NUMERICAL SIMULATION OF PLASTIC FLOW BY FINITE ELEMENT LIMIT ANALYSIS

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Abstract

Limit analysis has been rendered versatile in many problems such as structural problems and metal forming problems. In metal forming analysis, a slip-line method and an upper bound method have filled the role of limit analysis. As a breakthrough of the previous work, computational approach to limit solutions is considered as the most challenging areas.

In the present work, a general algorithm for limit solutions of plastic flow is developed with the use of finite element limit analysis. The algorithm deals with a generalized Holder inequality, a duality theorem, and a combined smoothing and successive approximation in addition to a general procedure for finite element analysis. The algorithm is robust such that from any initial trial solution, the first iteration falls into a convex set which contains the exact solution(s) of the problem. The idea of the algorithm for limit solution is extended from rigid/perfectly-plastic materials to work-hardening materials by the nature of the limit formulation, which is also robust with numerically stable convergence and highly efficient computing time.

1. Introduction

Limit analysis is known as the asymptotic approach in plasticity. Although the early theory of limit analysis was developed in an ad hoc manner, the current state of limit analysis have been able to be established on the deeper physical and mathematical foundation than previously attempted. A concise architecture of limit analysis is now emerged with new physical interpretation, rigorous mathematical formulation and efficient computational methodology. As a consequence, computational approach to limit analysis is often regarded as finite element limit analysis. With the aid of the finite element limit analysis, a new attempt to obtain the plastic flow field in various metal forming process becomes possible as a substitution of a slip-line method or an upper bound method which has been carried out by intuitions inspired from deep theoretical backgrounds and insights on the mechanics. The systematic approach is robust such that from any initial trial solution, the first iteration falls into a convex set which contains the exact solution(s) of the problem. Accordingly, it always converges to the exact solution from any arbitrary initial guess, and makes it possible to obtain the solutions of very complicated problems without a prior conjecture.

Although limit analysis studies the asymptotic behavior of elastoplastic materials, the idea of the algorithm for limit solutions can be extended from rigid/perfectly plastic materials to work-hardening materials. The present finite element limit analysis can be applied to the problem of plastic deformation for work-hardening materials by replacing the initial yield stress with the current flow stress. This idea is of no difference from that of the incremental analysis except it always guarantees numerical stable convergence and highly efficient computing time because there is no need to compute the elastoplastic tangential modulus.

In this paper, the framework in the finite element limit analysis is described as a mechanics problem, interpreting a model of asymptotic behavior of materials, stating a variational principle of duality, and developing a computational algorithm. Although the concept applies to general limit problems, examples and discussions pertain only to the class of plane strain problems and axisymmetric problems. Numerical examples of the plane strain problems are compared with the slip-line solutions and those of the axisymmetric problems deal with work-hardening materials as a real metal forming process.

2. Limit Analysis Theory

From a function space point of view, solutions of the mechanics problems lie in the intersection of three fundamental convex sets, the statically admissible set **S**, the constitutively admissible set **C**, and the kinematically admissible set **K**. If the intersection is empty, there exist no solution. If it contains a single element, the solution is unique. Otherwise, there is a set of feasible solutions of which one may be the most preferred optimal solution. The criterion of choosing the optimal solution is facilitated by an objective function. Since $\mathbf{S} \cap \mathbf{C} \cap \mathbf{K}$ is a subset of $\mathbf{S} \cap \mathbf{C}$, the optimal solution contained in the former is obviously in the latter. The primal formulation of a limit analysis problem seeks an extreme point in $\mathbf{S} \cap \mathbf{C}$ as its optimal solution and the dual formulation seeks an extreme point in **K** with the best choice of an objective function.

2.1. Primal Formulation

The primal formulation can be derived from the statically admissible conditions and the constitutively admissible conditions. The statically admissible conditions include the equilibrium equation $\nabla \cdot \sigma = 0$ in the domain D and the static boundary condition $\sigma \cdot n = t$ on the part of the boundary ∂D_s, where a given traction vector t is prescribed. The constitutively admissible conditions can be the yield condition with the implicit normality condition. A limit analysis problem seeks an extreme point in $S \cap C$, that maximizes the applied load in its proportional form, qt, where q is a positive, real scaling factor. The constrained maximization of the objective functional $q(\sigma)$ in the form,

$$\begin{aligned} &\text{maximize} && q(\sigma) && \\ &\text{subject to} && \nabla \cdot \sigma = 0 && \text{in } D \\ & && \sigma \cdot n = q \cdot t && \text{on } \partial D_s \\ & && \|\sigma\| \leq \sigma_o && \end{aligned} \tag{1}$$

defines the primal formulation of a general limit analysis problem. The problem (1) is a convex programming in the function space $R^{3\times3}(D)$, which seeks the maximum $q(\sigma)$, while the magnitude of stress σ is bounded by the yield condition in the form of a convex norm. It is also called the lower bound formulation in plasticity and $L = S \cap C$ is called the lower bound solution set since every point in L corresponds to a value of q either lower than or equal to the maximum value q* sought.

2.2. Dual Formulation

A convex programming problem has a dual problem whose minimum solution is equal to q*. To construct the dual problem of (1), it is started from the weak equilibrium equation

$$\int_D (\nabla \cdot \sigma) \cdot u \, d\Omega = 0 \tag{2}$$

where u is an arbitrary function in $R^3(D)$ with the physical interpretation of an admissible velocity function. An admissible which satisfies the kinematic conditions on the part of the boundary ∂D_k complement to ∂D_s and derives meaningful quantities under a generalized divergencetheorem will lead to the equivalent variational statement,

$$\int_D \sigma:\varepsilon \, d\Omega = q \int_{\partial D_s} t \cdot u \, d\Gamma \tag{3}$$

where $\varepsilon = \frac{1}{2}(\nabla u + \nabla u^T)$ is the 3×3 strain rate matrix and : denotes the inner product operator between two matrices. Since certain non-differentiable functions are admissible in (3), this relaxed variational principle greatly enlarges the kinematically admissible set K from the set of compatible strains defined in the theory of elasticity [7]. The boundary integral in (3) may be normalized such that

$$\int_{\partial D_s} t \cdot u \, d\Gamma = 1 \tag{4}$$

since u appears homogeneously in (3), explicitly in the right hand side and implicitly as ε in the left hand side. The normalization results in the alternative statement of (3) such that

$$q = \int_D \sigma:\varepsilon \, d\Omega \tag{5}$$

which also implies that the integral does not vanish and the integrand $\sigma:\varepsilon$ is always non-zero and positive. Accordingly, the term $\sigma:\varepsilon$ can be restated by a generalized Holder inequality [8] [9] as

$$\sigma:\varepsilon = |\sigma:\varepsilon| \leq \|\sigma\|_{(v)} \|\varepsilon\|_{(-v)} \tag{6}$$

where the (-v) norm is called the dual norm of the (v) norm which could be the von Mises norm or anyone else. When the von Mises norm is applied, the two norms in the right hand side of (6) can be expressed as

$$\begin{aligned} \|\sigma\|_{(v)} &= \sqrt{\tfrac{1}{2}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]} = \bar{\sigma} \\ \|\varepsilon\|_{(-v)} &= \sqrt{\tfrac{2}{9}[(\varepsilon_1-\varepsilon_2)^2+(\varepsilon_2-\varepsilon_3)^2+(\varepsilon_3-\varepsilon_1)^2]} = \bar{\varepsilon} \end{aligned} \tag{7}$$

The above two norm expressions are distinguished from each other by the subscripts (v) and (-v), since the nature of stress is different from that of strain. For explanation, the norm expressions in the case of the plane strain state become

$$\begin{aligned} \|\sigma\|_{(v)} &= \sqrt{\sigma_1^2-\sigma_1\cdot\sigma_2+\sigma_2^2} \\ \|\varepsilon\|_{(-v)} &= \sqrt{\frac{4}{3}[\varepsilon_1^2+\varepsilon_1\cdot\varepsilon_2+\varepsilon_2^2]} \end{aligned} \tag{8}$$

which are also valid for the plane stress state. The inequality (6) is sharp, meaning that equality holds when ε is chosen to be proportional to the gradient of the yield function. This sharpness condition,

$$\varepsilon = k \, \nabla \, \|\sigma\|_{(\beta)} \tag{9}$$

is the well known normality condition[10] in plasticity, where k is a proportional factor. Consequently, a sharp upper bound to the functional $q^{(\sigma)}$ can be established

$$q^{(\sigma)} = \int_D \sigma:\varepsilon \, d\Omega \le \int_D \|\sigma\|_{(v)}\|\varepsilon\|_{(-v)} \, d\Omega \le \sigma_o \int_D \|\varepsilon\|_{(-v)} \, d\Omega = \tilde{q}(u) \tag{10}$$

where the upper bound functional $\tilde{q}(u)$ depends only on the kinematically admissible function $u \in K$. The correct choice of K still needs to be studied in the thorough research of functional analysis and calculus of variation. Based on the inequalities in (10) and the existence of the absolute minimum of $\tilde{q}(u)$ [11], we may state the dual formulation,

$$\begin{aligned} &\text{minimize} \quad \tilde{q}(u) \\ &\text{subject to} \quad \tilde{q}(u) = \sigma_o \int_D \|\varepsilon\|_{(-v)} \, d\Omega \, , \\ &\qquad \int_{\partial D_s} t\cdot u \, d\Gamma = 1 \\ &\qquad Tr[\varepsilon]=1 \\ &\qquad \text{kinematic boundary conditions on } \partial D_k \end{aligned} \tag{11}$$

where $Tr[\varepsilon]=1$ is the incompressibility condition. While K is constructed with all kinematically admissible velocity fields, the exact solution is in the smallest function space whose elements satisfy the constraints in (11) and produce the absolute minimum of the objective functional. When the absolute minimum of $\tilde{q}(u)$ is attained, it is realized that the duality relation

$$\max q(\sigma) = q^* = \min \tilde{q}(u) \tag{12}$$

In real problems, general solutions of (11) could be obtained with the numerical method. In the next section, the upper bound functional is first discretized into finite elements, then a combined smoothing and successive approximation algorithm [9] [12] is used to solve the finite dimensional minimization problem.

3. Finite Dimensional Approximation and a Minimization Technique

The dual formulation is discretized with the aid of a finite element method and reduced into a convex programming problem in a finite dimensional space R^n where n is the total number of the discrete variables. The integral representing the upper bound functional $\tilde{q}(u)$ in (11) is then approximated by a sum

$$\tilde{q}(u) \approx \sum_{e=1}^{E} \sqrt{U^T A_e U} \tag{13}$$

where U is the discrete vector representation of the velocity field u, T transposes a vector, A_e is the element stiffness matrix, and the integer E is the total number of elements. The scalar product $U^T K_e U$ in each term of the sum is interpreted as a product formed in R^n where $U \in R^n$ is the global velocity vector and K_e is embedded in a $n \times n$ null matrix.

Similarly, the normalization equation $\int_{\partial D_t} t \cdot u \, d\Gamma = 1$ in (11) is approximated by $C^T U = 1$ where $C \in R^n$ is a constant vector. The finite dimensional approximation of (11) takes the form

$$\begin{aligned} &\text{minimize} \quad \tilde{q}(U) \\ &\text{subject to} \quad \tilde{q}(U) = \sum_{e=1}^{E} \sqrt{U^T A_e U} \end{aligned} \tag{14}$$

where the parameter σ_0 as well as the static and kinematic boundary conditions and the incompressibility condition is absorbed into the matrices A_e and the vector C. It can be easily shown that each A_e is positive definite or positive semi-definite and $\sqrt{U^T A_e U}$ is a convex function in R^n. Since the sum of convex functions is convex, $\tilde{q}(U)$ is convex and has a unique minimum value.

One last obstacle is still in the path of numerical solution of (14). Some matrices A_e are only positive semi-definite such that the product $U^T A_e U$ may vanish for some non-trivial vectors U, which may causes serious problems in the minimization procedure. To overcome this difficulties, the objective function is slightly perturbed with a small real number ε as a smoothing parameter[12] such that

$$\tilde{q}(U,\varepsilon) = \sum_{e=1}^{E} \sqrt{U^T A_e U + \varepsilon^2} \tag{15}$$

which is differentiable everywhere for $\varepsilon \neq 0$ and remains convex. The perturbed function recovers its original value as $\varepsilon \rightarrow 0$.

The constrained minimization problem (14) is converted, using an Lagrange multiplier λ, to an unconstrained one with the perturbed objective function in (15) such that

$$\text{minimize} \quad \Phi(U) = \tilde{q}(U,\varepsilon) - \lambda(C^T U - 1) \tag{16}$$

The minimum solution satisfies the conditions $\dfrac{\partial \Phi}{\partial U_i} = 0$, i = 1, 2, ...,n. It leads to the problem of solving a system of equations such that

$$AU_\varepsilon = \lambda C \tag{17}$$

in matrix notation, where the global stiffness matrix

$$A = \sum_{e=1}^{E} \frac{A_e}{\sqrt{U^T A_e U + \varepsilon^2}} \tag{18}$$

is regarded as a constant matrix in each iteration and is updated from iteration to iteration with the vector U obtained in the previous iteration. The above problem is treated in each iteration as a linear system to be solved repeatedly with an inner and an outer iterative sequences.

A solution of the problem may be symbolically expressed as

$$U_\varepsilon = \lambda A^{-1} C \tag{19}$$

where λ can be evaluated by the condition $C^T U = 1$ to obtain

$$\lambda = \frac{1}{C^T A^{-1} C} \tag{20}$$

The outer iteration is associated with a decreasing sequence of ε. At each fixed value of ε, an inner iteration begins with the previously obtained vector U as its initial vector. The solution of (17) in each inner iteration is used in a feedback loop to update A and λ. The converged solution of $\tilde{q}(U,\varepsilon)$ and U_ε under a suitable stopping criterion terminates an inner iteration loop. Then, is reduced for the outer iteration to start another inner iteration. During the inner and outer iterations, an initial vector $U^{(0)}$ is assumed only in the first inner iteration. From any initial vector $U^{(0)}$, the subsequent iterates are locked in a certain convex hall defined by the data of the discrete problem (14). Thus, in reality, only a few values of ε is needed to extrapolate to the limit, $\varepsilon \rightarrow 0$. This robust initial convergence and the rate of convergence for the subsequent iterations are discussed in [12].

4. Numerical Examples

The developed algorithm was demonstrated for its validity and versatility with various plane stress problems in [9]. In this section, the algorithm is to be demonstrated with several plastic flow problems in the plane strain case and the axisymmetric case. The finite dimensional approximation has been carried out by a finite element method with the three node linear triangular element or the four node bilinear quadrilateral element. Nevertheless the choice of the type of finite elements may depend on the nature of a problem, the linear element was rendered satisfactory in most cases.

4.1. Plane Strain Indentation by a Flat Punch

The first application of the present methodology must be a test of its ability to reproduce a known result. The plane strain indentation problem is selected as the bench mark for comparison since it is one of the typical problems solved by the slip-line method [2] [4]. Any solution of an indentation problem is immediately applicable, with only a change in sign, to the tension of a notched bar by superposing a uniform velocity to bring one end of the bar to have a uniform velocity. The estimated yield-point load for a plane semi-infinite medium indented by a smooth flat die is $P = 2ka(2+\pi)$ where 2a is the width of the die. Since the load was obtained from a lower bound formulation in terms of stress, it is a lower bound solution. A calculated dual solution for the yield-point load with the present algorithm is $P = 3.024901$, which is obtained after 20 iterations as a total sum of inner and outer iterations. The two upper and lower bound solutions shows the deviation of less than 2 %, which can be satisfactory when it is counted in that the numerical calculation approximates a semi-infinite medium as a finite medium with rough finite element meshes and allows continuity in a velocity field. A velocity field obtained for this problem is shown in Fig.1 with a schematic slip-line field. Fig. 1 demonstrates the obtained velocity field is almost same as the velocity field by the slip-line method except the former is continuous. It is a strong verification of the dual variational principle and the algorithm.

The present algorithm is readily applied to the constrained indentation problem which is an open problem with the slip-line method. The calculated yield-point load is $P = 3.438487$ after 20 iterations. This load is slightly larger than the upper bound solution $P = 2ak(2+\pi+\frac{b}{\sqrt{2}a})$ obtained with the velocity field drawn in Fig.2. The deviation of two solution is less than 2 % though. Fig. 2 shows the calculated velocity field and a schematic kinematically admissible velocity field. There seems a region with rigid body motion under the constrained boundary. It follows that the direct application of this velocity field to construct a slip-line field still needs a thorough study. The velocity field itself, however, provides a good information for the construction of a possible slip-line field.

Fig.3 shows the velocity fields in backward extrusion with the large and small extrusion ratio respectively as a modified problem of the indentation problem. The velocity fields explain that the plastic flow spreads over the entire region with the small extrusion ratio while the plastic flow is restricted within a part of the region with the large extrusion ratio. This tendency becomes obvious when the depth of a medium

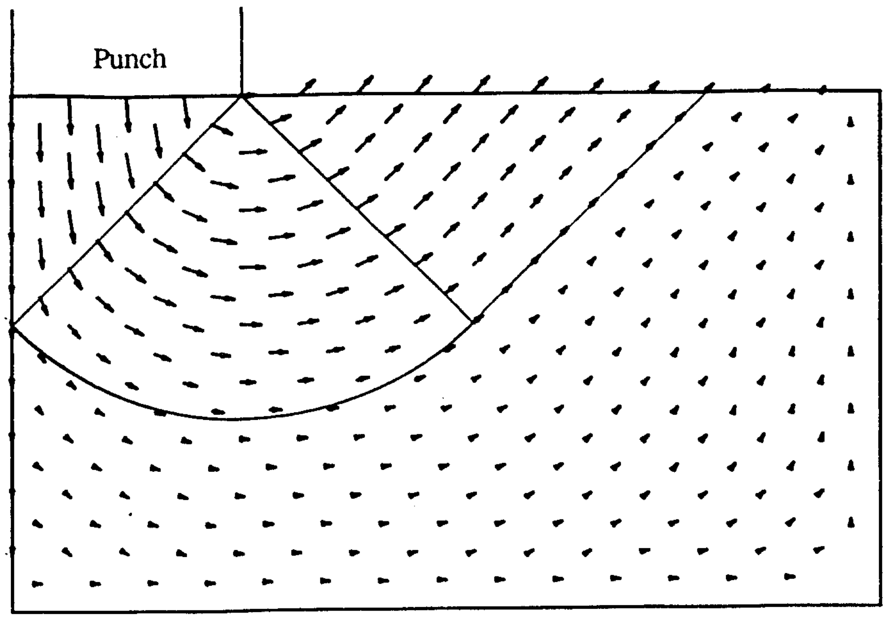


Fig. 1 Plastic flow in the indentation of a plane semi-infinite medium by a flat punch

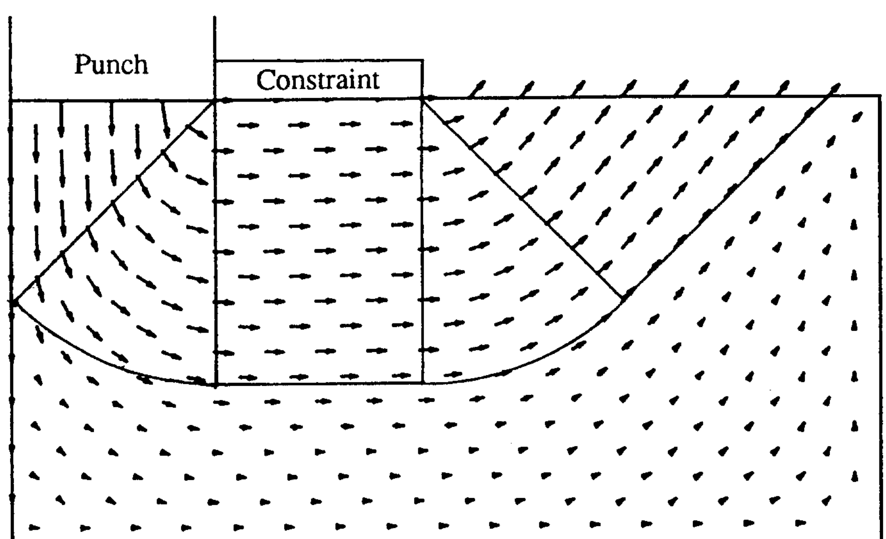
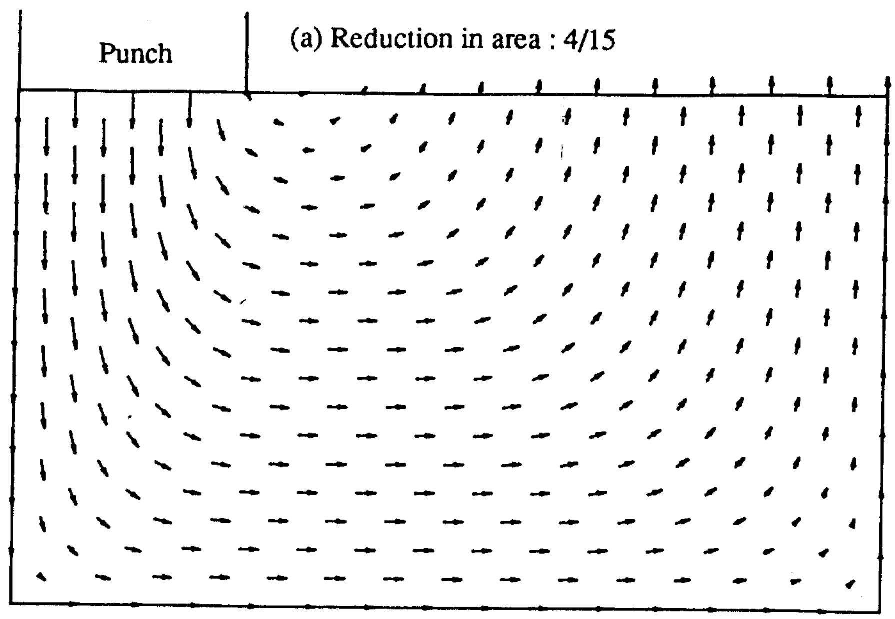


Fig. 2 Plastic flow in the constrained indentation of a plane semi-infinite medium by a flat punch



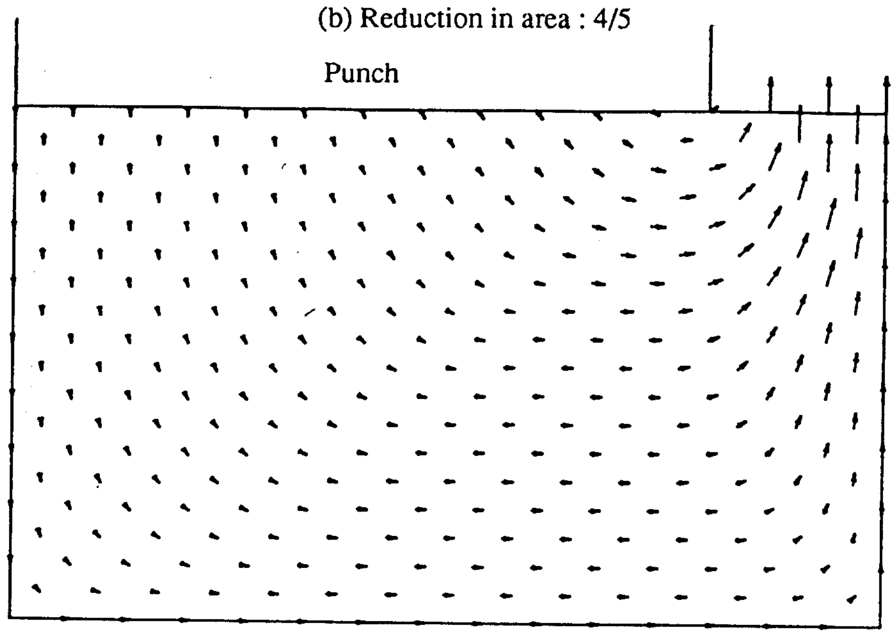


Fig. 3 Plastic flow in the backward extrusion with
(a) small extrusion ratio and
(b) large extrusion ratio

becomes larger than the domain in calculation, which is in good agreement with the result in [3].

4.2. Plane Strain Extrusion through a Square Die

The plane strain extrusion problem through a square die has been studied in great detail with the slip-line method [1] [3] and the upper bound method [5] [15]. The velocity field associated with the deformation region and the dead-metal region has been assumed in various ways and compared with the experimental data. This procedure needed intuitive techniques inspired from deep theoretical backgrounds and insights on the mechanics and mathematics. The present algorithm, on the other hand, offers the velocity field under the constraints of the minimum dual functional and the smallest kinematically admissible function space without any prior conjecture. The calculated extrusion load is plotted with the variation of the extrusion ratio in Fig.4, and compared with the slip-line solutions. The figure shows there is a narrow gap between the slip-line solutions and the calculated ones. The gap will be definitely narrower as the number of finite elements and iterations is increased.

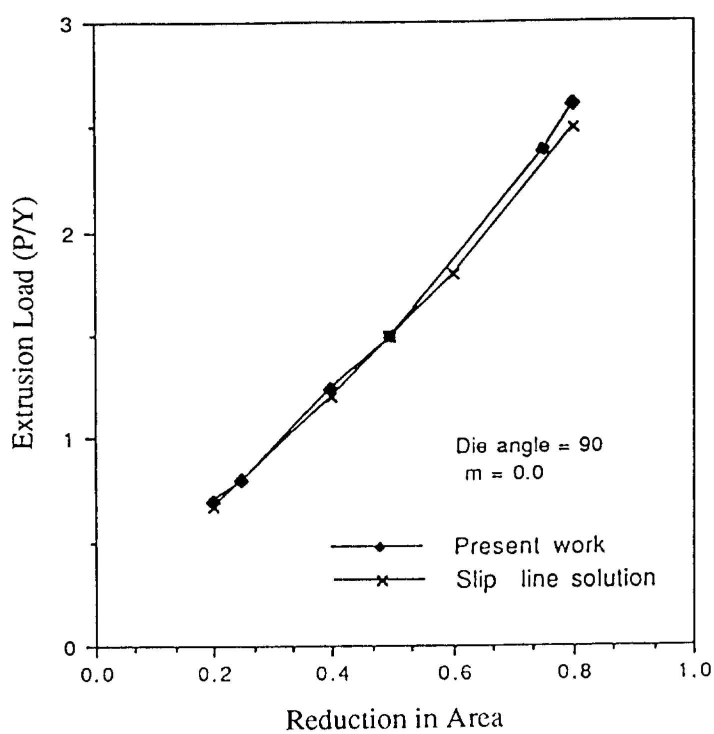
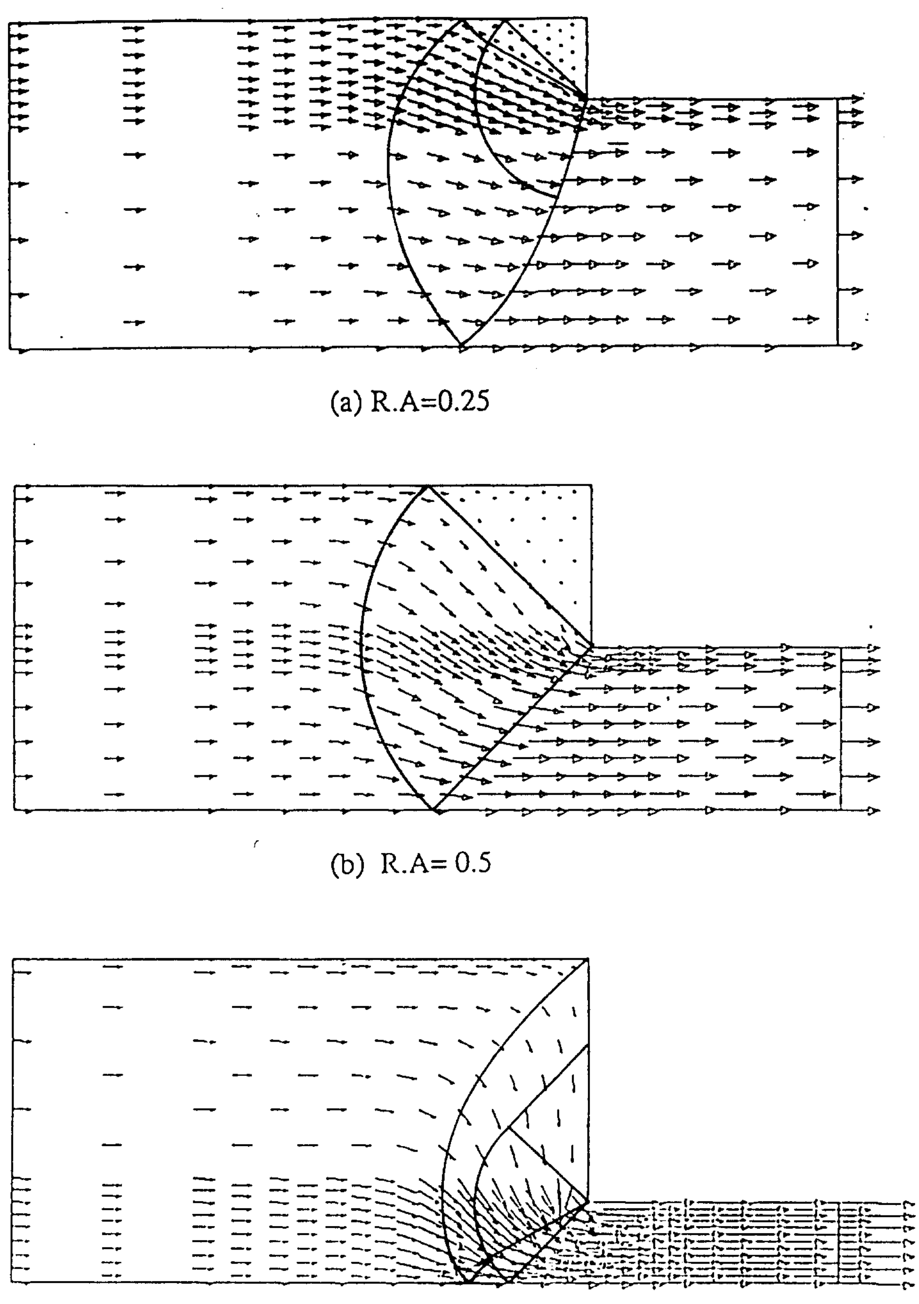


Fig. 4 Extrusion loads in the plane strain extrusion through a square die with the variation of reduction area



(a) R.A=0.25

(b) R.A= 0.5

(c) R.A= 0.75

Fig. 5 Plastic flow in the plane strain extrusion through a square die with the related schematic slip-line field

Fig.5 shows the velocity fields for the extrusion ratio of 0.25, 0.5, and 0.75. The figures indicate that the dead-metal region is formed with the small extrusion ratio and is reduced with the larger extrusion ratio. The figures also show there is only slight difference between the calculated velocity fields and the slip-line fields considering the former allows a continuous velocity field. With the extrusion ratio of 0.75, there observed difference to some extext between the two velocity fields. As a matter of fact, two slip-line solutions exists in this case [2] [3], one with the dead-metal region and the other without the dead-metal region. The obtained velocity field is a intermixture one and rather closer to the one without the dead-metal region. This result is in good agreement with the experimental result in [3].

4.3. Axisymmetric Extrusion with Work-hardening materials

In the previous two sections, the validity and versatility of the present algorithm is fully demonstrated by comparing the obtained result with the analytical and experimental result. To apply the present algorithm to a real metal forming process, however, the present methodology has to be furnished with the treatment of the friction between dies and materials, and the work-hardening behavior of materials. The objective function in the dual formulation (11) can be restated as

$$\tilde{q}(u) = \sigma_o \int_D \|\varepsilon\|_{(-v)} \, d\Omega + mk \int_{\partial D_f} |u_s| \, d\Gamma \qquad (21)$$

by including the term related to the frictional dissipation with the constant friction factor m. Of course, this objective function (21) can be derived from the weak equilibrium equation by applying the proper boundary condition without any difficulties. In the above objective function, the yield stress σ_o and the yield shear stress k can be replaced by the current yield stress as

$$\sigma_o = \overline{\sigma} = F(W_p) \text{ or } H(\overline{\varepsilon}) \qquad (22)$$

Then, the formulation can deal with not only the problem of a medium with different materials but the problem of a work-hardening medium as the aggregate of elements with different flow stresses. The current yield stress can be obtained from a typical and simple stress-strain relation

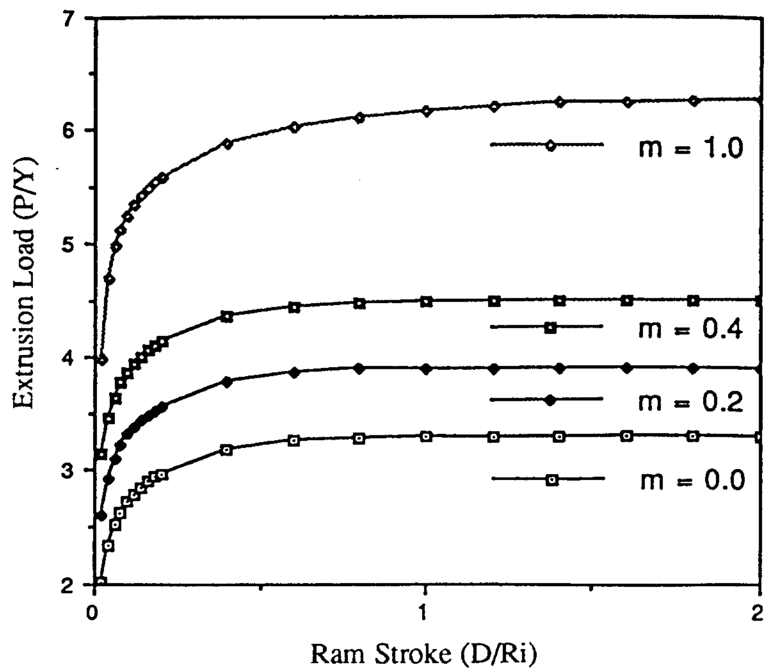


Fig. 6 Variation of the extrusion load for die friction in the axisymmetric extrusion through a conical die with the die angle of 30°

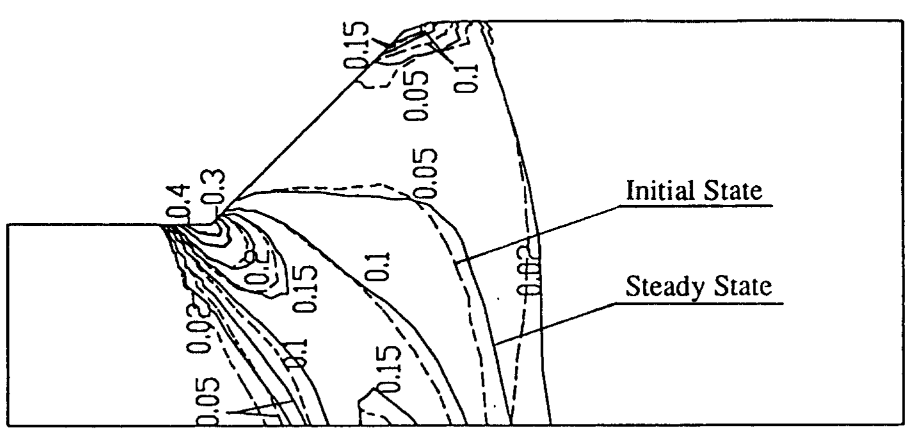


Fig. 7 Equivalent strain rates in the axisymmetric extrusion through a conical die with the die angle of 45° for the initial state and the steady state

$$\bar{\sigma} = a + b\bar{\varepsilon}^n$$

or

$$\bar{\sigma} = c(a + \bar{\varepsilon})^n \qquad (23)$$

where a,b, and c are constants for a given material.

The above idea extended from the conventional limit analysis makes it possible to simulate any three dimensional metal forming process with work-hardening materials. As an example, the present algorithm is applied to an axisymmetric extrusion problem with a work-hardening material obtaining extrusion loads, and distributions of the effective strain rate and the effective strain from the velocity field at each deformation stage. In this paper, only a part of the results is to be presented. A thorough study of this problem has been done in [16]. Fig.6 shows the increase of the extrusion load with the advance of a ram as a material is work-hardened with deformation.

The calculated load concerns only the deformation load neglecting the friction between a material and a container. This result is in good agreement with the result in [6]. Fig.7 illustrates the variation of the deformation region as the deformation proceeds. It is noted that thedeformation moves backward gradually during the deformation until the steady state is reached. That is because the material near the outlet is work-hardened more and more than the material near the inlet and the resistance against the deformation increases

5. Conclusions

A general algorithm for plastic flow analysis has been developed and successfully tested for plane strain problems and axisymmetric problems. The algorithm is built on sound physical, mathematical, and computational foundations. The duality theorem helps discern the direction and rate of convergence when an iterative approach is applied to either the primal or the dual formulation. When the primal and dual solutions are compared, the closing of the duality gap provides the true indicator of convergence especially in the case when the iterative solution of a velocity field wonders indefinitely between equally acceptable but non-unique solutions. The combined smoothing and successive approximation method homes in robustly on a correct optimal solution.

The numerical examples confirms the validity and versatility of the present algorithm with the results of very acceptable at a modest cost. Especially, a load and velocity field for the constrained indentation problem which is an open problem with the slip-line method is obtained without any prior conjecture. An axisymmetric extrusion problem with a work-hardening material is also analyzed with the idea extended from the conventional limit analysis. The result of the extrusion load and the distribution of the effective strain rate obtained from the related velocity fields is reasonable and in good agreement with those in literatures.

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