# Proportion Problem and Hilbertian Description

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**Abstract**

In this paper we present another solution the so-called proportion problem of donkey sentence based on the interpretation of an indefinite description, or existentially quantified term, as Hilbert's epsilon-term, a logical device invented for proof-theoretical consistency proof. Semantically, an epsilon-term can be thought of as representing an arbitrary object in the sense of Kit Fine (1985) and may be related to the non-existent individual (*Aussersein*) of Meinong. In our theory, an anaphoric pronoun is treated as a demonstrative term, as proposed by David Kaplan (1979) in his study on the logic of demonstratives.

Our solution to the proportion problem is more satisfactory in that it is general and seems to capture the intuition behind our use of quantification and anaphora. In presenting our theory with the epsilon-term, we stress the importance of dynamic aspects in interpretation in general and quantification in particular. A theory of semantic representation is suggested that emphasizes not only on denotational but also on the deductive and procedural side of meaning.

## 1 DRT and Proportion Problem

In Kamp's Discourse Representation Theory (DRT), the famous donkey sentence:

(1) Every farmer who owns a donkey beats it

is given the following Discourse Representation Structure:

```
Every farmer who owns a donkey beats it
  {u v}                  {}
  farmer(u)              u beats it
  u owns a donkey        u beats v
  donkey(v)
  u owns v
```

According to DRT's definition of semmantic interpretation, the universal sentence "*every* $\phi, \psi$" is satisfied by any assignment $f$ in the model $M$, given the DRS's $K$ and $K'$ for $\phi$ and $\psi$

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respectively, if and only if $K$ and $K'$ are such that $\|K\|_M \subseteq \|K\|_M \cap \|K'\|_M$. In other words,

$$\|\text{every}\|_M = \{\langle K, K'\rangle \mid \|K\|_M \subseteq \|K'\|_M\}.$$

If we define:

- $K = \{\text{farmer}(\mathbf{u}), \mathbf{u} \text{ owns } \mathbf{v}, \text{donkey}(\mathbf{v})\}$
- $K' = \{\mathbf{u} \text{ beats } \mathbf{v}\}$

then the condition above is clearly equivalent to the truth-condition of the following formula in first-order logic:

$$(\forall x)(\forall y)[(farmer(x) \wedge donkey(y) \wedge own(x, y)) \supset (beat(x, y))]$$

This analysis of the donkey sentence is intuitively correct. But is this way of analysis correct *in principle*, or does it just happen to be correct? We will see one possible problem which may beset DRT in principle.

## 1.1 Proportion problem

The proportion problem is a problem that arises when one extends the original vocabulary of DRT to a language containing a generalized quantifier "*most*". This problem is concerned with the following variant of the donkey sentence:

(2) Most farmers who own a donkey beat it

Presumably, DRT would give the following DRS for (2):

```
Most farmers who own a donkey beat it
 ---------------------------------------
|  {u v}              {}                |
|  farmer(u)          u beat it         |
|  u own a donkey     u beat v          |
|  donkey(v)                            |
|  u own v                              |
 ---------------------------------------
```

Following the standard treatment, one can define the truth-condition of the quantifier *most* in DRT as follows:

> Given the sentence "*Most* $\phi$ $\psi$" and DRS's $K$ and $K'$ for $\phi$ and $\psi$ respectively, the satisfaction condition in model $M$ for this sentence is
>
> 7. $\langle K, K'\rangle \in \|\text{most}\|_M \iff |\|K\|_M \cap \|K'\|_M| > |\|K\|_M - \|K'\|_M|$.

The notation $|A|$ means *the cardinality of the set A*. This satisfaction condition says, in effect, that the sentence "*most* $\phi, \psi$" is true in $M$, given $K$ and $K'$, if and only if the cardinality of the set of assignments that satisfy both $K$ and $K'$ is larger than that of the set of assignments that satisfy $K$ but not $K'$. Although this condition may not exactly be what our intuition would have for a *most*-sentence, it does capture the lower-bound of a condition that has to be satisfied by any *Most*-sentence.

Against the backdrop of this truth-condition, Mats Rooth (1987) defines the following model: Let $M_0$ be $\langle U_{M_0}, F_{M_0}\rangle$ such that $U_{M_0} = \{john, a_1, \ldots, a_{99}, d_1, d_2, \ldots, d_{1099}\}$;
$F_{M_0}(\text{farmer}) = \{john, a_1, \ldots, a_{99}\}$;
$F_{M_0}(\text{donkey}) = \{d_1, d_2, \ldots, d_{1099}\}$;
$F_{M_0}(\text{own}) = \{\langle john, d_1\rangle, \ldots, \langle john, d_{1000}\rangle, \langle a_1, d_{1001}\rangle, \ldots, \langle a_{99}, d_{1099}\rangle\}$;
$F_{M_0}(\text{beat}) = \{\langle john, d_1\rangle, \ldots, \langle john, d_{1000}\rangle\}$. In this model, there are 100 farmers: John and 99 others. John owns 1000 donkeys and beats every one of them. Each of the other farmers owns exactly one donkey, and none of them beats his donkey. The question is whether the interpretive mechanism of DRT, given the truth-condition of *most* and the DRS above, can give the truth-value *false* to this sentence.

The satisfaction condition for *Most* gives the following condition for the DRS's of (2).

$$\langle K, K'\rangle \in \|\text{most}\|_{M_0} \iff |\|K\|_{M_0} \cap \|K'\|_{M_0}| > |\|K\|_{M_0} - \|K'\|_{M_0}|,$$

where $K = \{\text{farmer}(\mathbf{u}), \mathbf{u} \text{ own } \mathbf{v}, \text{donkey}(\mathbf{v})\}$, and $K' = \{\mathbf{u} \text{ beat } \mathbf{v}\}$.

The satisfaction set of the antecedent DRS $K$, $\|K\|_{M_0} = \|\{\text{farmer}(\mathbf{u}), \mathbf{u} \text{ owns } \mathbf{v}, \text{donkey}(\mathbf{v})\}\|_{M_0}$, is the set $\{g | g \in \|\text{farmer}(\mathbf{u})\|_{M_0} \cap \|\mathbf{u} \text{ owns } \mathbf{v}\|_{M_0} \cap \|\text{donkey}(\mathbf{v})\|_{M_0}\}$. This set, in turn, is composed of those assignments $g$ such that $g \in \|\text{farmer}(\mathbf{u})\|_{M_0}$ and $g \in \|\mathbf{u} \text{ owns } \mathbf{v}\|_{M_0}$ and $g \in \|\text{donkey}(\mathbf{v})\|_{M_0}$. This means that these are assignments $g$ that satisfy $g(\mathbf{u}) \in F_{M_0}(\text{farmer})$, $g(\mathbf{v}) \in F_{M_0}(\text{donkey})$, and $\langle g(\mathbf{u}), g(\mathbf{v})\rangle \in F_{M_0}(\text{own})$.

In order to evaluate the whole DRS, we have also to have the satisfaction set for the consequent DRS, namely $K' = \{\mathbf{u} \text{ beat } \mathbf{v}\}$. $\|K'\|_{M_0}$ is the set $\{g | g \in \|\mathbf{u} \text{ owns } \mathbf{v}\|_{M_0}\}$, which is composed of assignments $g$ such that $\langle g(\mathbf{u}), g(\mathbf{v})\rangle \in F_{M_0}(\text{beat})$.

Since $U_{M_0} = \{john, a_1, \ldots, a_{99}, d_1, d_2, \ldots, d_{1099}\}$, there are 1099 assignments, namely $g_i : U_K \cup U_{K'} \mapsto U_{M_0}$ under the interpretation $F_{M_0}$. The first 1000 assignments $g_1, \ldots, g_{1000}$ give the same value *john* to the reference marker $\mathbf{u}$, but they differ from each other in the value they assign to $\mathbf{v}$. Notice that all of these 1099 assignments satisfy the conditions in $K$ so that it is the case $\|K\|_{M_0} = \{g_1, \ldots, g_{1099}\}$. On the other hand, only the assignments that assign *john* to $\mathbf{u}$, namely $g_1, \ldots, g_{1000}$, satisfy the condition in $K'$. So the satisfaction set for the latter DRS is $\|K'\|_{M_0} = \{g_1, \ldots, g_{1000}\}$. The intersection of these two sets is $\|K\|_{M_0} \cap \|K'\|_{M_0} = \{g_1, \ldots, g_{1000}\}$, which is the set of assignments that satisfy both the antecedent and the consequent of the sentence (2). The cardinality, therefore, of the set of assignments that satisfy both the antecedent and the consequent of (2) is

$$|\|K\|_{M_0} \cap \|K'\|_{M_0}| = 1000.$$

The cardinality of the set of assignments that satisfy the antecedent but not the consequent is, on the other hand,

$$|\|K\|_{M_0} - \|K'\|_{M_0}| = 99.$$

Obviously it is the case $1000 > 99$, so, according to the clause 7 above, we can conclude

$$\langle K, K' \rangle \in \|\text{most}\|_{M_0}.$$

Thus, in DRT, sentence (2) is true in the model $M_0$.

Intuitively, however, this result is strange. There is only one farmer that beats his donkeys, while 99 farmers do not beat theirs. If there are 99 farmers each of whom does not beat his donkey and there is only one farmer who does his, our intuition decrees that it not be the case that most farmers who beat their donkeys. Thus the DRT gives rise to a counter-intuitive result.

# 2 Hilbert's Epsilon-term as Description

In this section, we introduce David Hilbert's $\epsilon$-term. Hilbert's $\epsilon$-operator[1] is a term forming operator in the sense that if $\phi(x)$ is a formula with $x$ free then $\epsilon x.\phi(x)$ is a term. We call such a term *an $\epsilon$-term*. Proof-theoretically, an $\epsilon$-term is constrained by the following *$\epsilon$-axiom*:

$$\frac{\exists x.\phi(x)}{\phi(\epsilon x.\phi(x))}$$

That is, if one can prove that there is some $\phi$ then one can also show that the object which has been shown to be $\phi$ is indeed $\phi$. In a more semantics-oriented manner, one can characterize the $\epsilon$-term in terms of the following rule:

$$\frac{\vdash \exists x.\phi(x)}{\rhd \epsilon x.\phi(x)},$$

which says, intuitively, that if one has shown that there is something that satisfies $\phi$ then the $\epsilon$-term $\epsilon x.\phi(x)$ is indeed, semantically, well-defined.

Hilbert himself explains the informal interpretation of $\epsilon$-terms in the following manner[2]:

- If a proposition $\phi$ holds of one and only one object, then $\epsilon x.\phi(x)$ is the object of which $\phi(a)$ holds: *i.e.* $\epsilon$-operator plays the role similar to the $\iota$-operator.
- When there are more than one thing that $\phi$ is true of, then $\epsilon$ takes on the role of the *choice* function; $\epsilon x.\phi(x)$ is any *one* of the objects $a$ of which $\phi(a)$ holds.
- When $\exists x.\phi(x)$ does not hold, then $\epsilon$ chooses *arbitrarily anything*, any object whatsoever.

Notice the *arbitrary* character of an $\epsilon$-term. This is why Hilbert's $\epsilon$-calculus is closely related to some non-standard semantics based on arbitrary or non-existent (*Aussersein*) objects[3].

We can formalize Hilbert's informal ideas in the following model theory for $\epsilon$-calculus. We define[4] a (standard) model structure for (a calculus based on) the $\epsilon$-term as a triple $\langle D, I, \Phi \rangle$, where $D$ is the domain, or the universe, of the model and the $I$ the interpretation of basic expressions; thus the pair $\langle D, I \rangle$ may be taken as an ordinary model structure for the first-order language. $\chi$ is a set of choice functions, the component which we need in order to interpret

$\epsilon$-terms. More formally, $\chi$ is a set of choice functions on $D$ such that if $f \in \Phi$ then $f(N) \in N$ for any non-empty $N \subseteq D$, and, if $N = \emptyset$, $f(\emptyset)$ is an arbitrary member $d \in D$.

We can then give the denotation $[\![\alpha]\!]^{\theta}_{M}$ of an expression $\alpha$, of our formal language, in the model $M$ under the assignment $\theta$ to the variables, if any, in $\alpha$. For expressions other than the $\epsilon$-term, our semantics is standard. The only semantic clause worth mentioning here is the one for an $\epsilon$-term itself, which is:

$$[\![\epsilon x.\phi]\!]^{\theta}_{M} = f(\{d \in D | [\![\phi]\!]^{\theta^{x}_{d}}_{M} = \mathsf{True}\}), \text{ for } f \in \chi.$$

That is, an $\epsilon$-term is interpreted as a choice function which gives back as value an object which satisfies the condition of the term, if there is such, or an arbitrary object. Notice that this definition reflects not only the fact that an $\epsilon$-term is a CHOICE-FUNCTION but also the requirement that its denotation be an *arbitrarily* chosen object.

## 2.1 Introducing Generalized Quantifier

In order to treat sentences such as (2), we have to extend the original language of $\epsilon$-calculus to contain quantifier **Most** in addition to $\forall$ and $\exists$. Following the standard definition, as has been done in section 1.1, we assign the following truth-condition to the quantifier **most**:

$$[\![(\mathsf{Most}:x)(\phi,\psi)]\!]^{g}_{M} = \mu_{\mathsf{Most}}(m,n,p) = \mathsf{True} \iff m > n,$$

where

$m =$

$|\{d \in D | [\![\phi]\!]^{g^{x}_{d}}_{M} = \mathsf{True}\} \cap \{d \in D | [\![\psi]\!]^{g^{x}_{d}}_{M} = \mathsf{True}\}|,$

$n =$

$|\{d \in D | [\![\phi]\!]^{g^{x}_{d}}_{M} = \mathsf{True}\} \cap \{d \in D | [\![\neg\psi]\!]^{g^{x}_{d}}_{M} = \mathsf{True}\}|.$

Although the definition above looks a little different from the one given in section 1.1, it gives the same result. For example, if we are given a sentence

(3) Most men run.

Suppose we translate (3) to:

$$(\mathsf{Most}:x)(\mathsf{man}(x), \mathsf{run}(x)).$$

Then, according to the definition above, $[\![(\mathsf{Most}:x)(\mathsf{man}(x),\mathsf{run}(x))]\!]^{\theta^{x}_{d}}_{M} = \mathsf{True}$ if and only if $m = |\{d \in D | [\![\mathsf{man}(x)]\!]^{\theta^{x}_{d}}_{M} = \mathsf{True}\} \cap \{d \in D | [\![\mathsf{run}(x)]\!]^{\theta^{x}_{d}}_{M} = \mathsf{True}\}|$ is larger than $n = |\{d \in D | [\![\mathsf{man}(x)]\!]^{\theta^{x}_{d}}_{M} = \mathsf{True}\} \cap \{d \in D | [\![\neg\mathsf{run}(x)]\!]^{\theta^{x}_{d}}_{M} = \mathsf{True}\}|$, which is $|\{d \in D | [\![\mathsf{man}(x)]\!]^{\theta^{x}_{d}}_{M} = \mathsf{True}\} - \{d \in D | [\![\mathsf{run}(x)]\!]^{\theta^{x}_{d}}_{M} = \mathsf{True}\}|$. Thus, as far as the interpretation of the sentence (2) is concerned, our **Most** must be as sufficient as $\|\text{most}\|$.

## 2.2 Pronoun as a demonstrative term.

In order to treat anaphoric pronouns, we introduce the concept of *pro-terms*. We represent pro-terms as {**he**, **she**, **it**, *etc*}. Pro-terms are terms like names, but their semantic function is slightly different. The chief business of a pro-term is to stand for another name. Thus a pro-term is semantically similar to a variable.

We would like to think of a pro-term as an analog of what David Kaplan calls a demonstrative[5]: a term used to point to an object. Just as a demonstrative must be accompanied by an act of demonstration, a pro-term must be specified in place of what it is used by a sort of demonstration. Usually such demonstration is provided by an act of pointing that accompanies an utterance of a pro-term, or by an act of simply intending an object.

As far as our formal treatment is concerned, such demonstration can be modeled by a function that sends a pro-term to its intended object or a name. We call this demonstrating function simply *Dem*. We think of this *Dem* function (in a discourse) as a function that maps a pro-term to an object purported to be in the discourse. Those objects that comprise the range of the *Dem* function are the objects that are claimed to be well-defined in the contexts. Let us call these ***personae dramatis*** of a discourse. These function in the similar manner to the way discourse referents in DRT. But more specifically, we think of these personae dramatis as $\epsilon$-terms, introduced into the context by specific instances of the $\epsilon$-axiom. If, for example, in discourse $\delta$, the personae dramatis are defined as $\{\alpha, \beta, \gamma\}$, the *Dem* function may send pro-terms $\{\mathsf{he}_1, \mathsf{he}_2, \mathsf{it}\}$ in such a way that $Dem(\mathsf{he}_2) = \alpha$, $Dem(\mathsf{he}_2) = \beta$, and $Dem(\mathsf{it}) = \gamma$. But it may turn out, in the interpretation of the discourse, the personae dramatis $\alpha$ and $\beta$ are referring to the one and same object. In such a case, the pro-terms $\mathsf{he}_1$ and $\mathsf{he}_2$ can be called co-referential in the traditional sense.

## 3 Epsilon-term Analysis of the Proportion Problem

The procedure which translates (2) to a formula $\epsilon$-calculus is similar to a proof-procedure in logic or calculation in Montague Grammar, and can also be regarded as an impoverished version of the DRS construction algorithm in DRT.

Given sentence (2), the procedure first declares the type of the individual, whatever it is, that is to be designated by the noun phrase *most farmers*. This is represented as "$x$ : [Most]". In the context in which the variable $x$ is of type Most, the rest of the sentence is translated. the (pseudo-) noun phrase *farmers who own a donkey* is translated in the standard way, namely: $\langle \mathsf{farmer}(x) \wedge (\exists y)[\mathsf{own}(x,y) \wedge \mathsf{donkey}(y)\rangle$. The verb phrase *beat it* is translated as a formula that contains a pro-term it: $\langle \mathsf{beat}(x, \mathsf{it})\rangle$.

1. $x$ : [Most]
2. $[\langle \mathsf{farmer}(x) \wedge (\exists y)[\mathsf{own}(x,y) \wedge \mathsf{donkey}(y)\rangle, \langle \mathsf{beat}(x, \mathsf{it})\rangle]$

According to the $\epsilon$-axiom, an $\epsilon$-term is introduced in the following manner:

$$\frac{(\exists y)[\mathsf{own}(x,y) \wedge \mathsf{donkey}(y)]}{\rhd(\epsilon y)[\mathsf{own}(x,y) \wedge \mathsf{donkey}(y)]}$$

We could abbreviate this $\epsilon$-term thus: $\alpha = (\epsilon y)[\mathsf{own}(x,y) \wedge \mathsf{donkey}(y)]$. So the result of the procedure so far would be as follows:

3. $[\langle \mathsf{farmer}(x) \wedge \mathsf{own}(x,\alpha) \wedge \mathsf{donkey}(\alpha)\rangle, \langle \mathsf{beat}(x, \mathsf{it})\rangle]$.

This formula is not yet in a proper form for it still contains the pro-term **it**, and we need to eliminate it. Since $\alpha$ is the only persona dramatis introduced in the sentence, it is natural to define the *Dem* function to be such that $Dem(\mathsf{it}, \alpha)$. The result is:

4. $[\langle \mathsf{farmer}(x) \wedge \mathsf{own}(x,\alpha) \wedge \mathsf{donkey}(\alpha)\rangle, \langle \mathsf{beat}(x, \alpha)\rangle]$.

Finally, the context of variable type is resolved, giving:

(4) $$(\mathsf{Most}: x)[\mathsf{farmer}(x) \wedge \mathsf{own}(x,\alpha) \wedge \mathsf{donkey}(\alpha), \mathsf{beat}(x,\alpha)],$$

where $\alpha = (\epsilon y)[\mathsf{own}(x,y) \wedge \mathsf{donkey}(y)]$. This last result is the translation of (2) within our theory of $\epsilon$-terms.

Now the real problem of proportion; is (4) true or false in the model we presented in section 1.1?

In order to compute the value of

$$[\![(\mathsf{Most}: x)[\mathsf{farmer}(x) \wedge \mathsf{own}(x,\alpha) \wedge \mathsf{donkey}(\alpha), \mathsf{beat}(x,\alpha)]]\!]^{\theta}_{M_0}$$

we need to calculate the cardinalities of the set

$$\{u \in D_{M_0} | [\![\mathsf{farmer}(x) \wedge \mathsf{own}(x,\alpha) \wedge \mathsf{donkey}(\alpha)]\!]^{\theta^x_u}_{M_0} = \mathsf{True}\}$$

and the set

$$\{u \in D_{M_0} | [\![\mathsf{beat}(x,\alpha)]\!]^{\theta^x_u}_{M_0} = \mathsf{True}\}.$$

For this, in turn, we have to know the value of

$$[\![\mathsf{farmer}(x) \wedge \mathsf{own}(x,\alpha) \wedge \mathsf{donkey}(\alpha)]\!]^{\theta}_{M_0}$$

and

$$[\![\mathsf{beat}(x,\alpha)]\!]^{\theta}_{M_0}.$$

Now, in order to know the values of $[\![\mathsf{farmer}(x) \wedge \mathsf{own}(x,\alpha) \wedge \mathsf{donkey}(\alpha)]\!]^{\theta^x_u}_{M_0}$, we first have to know the value of the $\epsilon$-term $\alpha := (\epsilon y)[\mathsf{own}(x,y) \wedge \mathsf{donkey}(y)]$.

The value of $[\![(\epsilon y)[\mathsf{own}(x,y) \wedge \mathsf{donkey}(y)]]\!]^{\theta^x_u}_{M_0}$ is the value of the choice function $f$ taking as argument the set $\{d \in D_{M_0} | [\![\mathsf{own}(x,y) \wedge \mathsf{donkey}(y)]\!]^{\theta^{x,y}_{u,d}}_{M_0} = \mathsf{True}\}$. Hence we have to know what constitutes this latter set. The interpretation $[\![\mathsf{own}(x,y) \wedge \mathsf{donkey}(y)]\!]^{\theta^{x,y}_{u,d}}_{M_0}$ has the value true iff both $[\![\mathsf{own}(x,y)]\!]^{\theta^{x,y}_{u,d}}_{M_0} = \mathsf{True}$ and $[\![\mathsf{donkey}(y)]\!]^{\theta^{x,y}_{u,d}}_{M_0} = \mathsf{True}$. And $[\![\mathsf{own}(x,y)]\!]^{\theta^{x,y}_{u,d}}_{M_0} = \mathsf{True}$ iff $\langle \theta^{x,y}_{u,d}(x), \theta^{x,y}_{u,d}(y)\rangle \in I_{M_0}(\mathsf{own})$, while $[\![\mathsf{donkey}(y)]\!]^{\theta^{x,y}_{u,d}}_{M_0} = \mathsf{True}$ iff $\theta^{x,y}_{u,d}(y) \in I_{M_0}(\mathsf{donkey})$. That is, in a more informal notation, $\langle u, d\rangle \in I_{M_0}(\mathsf{own})$ and $d \in I_{M_0}(\mathsf{donkey})$. What constitutes the set $\{d \in D_{M_0} | [\![\mathsf{own}(x,y) \wedge \mathsf{donkey}(y)]\!]^{\theta^{x,y}_{u,d}}_{M_0} = \mathsf{True}\}$ varies according as to what exactly our meta-variable $u$ is.

If the value is actually $j$, that is $\theta^{x,y}_{u,d}(x) = j$, then the set $\{d|[\![\mathsf{own}(j,y) \wedge \mathsf{donkey}(y)]\!]^{\theta^y_d}_{M_0} = \mathsf{True}\}$ is composed of those individuals $d$ in $D$ such that $d \in I_{M_0}(\mathsf{donkey})$ and $\langle j,d\rangle \in I_{M_0}(\mathsf{own})$. Since $\langle j,d_1\rangle,\ldots,\langle j,d_{1000}\rangle, \in I_{M_0}(\mathsf{own})$, we see that

$$\{d \in D_{M_0}|[\![\mathsf{own}(j,y) \wedge \mathsf{donkey}(y)]\!]^{\theta^y_d}_{M_0} = \mathsf{True}\} = \{d_1,\ldots,d_{1000}\}.$$

If, on the other hand, $u$ is actually $a_1$, that is $\theta^{x,y}_{u,d}(x) = a_1$, then the set $\{d \in D_{M_0}|[\![\mathsf{own}(a_1,y) \wedge \mathsf{donkey}(y)]\!]^{\theta^y_d}_{M_0} = \mathsf{True}\}$ is composed of those individuals $d \in D_{M_0}$ such that $d \in I_{M_0}(\mathsf{donkey})$ and $\langle a_1,d\rangle \in I_{M_0}(\mathsf{own})$. We know that $\langle a_1,d_{1001}\rangle$ is the only pair that satisfies this. Hence $\{d \in D_{M_0}|[\![\mathsf{own}(a_1,y) \wedge \mathsf{donkey}(y)]\!]^{\theta^y_d}_{M_0} = \mathsf{True}\} = \{d_{1001}\}$. Similarly for other $a_i \in I_{M_0}(\mathsf{farmer})$. In sum, for each $u \in D_{M_0}$, there are 100 different values of $\{d \in D_{M_0}|[\![\mathsf{own}(x,y) \wedge \mathsf{donkey}(y)]\!]^{\theta^{x,y}_{u,d}}_{M_0} = \mathsf{True}\}$.

Now let us compute the value of the $\epsilon$-term $(\epsilon y)[\mathsf{own}(x,y) \wedge \mathsf{donkey}(y)]$. Recall that in our interpretation:

$$[\![(\epsilon y)[\mathsf{own}(x,y) \wedge \mathsf{donkey}(y)]]\!]^{\theta^x_u}_{M_0} = f(\{d \in D_{M_0}|[\![\mathsf{own}(x,y) \wedge \mathsf{donkey}(y)]\!]^{\theta^{x,y}_{u,d}}_{M_0} = \mathsf{True}\})$$

of the choice function $f \in \chi$. Since the argument of this function is dependent on the specific value $u$ of the variable $x$, the exact denotation of this $\epsilon$-term is also dependent on the value of $x$. For $a_1,\ldots,a_{99} \in D_{M_0}$ the computation is simple, since the extensions of $\{d \in D_{M_0}|[\![\mathsf{own}(x,y) \wedge \mathsf{donkey}(y)]\!]^{\theta^{x,y}_{u,d}}_{M_0} = \mathsf{True}\}$ for these individuals are unit sets.

The case in which $\theta^{x,y}_{u,d}(x) = j \in D_{M_0}$ is where the choice-function aspect of $\epsilon$-terms comes in. There are 1000 members in the set $\{d \in D|[\![\mathsf{own}(j,y) \wedge \mathsf{donkey}(y)]\!]^{\theta^y_d}_{M_0} = \mathsf{True}\}$, namely $\{d_1,\ldots,d_{1000}\}$. The choice-function $f$ arbitrarily selects exactly one member from this set; the value could be $d_{259}$ or $d_{713}$ or $d_1$. We know that there is exactly one member that answers to this function but do not know which. Since it does not matter which one these 1000 individuals the choice-function $f$ selects, we can just name it $d_@$; $d_@$ is the meta-variable for the individual that $f$ arbitrarily selects from $\{d_1,\ldots,d_{1000}\}$. Hence

- for $j$,

  $f(\{d \in D|[\![\mathsf{own}(j,y) \wedge \mathsf{donkey}(y)]\!]^{\theta^y_d}_{M_0} = \mathsf{True}\}) = d_@$

This is an interesting result, because, although $I_{M_0}(\mathsf{own})$ consists of 1099 pairs, namely $\{\langle j,d_1\rangle,\ldots,\langle j,d_{1000}\rangle,\langle a_1,d_{1001}\rangle,\ldots,\langle a_{99},d_{1099}\rangle\}$, as far as the extension, so to speak, of the whole antecedent *farmers who own a donkey* is concerned, there are only 100: $\{\ \langle j,d_@\rangle,\langle a_1,d_{1001}\rangle,\langle a_2,d_{1002}\rangle,\ldots,\langle a_{99},d_{1099}\rangle\}$. So the set of values for $x$ that are in this extension is now known to be:

$$\{u \in D_{M_0}|[\![\mathsf{farmer}(x) \wedge \mathsf{own}(x,\alpha) \wedge \mathsf{donkey}(\alpha)]\!]^{\theta^x_u}_{M_0} = \mathsf{True}\} = \{j,a_1,\ldots,a_{99}\}.$$

The "extension" of the consequent phrase *beat it* can be considered in the similar manner to that of the antecedent phrase *farmers who own a donkey*. Although all 1000 of $\langle j,d_1\rangle,\ldots,\langle j,d_{1000}\rangle$

are in $I_{M_0}(\mathsf{beat})$, the phrase *beat it* can be interpreted as containing only $\langle j, d_@\rangle$, because the phrase is

$$[\![\mathsf{beat}(x, (\epsilon y)[\mathsf{own}(x, y) \wedge \mathsf{donkey}(y)])]\!]^{\theta^x_u}_{M_0}.$$

This is satisfied by those assignments $\theta'(x) = j$ and the value of $[\![(\epsilon y)[\mathsf{own}(x, y) \wedge \mathsf{donkey}(y)]]\!]^{\theta^x_u}_{M_0}$ at $j$ is $d_@$. The set of the individuals that are the values of $x$ and satisfy this condition is, therefore, the set $\{u \in D_{M_0} | [\![\mathsf{beat}(x, \alpha)]\!]^{\theta^x_u}_{M_0} = \mathsf{True}\} = \{j\}$.

Now the truth definition clause for the generalized quantifier **Most** tells us that:

$$[\![(\mathsf{Most} : x)(\phi, \psi)]\!]^{\theta}_{M} = \mu_{\mathsf{Most}}(m, n, p) = \mathsf{True} \iff m > n,$$

where

- $m = |\{d \in D | [\![\phi]\!]^{\theta^x_d}_{M} = \mathsf{True}\} \cap \{d \in D | [\![\psi]\!]^{\theta^x_d}_{M} = \mathsf{True}\}|$, and
- $n = |\{d \in D | [\![\phi]\!]^{\theta^x_d}_{M} = \mathsf{True}\} \cap \{d \in D | [\![\neg\psi]\!]^{\theta^x_d}_{M} = \mathsf{True}\}|$.

Let

- $\Phi = \{d \in D_{M_0} | [\![\mathsf{farmer}(x) \wedge \mathsf{own}(x, \alpha) \wedge \mathsf{donkey}(\alpha)]\!]^{\theta^x_d}_{M_0} = \mathsf{True}\}$ and
- $\Psi = \{d \in D_{M_0} | [\![\mathsf{beat}(x, \alpha)]\!]^{\theta^x_d}_{M_0} = \mathsf{True}\}$.

Then $\Psi' = \{d \in D_{M_0} | [\![\neg\mathsf{beat}(x, \alpha)]\!]^{\theta^x_d}_{M_0} = \mathsf{True}\} = \{d \in D_{M_0} | [\![\mathsf{beat}(x, \alpha)]\!]^{\theta^x_d}_{M_0} = \mathsf{False}\}$. And therefore we have:

$$m = |\Phi \cap \Psi| = |\{j, a_1, \ldots, a_{99}\} \cap \{j\}| = |\{j\}| = \mathsf{True}$$

$$n = |\Phi \cap \overline{\Psi}| = |\Phi - \Psi| = |\{j, a_1, \ldots, a_{99}\} - \{j\}| = |\{a_1, \ldots, a_{99}\}| = 99$$

Thus $m = \mathsf{True}$ and $n = 99$ in the model $M_0$. Since $\mu_{\mathsf{Most}}(m, n, p) = \mathsf{True} \iff m > n$ and clearly $1 \not> 99$. Thus

$$\mu_{\mathsf{Most}}(m, n, p) = 0.$$

The sentence (2) is clearly false in the model $M_0$ with respect to our $\epsilon$-term analysis.

## 3.1 Remarks and Conclusion

We notice that there are a few things that clearly distinguish our $\epsilon$-term analysis from DRT. These are:

1. Quantification is over individuals.
2. The value of *a donkey* gives us just one individual which is chosen arbitrarily from the extension of *donkey*. This individual plays the role of the representative donkey that is the denotation of the indefinite noun phrase *a donkey*.

3. The dependence of "a donkey" on "a farmer" which is expressed by the phrase *who owns a donkey* is interpreted by means of a function that depends on the denotation of *farmer*.

The first point is in contrast with the quantification over assignments of DRT. The second point shows a sharp contrast between our interpretation of indefinite noun phrases and that of DRT. The third point is what is completely missed in DRT. These last two points are extremely important and are what make a difference in the treatment of the donkey sentence containing *most*.

To make these points more conspicuous and concrete, let us reexamine our treatment of the problematic sentence (2).

The truth value of the interpretation of (2):

$$[\![(\mathsf{Most} : x)[\mathsf{farmer}(x) \wedge \mathsf{own}(x, \alpha) \wedge \mathsf{donkey}(\alpha), \mathsf{beat}(x, \alpha)]\,]\!]^{\theta}_{M_0},$$

where $\alpha = (\epsilon y)[\mathsf{own}(x, y) \wedge \mathsf{donkey}(y)]$, is dependent on the cardinalities of the set

$$\{u \in D_{M_0} | [\![\mathsf{farmer}(x) \wedge \mathsf{own}(x, \alpha) \wedge \mathsf{donkey}(\alpha)]\!]^{\theta^x_u}_{M_0} = \mathsf{True}\}$$

and the set

$$\{u \in D_{M_0} | [\![\mathsf{beat}(x, \alpha)]\!]^{\theta^x_u}_{M_0} = \mathsf{True}\}.$$

This shows that our interpretation of the quantifier *most* is not taken as a quantifier over assignments but one over individuals.

One of the most important points in our treatment is the fact that the translation, so to speak, of the noun phrase *a donkey* gives rise to the $\epsilon$-term:

$$(\epsilon y)[\mathsf{own}(x, y) \wedge \mathsf{donkey}(y)].$$

This can be seen, of course, from the fact that in the translation above *a donkey* corresponds to the translation:

$$\mathsf{donkey}((\epsilon y)[\mathsf{own}(x, y) \wedge \mathsf{donkey}(y)])$$

This is in sharp contrast with the representation given in DRT where *a donkey* is simply translated as:

$$\text{donkey}(\mathbf{v}),$$

where $\mathbf{v}$ is simply a free variable.

The point of $\epsilon$-term is that although the set

$$\{d \in D_{M_0} | [\![\mathsf{own}(x, y) \wedge \mathsf{donkey}(y)]\!]^{\theta^{x,y}_{j,d}}_{M_0} = \mathsf{True}\}$$

of the donkeys that are owned by John is not a singleton set in $M_0$, the interpretation of this $\epsilon$-term:

$$[\![(\epsilon y)[\mathsf{own}(x, y) \wedge \mathsf{donkey}(y)]\,]\!]^{\theta^x_j}_{M_0}$$

is a single individual. In other words, even though there are 1000 donkeys that satisfy the formula:

$$\mathsf{own}(x, y) \wedge \mathsf{donkey}(y)$$

when $x$ is assigned to $j$, only one of them counts as the denotation of the $\epsilon$-term and hence of the noun phrase ***a donkey*** as is translated as:

$$\mathsf{donkey}((\epsilon y)[\mathsf{own}(x,y) \wedge \mathsf{donkey}(y)]).$$

Any one of $d_1, \ldots, d_{1000}$ can be the denotation of the $\epsilon$-term. Which one of the donkeys is actually chosen is not important; what is important is that one such individual is chosen. This is why we can think of the denotation of the $\epsilon$-term as the *representative* of the extension of the noun phrase *a donkey*[6]. Our truth condition for the formula and the pro-term elimination make it clear that if John beats this representative donkey, then he counts as a farmer who beats a donkey he owns. This representative characteristic of the $\epsilon$-term is a consequence of the $\epsilon$-axiom:

$$\frac{\vdash \exists x \phi(x)}{\rhd \epsilon x.\phi(x)}$$

and its interpretation of an $\epsilon$-term in terms of a choice function that we gave at the end of section 2. The reader may notice the intuitive way our theory interprets the indefinite noun phrase.

Notice also that the interpretation of the indefinite is not bound by a quantifier directly. So it may be natural to ask the question: "How can the quantificational force of an indefinite be derived, when such is needed?" The answer to this question is directly related to the third characteristic above of our analysis.

Notice that in the translation of the whole sentence (2) the $\epsilon$-term:

$$(\epsilon y)[\mathsf{own}(x,y) \wedge \mathsf{donkey}(y)]$$

has the free-variable $x$, which is bound in turn by the quantifier $(\mathsf{Most} : x)$. So the interpretation:

$$[\![(\epsilon y)[\mathsf{own}(x,y) \wedge \mathsf{donkey}(y)]]\!]^{\theta^x_u}_{M_0}$$

of this term is not just an individual but, in fact, is a function that gives a value depending on the assignment to the variable $x$; the denotation of the $\epsilon$-term is dependent on the value of the assignment to the variable that stands for a farmer. Put informally, the representative donkey is always chosen with respect to the farmer that owns the donkey. When we abbreviate the $\epsilon$-term, we could use the more suggestive abbreviation that shows the dependence of the $\epsilon$-term on the variable $x$. Namely, thus:

$$[\![\mathsf{farmer}(x) \wedge \mathsf{own}(x,\alpha_x) \wedge \mathsf{donkey}(\alpha_x)]\!]^{\theta^{x,y}_{u,d}}_{M_0}.$$

Now this is an important point. Because of this dependence interpreted as a function, we can give both the original donkey sentence and the problematic (2) the intuitively correct truth conditions, even though there is only one quantifier that quantifies over individuals. In other words, we would not need to consider a set of pairs such as:

$$\{\langle u,d\rangle \mid [\![\mathsf{farmer}(x) \wedge \mathsf{own}(x,\alpha_x) \wedge \mathsf{donkey}(\alpha_x)]\!]^{\theta^{x,y}_{u,d}}_{M_0} = \mathsf{True}\},$$

because our translation makes it clear that, for any $u$, $d$ is chosen depending on $u$. Notice, however, that even if we were to think of a set of pairs we would not be troubled by the proportion problem any more than in the original treatment of (2) above. Not only is $d$ chosen depending on $u$ in $\langle u, d\rangle$, it is also chosen arbitrarily as the representative.

One may notice in passing that the function-like dependence of the $\epsilon$-term on the universally bound variable and the representative character of its interpretation can account for the non-exhaustive reading of the original donkey sentence. Thanks to the dependence, our theory does not have to resort to the quantification over pairs. The interpretation of the donkey sentence which incorporates the representative individual can then give a weaker truth-condition than the one given by DRT. The DRT analyses, on the other hand, make it mandatory that the donkey sentence be treated in such a way that the quantification is over the set of pairs of assignments, forcing upon the interpretation the truth-condition in which the *beat*-relation is exhaustive over the farmer-donkey pairs that satisfy the *own*-relation.

In sum, our theory has the following characteristics that give it the advantage over the other theories.

a. For each farmer there is a donkey that depends on the denotation of the farmer.

b. If there is at least one donkey in the extension of *donkey* then any of the individuals in the extension counts as the representative of the donkeys.

c. The pronoun *it* is substituted for the term whose reference is this representative. In other words, it refers to whatever individual is denoted by this representative.

As we have just seen, these characteristics are all important in the intuitive treatment of the donkey sentence, especially in the form of (2).

The success with which DRT has treated the original donkey sentence seems to be rather accidental and contingent upon the specific logical form, and interpretation, the donkey sentence requires; the donkey sentence in its original form requires that the indefinite noun phrase be interpreted as a universal quantifier, which is the traditional translation of the noun phrase with *every*. Hence the two quantifiers that head the translated formula have the same quantificational force: that of the universal quantifier. This fact does not interfere with the quantification over pairs of assignments, since a universal quantifier binding a pair of variables (or assignments) gives the same truth condition as two universal quantifiers on respective components of the pair;

$$(\forall x)(\forall y)\phi(x, y) \equiv (\forall xy)\phi(x, y).$$

However, when the quantifiers have different quantificational force, as in the proportion problem, such a treatment does not work and gives the wrong truth condition. This fact shows that DRT lacks something essential in the treatment of the donkey sentence, something which is at the heart of the problem. What is at the heart of the problem requires a treatment of anaphora and reference that somehow incorporates the characteristics which the $\epsilon$-term analysis possesses, as has been made clear with respect to our approach to the proportion problem.

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## Notes


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[1]See Hilbert and Bernays (1939) and Leisenring (1969) for detail.

[2]Hilbert and Bernays (1939).

[3]See for example Fine (1985) and Chisholm (1960) for these ideas.

[4]Definitions are basically as in Leisenring (1969).

[5]See Kaplan (1989), pp.489–490.

[6]In this respect, we could even think of the $\epsilon$-term itself as a semantic object, which represents an arbitrary individual.