627 reported by Cinquini et al. [16]. The plastic regime corresponding to the optimal solution is shown in Fig. 8b where the numbers refer to node points. It is clear that hinge circles (m=1) developed at stations 6 and 9, and of the nine nodes only node 1 is unyielded. For this case, two other meshes, h = 0.25 and h = 0.0625, were also used; the computed collapse multipliers were 2.5887 (-1.5%) and 2.6276 (0.02%), respectively.

The finite difference discretization with h = 0.125 was considered to be sufficiently accurate, and was therefore adopted for performing a series of analyses of hydrostatically loaded shell with varying upper edge supports (i.e. free, simply supported, or fixed) and shell parameter α. These results are shown in Figs. 9(a-c) as discrete points superimposed on the analytically derived curves of Cinquini et al. [16]; close correlations between the LP solutions and the analytical results are obtained in all cases. Note that the collapse loads for long shells (large α) tend to be the same for all three boundary conditions. This was expected as the limit loads for long, hydrostatically loaded shells depend on the support conditions at their remote upper edges.

5. CONCLUSIONS

(a) The mathematical programming approach to limit analysis problem is simple and can be systematically set up.

(b) Easy extensions of the present study include the calculations of collapse loads for shells and plates with other boundary conditions, different axisymmetric loadings, varying piecewise constant thicknesses, and even [16] some degree of material anisotropy. Extension to non-zero axial force for the case of the cylindrical shell is simple, however, a three-dimensional yield surface needs to be be used; yield vertices become three-component vectors instead of two as with the two-dimensional case.

(c) The mathematical programming formulation eliminates the difficult step of constructing admissible stress fields and feasible collapse mechanism, as would be required in analytical techniques.

(d) A disadvantage of the method stems from the inaccuracy produced by the discretization process. The computed collapse load may not be a lower bound solution since unchecked points may have yielded. It is therefore important to carry out sensitivity analyses to obtain indication of the accuracy obtained.

(e) The adoption of a piecewise linear approximation to the yield surface is not a major shortcoming since better yield polygons, involving more vertices, can be used without significant computational cost. With vertex formulation, the number of variables increase as a result but not the number of constraints.

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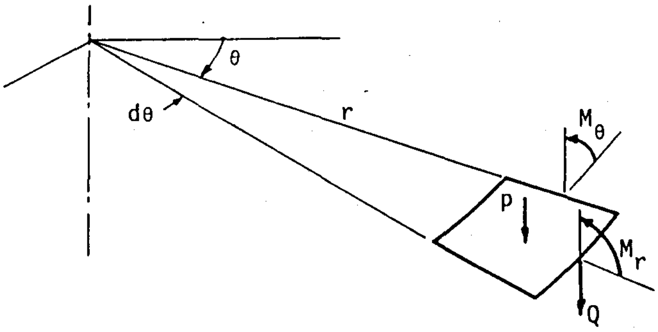

Fig. 1 Element of circular plate.

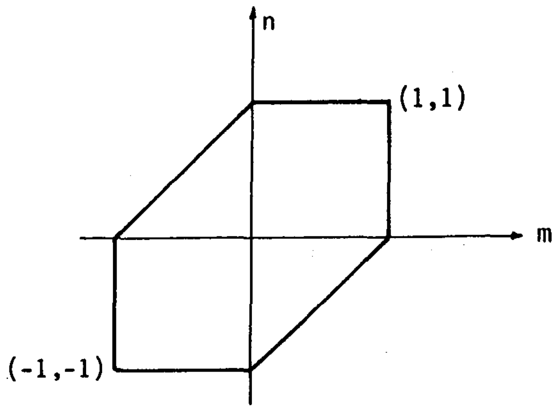

Fig. 2 Tresca hexagon

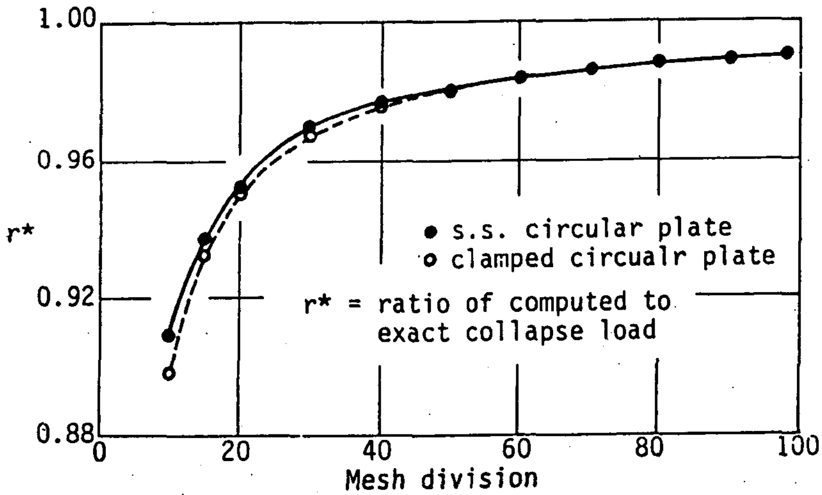

Fig. 3 Variation of collapse load with mesh division

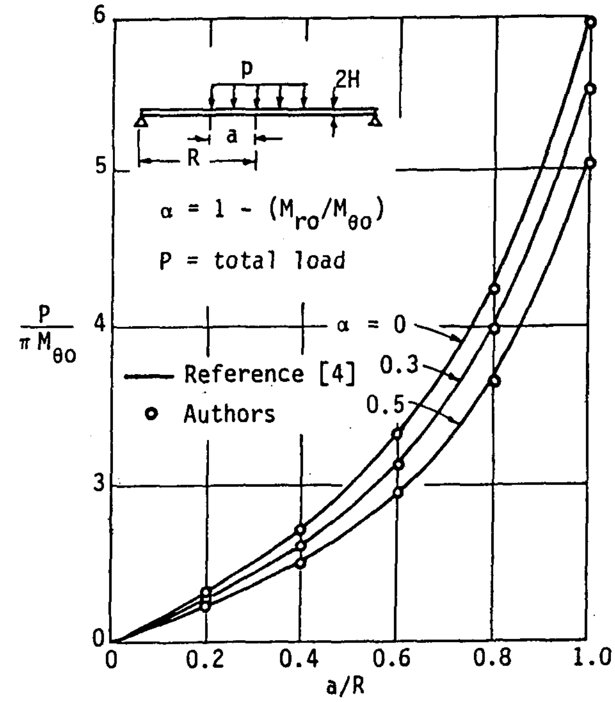

Fig. 4a Collapse loads of simply supported circular plate under uniformly distributed load.

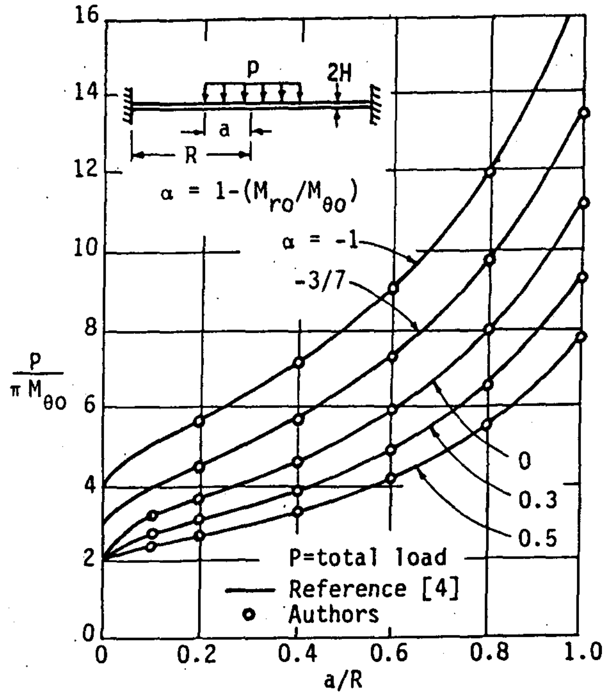

Fig. 4b Collapse loads of clamped circular plate under uniformly distributed load.

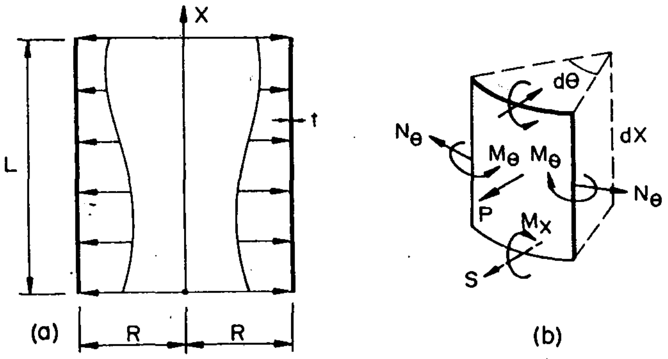

Fig. 5 Cylindrical shell: (a) geometry (b) element

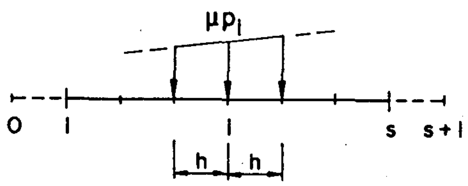

Fig. 6 Finite difference scheme

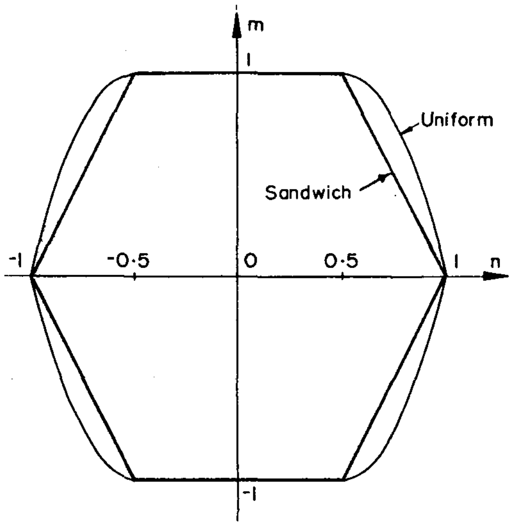

Fig. 7 Yield curves

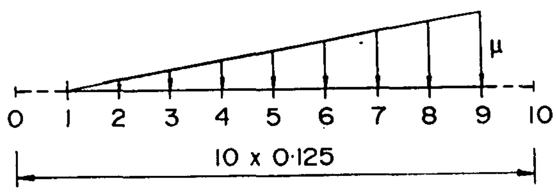

Fig. 8a Finite difference model

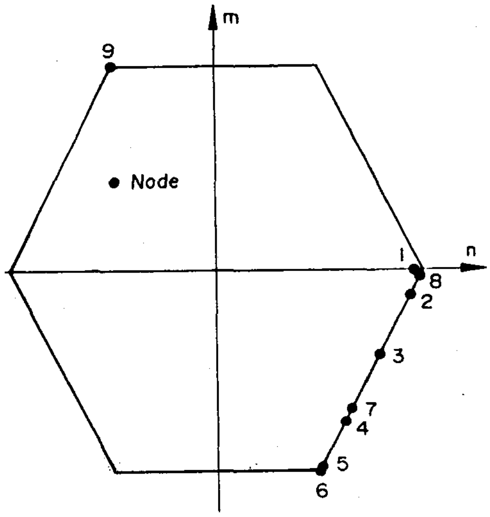

Fig. 8b Plastic regime for free-fixed shell example

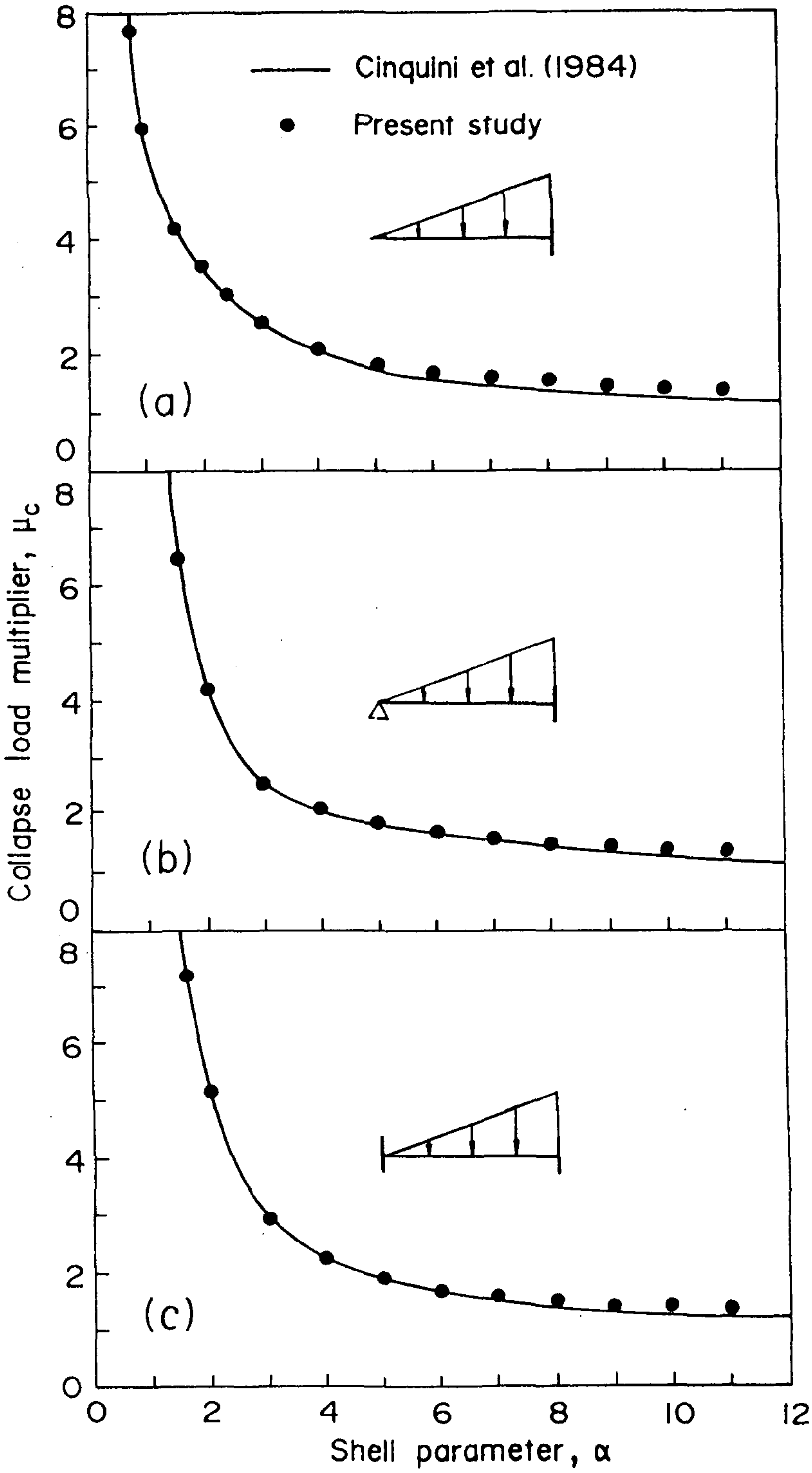

Fig. 9 Variation of collpase load multiplier with shell parameter:
(a) free-fixed (b) simply supported-fixed (c)fixed-fixed.