Frequency Window Method 에 의한 Secondary 구조 시스템의 진동특성

Frequency Window Method for the Vibration of Secondary Structural Systems

민 경원 *　　　Takeru Igusa **　　　Jan D. Achenbach ***

Kyung-Won Min

1. Introduction

Recently, demands on light weight, high strength, and low noise or vibration have led to the design of complicated structural systems. Although finite elements [1], mode synthesis [2], and statistical energy analysis [3] can be used to compute the dynamic response of such systems, the structural complexity has made the interpretation of the results of such analyses difficult. Many researchers in dynamic analysis have sought to further develop existing theories or develop alternate methods to obtain greater insight in the behavior of large massive primary systems (P systems) with connected light secondary systems (S systems). Some recent research includes work by Sackman and Kelly [4], Sackman et al. [5] , Der Kiureghian et al. [6], and Igusa and Der Kiureghian [7-9] who have combined mode synthesis concepts, matrix algebraic theory, and perturbation methods for characterizing weakly-coupled structural systems. A major limitation of these works are that they are limited to lumped mass S systems.

In this paper, the general ideas in the Refs. [4-9] are used to study continuous S systems and the method to reduce the complexity, studied in the works by Igusa, Achenbach, and Min [10,11], is developed into the frequency window method.

2. Lagrange's Equations

Consider the system illustrated in Fig. 1 composed of one P system with domain Ω and n S systems with domain Ω_k, $k = 1, \ldots, n$. Each S system is rigidly connected to the P system at a single point, $\mathbf{x}_k$.

The P system without S systems can be described by mode shapes $\mathbf{\Phi}_i(\mathbf{x})$, natural frequencies, ω_i, and mass density $\rho(\mathbf{x})$, where $\mathbf{x}$ is the coordinate vector and the mode shapes are normalized with respect to the mass density. Similarly, each S system k with fixed boundary at the support point $\mathbf{x}_k$ can be described by normalized mode shapes $\mathbf{\Psi}_{kj}(\mathbf{x})$, natural frequencies, ω_{kj}, and mass density $\rho_k(\mathbf{x})$.

For harmonic response, the free vibration displacement fields are given by

$$\mathbf{w}(\mathbf{x},t) = \sum_i a_i \mathbf{\Phi}_i(\mathbf{x})\, e^{-i\omega t} \tag{1}$$

for the P system and

$$\mathbf{u}_k(\mathbf{x},t) = \left[\sum_j b_{kj} \mathbf{\Psi}_{kj}(\mathbf{x}) + \mathbf{c}_k + \mathbf{d}_k \times (\mathbf{x}-\mathbf{x}_k)\right] e^{-i\omega t} \tag{2}$$

for S system k, where a_i and b_{kj} are modal coordinates and $\mathbf{c}_k$ and $\mathbf{d}_k$ represent the magnitudes and directions of the support translation and rotation, respectively. In the following, the omnipresent harmonic term $e^{-i\omega t}$ will not be explicitly included in the equations.

The constraints are specified by the displacements and rotations at the support locations $\mathbf{x}_k$. Equating the displacements and rotations of S system k with those of the P system

* 공학박사, 삼성종합건설 기술연구소
** Professor, Northwestern University
*** Professor, Northwestern University

yields

$$\mathbf{f}_k \equiv \begin{bmatrix} \mathbf{c}_k \\ \mathbf{d}_k \end{bmatrix} = \sum_i a_i \hat{\mathbf{\Phi}}_{ik} \tag{3}$$

where $\hat{\mathbf{\Phi}}_{ik}$ are the displacements and rotations corresponding to mode i of the P system at attachment point $\mathbf{x}_k$ given by

$$\hat{\mathbf{\Phi}}_{ik} \equiv \begin{bmatrix} \mathbf{\Phi}_i(\mathbf{x}_k) \\ \nabla\times\mathbf{\Phi}_i(\mathbf{x}_k) \end{bmatrix} \tag{4}$$

Using a vector of Lagrange multipliers $\boldsymbol{\lambda}_k$ for each S system k corresponding to the constraints in Eq. (3), the Lagrange's equations of motion are

$$\left(\omega_i^2 - \omega^2\right)a_i + \sum_k \boldsymbol{\lambda}_k^T \hat{\mathbf{\Phi}}_{ik} = 0 \tag{5}$$

$$\left(\omega_{kj}^2 - \omega^2\right)b_{kj} - \omega^2 \mathbf{f}_k^T \widehat{\mathbf{M}}_{kj} = 0 \tag{6}$$

$$\omega^2 \left\{ \widehat{\mathbf{M}}_k \mathbf{f}_k + \sum_j b_{kj} \widehat{\mathbf{M}}_{kj} \right\} + \boldsymbol{\lambda}_k = 0 \tag{7}$$

where $\widehat{\mathbf{M}}_{kj}$ and $\widehat{\mathbf{M}}_k$ are mass vector and matrices, respectively, which represent coupling between modes of P and S systems.

3. Reformulation in terms of Mobilities and Impedances

Reformulation of Eqs. (5)-(7) yields the impedance $\mathbf{Z}_k(\omega)$ of S system k with translational and angular velocities applied at its attachment point and the mobility $\mathbf{N}_{kl}(\omega)$ of the P system, without S systems, for input force at support location l and response velocity at support location k. The modal expressions are, respectively,

$$\mathbf{Z}_k(\omega) = -i\omega \left\{ \widehat{\mathbf{M}}_k + \sum_j \frac{\omega^2}{\omega_{kj}^2 - \omega^2} \widehat{\mathbf{M}}_{kj} \widehat{\mathbf{M}}_{kj}^T \right\} \tag{8}$$

$$\mathbf{N}_{kl}(\omega) \equiv \sum_i \frac{-i\omega}{\omega_i^2 - \omega^2} \hat{\mathbf{\Phi}}_{ik} \hat{\mathbf{\Phi}}_{il}^T \tag{9}$$

Recasting the results in matrix form yield

$$[\mathbf{I} + \mathbf{N}(\omega)\,\mathrm{diag}\{\mathbf{Z}_1(\omega) \cdots \mathbf{Z}_n(\omega)\}]\mathbf{f} = 0 \tag{10}$$

where $\mathbf{N}(\omega)$ is the mobility matrix of the P system given by

$$\mathbf{N}(\omega) \equiv \begin{bmatrix} \mathbf{N}_{11}(\omega) & \cdots & \mathbf{N}_{1n}(\omega) \\ \vdots & \ddots & \vdots \\ \mathbf{N}_{n1}(\omega) & \cdots & \mathbf{N}_{nn}(\omega) \end{bmatrix} = \sum_i \frac{-i\omega}{\omega_i^2 - \omega^2} \hat{\mathbf{\Phi}}_i \hat{\mathbf{\Phi}}_i^T \tag{11}$$

and $\mathbf{f}$ is the $6n$-vector of support displacements and rotations.

The generalization to damped systems is mathematically performed by introducing the concept of complex frequencies. It is assumed that the free P system and the fixed-base S systems are classically damped, i.e., their mode shapes are real valued.

4. Frequency Window Method

The complexity of the dynamic analysis problem can be measured by the polynomial degree of the eigenvalue problem in Eq (10). It can be seen that if there are P modes in the P system and $Q(k)$ modes in S system k, then the complexity Θ of the problem is

$$\Theta = 2\left[P + \sum_k Q(k)\right] \tag{12}$$

Frequency window method is to separate dominant and non-dominant terms in the characteristic equation and to make approximations in the non-dominant expressions.

On examining the summations in Eqs (8) and (9), it can be seen that certain terms become relatively large when the natural frequencies of the systems are close to ω, or equivalently, in a neighborhood of ω_0. Sets of indices can be defined which correspond to these system modes

$$I(\omega_0, \delta) \equiv \{\text{all } i \text{ such that } |\omega_i - \omega_0| < \delta\} \tag{13}$$

$$I(\omega_0, \delta, k) \equiv \{\text{all } kj \text{ such that } |\omega_{kj} - \omega_0| < \delta\} \tag{14}$$

where it is understood that single indices i refer to the P system and double indices kj refer to S system k. and δ and ω_0 are the size and central value of window, respectively.

Using this classification of system modes and approximating the non-dominant term by substituting ω_0 for ω yield

$$\mathbf{Z}_k(\omega) \approx \mathbf{Z}_{k0}(\omega, \omega_0, \delta) + \mathbf{Z}_{k1}(\omega_0, \delta) \tag{15}$$

$$\mathbf{N}(\omega) \approx \mathbf{N}_0(\omega, \omega_0, \delta) + \mathbf{N}_1(\omega_0, \delta) \tag{16}$$

where $\mathbf{Z}_{k0}(\omega, \omega_0, \delta)$ and $\mathbf{N}_0(\omega, \omega_0, \delta)$ are dominant terms and $\mathbf{Z}_{k1}(\omega_0, \delta)$ and $\mathbf{N}_1(\omega_0, \delta)$ are non-dominant terms.

Substituting Eqs (15) and (16) into the characteristic equation (10) yields an eigenvalue problem of lower order, and hence reduced complexity. The complexity is determined by the numbers of elements of the index sets. If an operator $\Theta|\cdot|$ is defined for counting the size of sets, then the complexity of the eigenvalue problem is

$$\Theta = \Theta|I(\omega_0, \delta)| + \sum_k \Theta|I(\omega_0, \delta, k)| \quad (17)$$

which can be considerably smaller than the complexity of the original problem given in Eq (12).

5. One-Mode Windows

A *tuning window of order δ at ω_0* is defined by the index set

$$J(\delta, \omega_0) \equiv \{\text{all indeces } i \text{ and } kj \text{ such that } |\omega_i - \omega_0| < \delta, |\omega_{kj} - \omega_0| < \delta\}$$

The simplest windows are one-mode windows, where $J(\delta, \omega_0)$ has only a single element.

There are two possible types of one-mode windows given by the following representations of $J(\delta, \omega_0)$:

$\{i\}$ and $\{kj\}$

5.1 Mode {kj}

For notational convenience, and without any loss of generality, it is assumed that $k = 1$. It is natural to assign the central frequency to the $1j$ frequency of S system one, $\omega_0 = \omega_{1j}$. The index sets defined in Eqs (13) and (14) are simply $I(\omega_0, \delta) = \{\ \}, I(\omega_0, \delta, 1) = \{1j\}$, and $I(\omega_0, \delta, l) = \{\ \}$ for $l \neq 1$. Substituting into Eqs (15) and (16) for the mobilities and impedances and using the simplified approximate results in Eq (10) yields the following eigenvalue problem

$$\left[\mathbf{I} + \mathbf{N}(\omega_0)\,\mathrm{diag}\{\mathbf{Z}_{10}(\omega, \omega_0, \delta) + \mathbf{Z}_{11}(\omega_0, \delta)\ \ \mathbf{Z}_2(\omega_0) \cdots \mathbf{Z}_n(\omega_0)\}\right]\mathbf{f} = 0 \quad (18)$$

Since $\mathbf{Z}_{10}(\omega, \omega_0, \delta)$ is one order of magnitude larger than all the remaining impedance terms, the first-order reduction of the eigenvalue problem in Eq (18) is

$$\left[\mathbf{I} + \mathbf{N}(\omega_0)\,\mathrm{diag}\{\mathbf{Z}_{10}(\omega, \omega_0, \delta)\ \ 0 \cdots 0\}\right]\mathbf{f} = 0 \quad (19)$$

Considering only $\mathbf{f}_1$ in the eigenvalue problem leads to

$$\left[\mathbf{I} + \mathbf{N}_{11}(\omega_0)\mathbf{Z}_{10}(\omega, \omega_0, \delta)\right]\mathbf{f}_1 = \left[\mathbf{I} + \frac{-i\omega_0^3}{\omega_{1j}^2 - \omega^2}\mathbf{N}_{11}(\omega_0)\widehat{\mathbf{M}}_{1j}\widehat{\mathbf{M}}_{1j}^T\right]\mathbf{f}_1 = 0 \quad (20)$$

Although this problem is of order 6×6, the outer product indicates that the second matrix term in brackets is of rank one, implying that the eigenvalue problem is actually of order 1×1. This reduced eigenvalue problem can be extracted from Eq. (20) by pre-multiplying by $\widehat{\mathbf{M}}_{1j}^T$ and defining

$$m_1 \equiv \frac{\omega_0^2}{\omega_{1j}^2 - \omega^2}\widehat{\mathbf{M}}_{1j}^T\mathbf{f}_1 \quad (21)$$

to yield the scalar equation

$$\left[\omega_{1j}^2 - \omega^2 - i\omega_0^3\widehat{\mathbf{M}}_{1j}^T\mathbf{N}_{11}(\omega_0)\widehat{\mathbf{M}}_{1j}\right]m_1\,\omega_0^{-2} = 0 \quad (22)$$

The solution for the frequency is

$$\omega = \omega_{1j}\sqrt{1 - \gamma_1} \quad (23)$$

where

$$\gamma_1 = i\omega_0\widehat{\mathbf{M}}_{1j}^T\mathbf{N}_{11}(\omega_0)\widehat{\mathbf{M}}_{1j} \quad (24)$$

The constant γ_1 is dimensionless and is a physical measure of the effect of the flexibility of the P system at the attachment location on the natural frequency of the combined system. If $\gamma_1 = 0$, then the P system acts as a rigid support and the natural frequency of the combined system is identical to that of the fixed-end S system. Positive real components for γ_1 corresponds to a flexible support and results in a reduction of the natural frequency. The term *support flexibility parameter* is applied to γ_1

The mode shape component can be determined by substituting the result for the natural frequency into the eigenvalue equation. The result is

$$\mathbf{f}_k = i\omega_0\mathbf{N}_{k1}(\omega_0)\widehat{\mathbf{M}}_{1j} \quad (25)$$

5.2 Mode {i}

It is natural to assign the central frequency to the i-th frequency of the P system, $\omega_0 = \omega_i$. Following the same procedure in mode {kj} yields the first-order problem

$$\left[\mathbf{I} + \mathbf{N}_0(\omega, \omega_0, \delta)\,\mathrm{diag}\{\mathbf{Z}_1(\omega_0) \cdots \mathbf{Z}_n(\omega_0)\}\right]\mathbf{f} = 0 \quad (26)$$

The reduced eigenvalue problem can be extracted from Eq. (26) by pre-multiplying by $i\omega_0^{-1}\hat{\mathbf{\Phi}}_i^T\,\mathrm{diag}\{\mathbf{Z}_1(\omega_0) \cdots \mathbf{Z}_n(\omega_0)\}$ and defining

$$p_i \equiv \frac{i\omega_0}{\omega_0^2 - \omega^2}\hat{\mathbf{\Phi}}_i^T\,\mathrm{diag}\{\mathbf{Z}_1(\omega_0) \cdots \mathbf{Z}_n(\omega_0)\}\mathbf{f} \quad (27)$$

to yield the scalar equation

$$\left[\omega_i^2 - \omega^2 - i\,\omega_0\,\hat{\Phi}_i^T\,\mathrm{diag}\{\mathbf{Z}_1(\omega_0)\ \cdots\ \mathbf{Z}_n(\omega_0)\}\hat{\Phi}_i\right]p_i\,\omega_0^{-2} = 0 \qquad (28)$$

The solution for the frequency is

$$\omega = \omega_i\sqrt{1-\mu_i} \qquad (29)$$

where

$$\mu_i = i\,\omega_0^{-1}\,\hat{\Phi}_i^T\,\mathrm{diag}\{\mathbf{Z}_1(\omega_0)\ \cdots\ \mathbf{Z}_n(\omega_0)\}\hat{\Phi}_i \qquad (30)$$

The constant μ_i is dimensionless and is a physical measure of the effect of the impedance of all S systems on the natural frequency of the combined system. If $\mu_i = 0$, then the system acts as massless, decoupled S systems and the natural frequency of the combined system is identical to that of the free P system. Positive real components for μ_i correspond to a mass effect and result in a reduction of the natural frequency. The term *S system mass parameter* is applied to μ_i,

Following the same procedure in mode $\{kj\}$ yields the mode shape

$$\mathbf{f} = \hat{\Phi}_i \qquad (31)$$

6. Example Studies

To illustrate the results of this paper, a simple example is studied. The system, shown in Fig. 2, consists of an Euler-Bernoulli beam with free ends supporting two smaller cantilever beams at one third span intervals. The cantilevers are the S systems and can represent ribs on the P system. The modulus of elasticity, E, is the same for all beams, and the moments of inertia and masses per unit length are denoted I and m for the P system and I_s and m_s for the two S systems, respectively. It is useful to define nondimensional parameters

$$\lambda \equiv \frac{m_s}{m} \quad \text{and} \quad \tau\lambda \equiv \frac{EI_s}{EI} \qquad (32)$$

where λ is a mass density ratio and $\tau\lambda$ is a rigidity ratio. For simplicity, vibrations corresponding to axial deformations are not considered and damping is neglected. The parameter l_k denotes the length of S system k normalized by the length of the P system.

One set of system are considered, corresponding to window $\{kj\}$. Three mass density ratios, $\lambda = 0.05$, 0.15, and 0.5, are considered and the parameter τ is held constant at 0.25.

The natural frequencies and mode shapes corresponding to the tuning window are obtained using an exact and a frequency window analyses. The exact results are determined by numerically solving the original eigenvalue problem using a sufficiently large number of system modes to produce convergent values. The frequency window method results are obtained by using the closed-form expressions derived in this paper. The mode shape components $\mathbf{f}$ are scaled so that the vector norm is $|\mathbf{f}| = 1$.

For the example study of window $\{kj\}$, the rib lengths are $l_1 = 0.092$ and $l_2 = 0.083$, and the modal properties of the combined system corresponding to the first fixed-base mode of rib one is examined, i.e., $kj = 1,1$. The approximate frequencies, normalized with respect to $\omega_0 = \omega_{11}$, and the mode shape components $\mathbf{f}$ are computed using Eqs. (23) and (25). The numerical results are compared in Table 1 with exact results. The frequencies are also plotted in Fig. 3 to illustrate the effect of varying rib mass density.

The numerical results show several features predicted by the frequency window method. First, as the mass density of the ribs decreases, the normalized frequency approaches unity, i.e., the natural frequency of the combined system approaches that of the fixed-base rib. This can be explained in terms of the parameter γ_i defined in Eq. (24). For decreasing mass density, γ_i become small relative to unity, and Eq. (23) shows that the natural frequency of the combined system approaches that of the fixed-base S system. Second, the mode shape components $\mathbf{f}$ are nearly independent of the S system mass densities. Equation (25) confirm this observation, and also indicate that the mode shape is proportional to the vibration shape of the free-free beam subjected to a reaction force of magnitude $\hat{\mathbf{M}}_{1j}$ at the base of rib one.

7. Conclusion

Lagrange's equations were used to develop a characteristic equation for connected structural systems in terms of S system impedances and P system mobilities. Next, methods of reducing the problem complexity were developed by decomposing the impedance and mobility rational expressions into dominant and secondary terms. The reduced problem was examined for the simplest class of systems characterized by one dominant mode. Closed-form expressions for the modal properties were found to contain parameters with direct physical interpretations.

The example studies show that the

approximate results by the frequency window method are accurate, particularly for small S system masses. This is remarkable in view of the fact that the example system has three components, each with an infinite number of modes, while the frequency window analysis uses a single variable equation, appropriately chosen to capture the dominant characteristics of the system.

8. References

1. O. C. ZIENKIEWICZ, 1977*The finite element method*, London, McGraw-Hill.
2. W. C. HURTY and M. F. RUBINSTEIN, 1964 *Dynamics of Structures*, Englewood Cliffs: Prentice-Hall.
3. R. H. LYON, 1975 *Statistical Energy Analysis of Dynamical Systems: Theory and Applications*. Cambridge: MIT Press.
4. J. L. SACKMAN AND J. M. KELLY, 1979 *Engineering Structures* 1, 179-190. "Seismic analysis of internal equipment and components in structures".
5. J. L. SACKMAN, A. DER KIUREGHIAN, and B. NOUR-OMID, 1983 *Journal of the Engineering Mechanics Division* ASCE **109**, 73-89. "Dynamic analysis of light equipment in structures: Modal properties of the combined system".
6. A. DER KIUREGHIAN, J. L. SACKMAN, and B. NOUR-OMID, 1983 *Journal of the Engineering Mechanics Division* ASCE **109**, 90-110. "Dynamic analysis of light equipment in structures: Response to stochastic input".
7. T. IGUSA and A. DER KIUREGHIAN, 1985 *Journal of the Engineering Mechanics Division* ASCE **111**, 1-19. "Dynamic characterization of two-degree-of-freedom equipment-structure systems".
8. T. IGUSA and A. DER KIUREGHIAN, 1985 *Journal of the Engineering Mechanics Division* ASCE **111**, 20-41. "Dynamic response of multiply supported secondary systems'.
9. T. IGUSA and A. DER KIUREGHIAN, 1983 July "Dynamic analysis of multiply tuned and arbitrarily supported secondary systems", Report No. UCB/EERC-83/07, Earthquake Engineering Research Center, University of California, Berkeley, Calif.
10. T. IGUSA, J. D. ACHENBACH, and K.-W. MIN, April, 1990 "Resonance characteristics of connected subsystems: theory and simple configurations" *Journal of Sound and Vibration*, Accepted for publication.
11. T. IGUSA, J. D. ACHENBACH, and K.-W. MIN, April, 1990, "Resonance characteristics of connected subsystems: general configurations" *Journal of Sound and Vibration*, Accepted for publication.

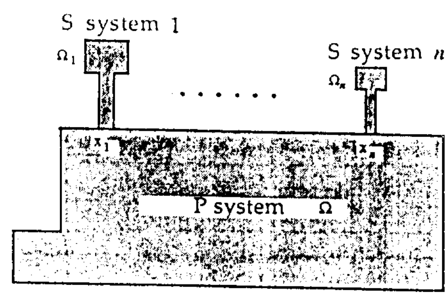

Figure 1 Structural system with rigid, single-point connections

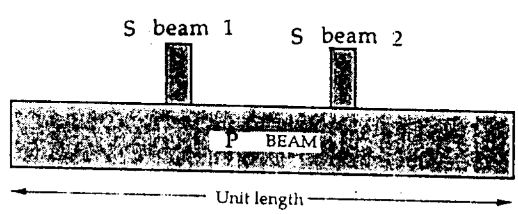

Figure 2 Connected beams

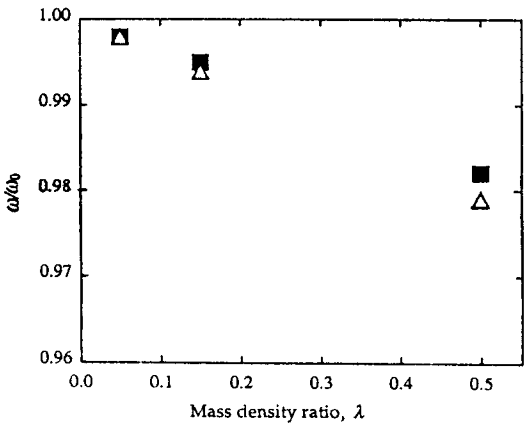

Figure 3 Frequency ratio, ω/ω_0, *vs.* rib mass density ratio, λ, for mode (kj) by both exact (■) and frequency window (△) method.

	λ=0.05		λ=0.15		λ=0.5	
	Exact	Apprx.	Exact	Apprx.	Exact	Apprx.
Freq.ratio	0.998	0.998	0.995	0.994	0.982	0.979
Mode						
x1	0.20	0.19	0.20	0.19	0.22	0.19
y1	0.28	0.24	0.26	0.24	0.20	0.24
r1	0.06	0.05	0.03	0.05	-0.03	0.05
x2	0.20	0.19	0.20	0.19	0.22	0.19
y2	-0.65	-0.63	-0.63	-0.63	-0.60	-0.63
r2	-0.65	-0.68	-0.67	-0.68	-0.71	-0.68

Table 1 Exact and approximate frequency ratios and mode shape components for mode (kj)