**Comparison of the results by the Matrix method and the Coupled wave method in analyzing Bragg Reflector structures**

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**Abstract**

We compare the reflectivity spectrum and phase change of Bragg reflectors obtained by the matrix method and the coupled wave method. We show that the results obtained by the two methods agree well generally and the discrepancy between the results obtained by the two methods increases as the fractional refractive index difference between adjacent layers increases and/or the absorption loss increases, due to the approximations inherent in the coupled wave method for the analysis of multiple dielectric layers.

## 1 Introduction

Periodic optical media and stratified periodic optical structures play an important role in a number of applications in optics. Recent developments in crystal-growing techniques have made it possible to monolithically grow high reflectance Bragg mirrors [1]. Integration of Bragg reflectors has enabled important optical devices such as vertically emitting lasers [2] and microresonators to be monolithically grown [3].

There are two important theoretical methods to analyze the distributed feedback (DFB) structures. One is the matrix method and the other is the coupled-wave method. The use of the matrix method in studying the propagation of plane electromagnetic waves through a stratified medium is well known in optics [4]. However, it is very difficult to obtain simple analytic expressions of the reflectance spectrum for stratified periodic structures when using a matrix method. The coupled-wave method has long been used to analyze the characteristics for a periodic layered medium because it gives simple analytic expressions for the reflectivity spectrum of Bragg reflectors [5]. However, the coupled-wave method assumes that the beam is incident into the DFB structure from a medium with an average refractive index of the DFB structure. This means that the coupled-wave method does not consider the effect of end reflections occurring from the boundaries of the DFB structure.

We present theories for Bragg reflectors in the simultaneous presence of refractive index modulation and absorption modulation using both the matrix method and the coupled-wave method. For the sake of simplicity, we will only consider the case of normal incidence. We will show that in general the results obtained by the matrix method and the coupled-wave method agree well for the reflectance spectrum and the phase change from a Bragg reflector as a function of detuning, $\delta L$. Also, we will see that the difference between the results by the two methods increases as the fractional refractive index difference between adjacent layers, $\Delta n/n_L$, increases and/or the absorption loss increases, due to the approximations inherent in the coupled wave method.

## 2 The Matrix Method

Consider the structure as shown in Fig.1 with an electric field travelling in the yz plane. The electric field in the ith layer can be represented by

$$\begin{aligned}\vec{E}_i &= \{\hat{e}_i^+ E_i^+ exp[-ik_i(z-d_{i-1})] + \\ &\quad \hat{e}_i^- E_i^- exp[ik_i(z-d_{i-1})]\} exp[i(\omega t)], \end{aligned} \tag{1}$$

where $\hat{e}_i^+$ and $\hat{e}_i^-$ represent the unit vectors along the direction of the electric field, and $E_i^+$ and $E_i^-$ represent the electric field amplitudes of waves propagating in the forward and backward directions, respectively. Imposing the continuity of tangential components of $\vec{E}$ and $\vec{H}$ at the interface separating ith and (i+1)th layers and after some matrix manipulations, we obtain the following matrix equation [6]

$$\begin{bmatrix} E_i^+ \\ E_i^- \end{bmatrix} = \frac{1}{2k_i}\begin{bmatrix} (k_i+k_{i+1})e^{i\delta_i} & (k_i-k_{i+1})e^{i\delta_i} \\ (k_i-k_{i+1})e^{-i\delta_i} & (k_i+k_{i+1})e^{-i\delta_i} \end{bmatrix}\begin{bmatrix} E_{i+1}^+ \\ E_{i+1}^- \end{bmatrix} \tag{2}$$

where $\Delta_i = d_i - d_{i-1}$, $\Delta_0 = 0$, $\delta_i = k_i\Delta_i$, $k_i = k_0 n_i$, $n_i$ is the complex refractive index of ith layer, $k_0$ is the propagation constant in free space. We could represent eq. (2) as

$$\begin{bmatrix} E_i^+ \\ E_i^- \end{bmatrix} = S_i \begin{bmatrix} E_{i+1}^+ \\ E_{i+1}^- \end{bmatrix}$$

where

$$S_i = \frac{1}{t_i}\begin{bmatrix} e^{i\delta_i} & r_i e^{i\delta_i} \\ r_i e^{-i\delta_i} & e^{-i\delta_i} \end{bmatrix} \tag{3}$$

and $r_i$ and $t_i$ represent the amplitude reflection and transmission coefficients at the interface separating ith and (i+1)th layers, respectively.

These matrix relations can be used to determine the transmission and reflection from a layered structure as follows. The electric fields across a layer or boundary are related, so that we can determine the matrix relation across the entire structure by multiplying the relevant matrices together. Assuming that light is incident from only one side of the structure, we could put $E^-_{final} = 0$. This boundary condition allows us to determine the incident fields $E^-_{in}$ and $E^+_{in}$ in terms of $E^+_{final}$

$$\begin{bmatrix} E^+_{in} \\ E^-_{in} \end{bmatrix} = S_0 S_1 \cdots S_N \begin{bmatrix} E^+_{final} \\ 0 \end{bmatrix} \quad (4)$$

Since the incident electric fields $E^+_{in}$ and $E^-_{in}$ are expressed in terms of $E^+_{final}$, we can calculate the complex reflection coefficient of a multilayer structure, $r = E^-_{in}/E^+_{in}$ if we know the refractive index and absorption coefficient in each layer. The reflectivity of a multilayer structure is the square magnitude of the complex reflection coefficient of a multilayer structure because the intensity $I$ is given by $I \propto n \mid E \mid^2$. Also, we can calculate the phase change of electric fields in a multilayer structure from the complex reflection coefficient.

## 3 The Coupled-wave method

Bragg reflectors consist of alternating periodic quarter wavelength layers with different refractive index and absorption coefficient. In GaAs based materials, the material with high refractive index has the narrow bandgap. Consider the Bragg reflectors shown in Fig.2. Since the optical thickness of each layer is a quarter of the Bragg wavelength, the ratio of the thickness of the layer with high refractive index to the period of the Bragg reflector, $a$, is given by $a = n_L/(n_h+n_L)$, where $n_L$ and $n_h$ represent the low refractive index and high refractive index, respectively. For the sake of simplicity, we will assume that the layer with high refractive index is lossy and the layer with low refractive index is transparent ($\alpha = 0$).

Since the coupled-wave method considers the periodic variation of the dielectric constant and absorption coefficient as a perturbation that couples the unperturbed normal waves of the structure, we have to calculate the average refractive index, $n_{av}$, of the structure. Since Bragg reflectors are composed of alternating periodic quarter wavelength layers with different refractive index at the Bragg wavelength, the period corresponds to the half wavelength of the medium with $n_{av}$ at the Bragg wavelength. Thus, the average refractive index of a Bragg reflector is given by

$$n_{av} = \frac{\lambda_B}{2\Lambda} = \frac{2n_h n_L}{n_h + n_L}, \quad (5)$$

where $\lambda_B$ is the Bragg wavelength. The complex dielectric constant, $\epsilon(z)$, as a function of $z$ is written as

$$\epsilon(z) = \epsilon_{av} + \Delta\epsilon(z) - i\sigma(z), \quad (6)$$

where $\epsilon_{av}$ is the unperturbed part of the dielectric constant, $\sigma(z)$ is given by $n_{av}\epsilon_0\lambda_0\alpha(z)/2\pi$, $\lambda_0$ is the free space wavelength of the incident light. Both $\sigma(z)$ and $\Delta\epsilon(z)$ are periodic in the $z$ direction.

The starting point of the analysis is Maxwell's wave equation

$$\{\nabla^2 + \omega^2\mu\epsilon(z)\}\vec{E}(z) = 0. \quad (7)$$

Since we are interested in the coupling of two waves, the incident and the reflected, in a Bragg reflector interacting through the first order spatial Fourier component of complex dielectric constant, it is convenient to express the complex amplitude of the field $E(z)$ as a sum of the wave $E_f$ propagating in the positive direction of the $z$ axis and the wave $E_b$ propagating in the negative direction of the $z$ axis:

$$E(z) = E_f(z)e^{-i\beta z} + E_b(z)e^{i\beta z} \quad (8)$$

where $E_f(z)$ and $E_b(z)$ are slowly varying complex amplitudes.

We choose the origin of coordinate system $z = 0$ in the middle of the layer with low refractive index [7]. Putting eq.(8) into eq.(7), using slowly varying envelope approximation ($\mid d^2E/dz^2 \mid \ll \mid \beta dE/dz \mid$), and performing the usual manipulations [5], we obtain the coupled wave equations

$$\frac{d}{dz}E_f(z) = (iK + F)E_b(z)e^{i2\delta z} - a\frac{\alpha}{2}E_f(z),$$

$$\frac{d}{dz}E_b(z) = (-iK - F)E_f(z)e^{-i2\delta z} + a\frac{\alpha}{2}E_b(z), \quad (9)$$

where $K = sin(a\pi)(n_h^2 - n_L^2)/(n_{av}\lambda)$, $F = sin(a\pi)\alpha/(2\pi)$, and $\delta = \beta - (\pi/\Lambda)$. $K$ is the coupling coefficient due to refractive index modulation, $F$ is the coupling coefficient due to absorption modulation, and $a$ is a measure of the difference of adjacent layer thickness of a Bragg reflector. The wavelength dependence of $K$ can be ignored because it is of second order. This is consistent with the approximations inherent in the coupled wave method [8]. This means that $K = sin(a\pi)(n_h^2 - n_L^2)/(n_{av}\lambda_B)$.

Imposing the boundary condition $E_b(z) = 0$ at $z$ = L, we obtain an analytic expression for the reflection coefficient

$$r = \frac{E_b(0)}{E_f(0)} = i(K - iF)\frac{sinh(sL)}{s\,cosh(sL) + i(\delta - ia\frac{\alpha}{2})sinh(sL)}, \quad (10)$$

where

$$s = \sqrt{\{(K^2 - F^2) - \delta^2 + \frac{a^2\alpha^2}{4}\} + i\{\delta a\alpha - 2KF\}}. \quad (11)$$

This reflection coefficient of eq.(10) is obtained when a half of a quarter wavelength layer with low refractive index is on the top of a Bragg reflector because we chose the origin of coordinate system $z = 0$ in the middle of the layer with low refractive index to calculate the Fourier components of complex dielectric constant.

We could think this low refractive index layer with a thickness of one-eighth of an optical wavelength of the Bragg wavelength as a phase shifter. Then, we could obtain the reflection coefficient of a Bragg reflector when the outermost layer has the high refractive index, $r_h$, as

$$r_h = \frac{E_i(0)e^{i\frac{\pi}{4}}}{E_f(0)e^{-i\frac{\pi}{4}}} = re^{i\frac{\pi}{2}} = ir. \tag{12}$$

If the layer sequence is changed (outermost layer has the low refractive index), the reflection coefficient, $r_L$, is given by

$$r_L = \frac{E_i(0)e^{-i\frac{\pi}{4}}}{E_f(0)e^{i\frac{\pi}{4}}} = re^{-i\frac{\pi}{2}} = -ir = -r_h. \tag{13}$$

When the layer sequence is changed, the phase of a reflection coefficient changes by $\pi$ radian; however the magnitude of a reflection coefficient is the same.

## 4 Comparison of the Results by the Matrix Method and the Coupled-Wave method

We calculated the reflectivity as a function of detuning by the two methods to compare the results. We used eq.(2) in the matrix method case. In the coupled wave method case, we used eq.(13). Figs.3 and 4 show the reflectivity spectrum obtained by the two methods as a function of detuning in the case of absorption loss $(\alpha L) = 1$ for $KL = 1.1$. $\Delta n/n_L = 0.2$ and N = 6 were used in Fig.3, and $\Delta n/n_L = 0.067$ and N = 17 were used in Fig.4, where N is the number of periods. Since we used different values of $\Delta n/n_L$ to obtain the reflectance spectrum of Figs.3 and 4 for same $KL$, the number of layers used in each figure is different. We can clearly see that the difference between reflectance spectrum obtained by the two methods is small when the fractional refractive index difference is small and is larger when the fractional refractive index difference is larger. We may attribute the rapid decrease of the differences in Fig.4 compared to those in Fig.3 to the approximations inherent in the coupled wave method which assume $\Delta n/n_L \ll 1$ [5,8].

Figs.3 and 5 show the reflectance spectrum obtained by the two methods in the case of same $KL = 1.1$ and $\Delta n/n_L$, but two different absorption loss. We can clearly see that the difference between reflectance spectrum obtained by the two methods is small when the absorption loss is small and is larger when the absorption loss is larger. We may attribute the rapid decrease of the differences in Fig. 5 compared to those in Fig.3 to the approximations inherent in the coupled wave method which assume $(\alpha L) \ll 1$.

We calculated the phase change as a function of detuning by the two methods when the absorption loss and $KL$ remain the same but $\Delta n/n_L$ changes. We can see the difference of the phase change spectrum obtained by the two methods also decreases, as $\Delta n/n_L$ decreases. We compared the phase change obtained in the presence of same $KL$ and $\Delta n/n_L$, but two different absorption loss. We can see the difference of the phase change spectrum obtained by the two methods also decreases, as absorption loss decreases.

## 5 Conclusions

We compared the results obtained by the matrix method and the coupled-wave method. We see that in general these two results agree well for the reflectance spectrum and the phase change from a Bragg reflector as a function of detuning, $\delta L$. Also, we see that the difference between the results by the two methods increases as the fractional refractive index difference between adjacent layers, $\Delta n/n_L$, increases and/or the absorption loss increases, due to the approximations inherent in the coupled wave method for the analysis of multiple dielectric layers. Since the coupled wave equation is a differential equation, however the multiple dielectric layers is a discrete system so that the inherent approximations in the coupled wave method for the analysis of multiple dielectric layers are the fractonal refractive index difference between adjacent layers is small and the absorption loss is small.

## References

[1] P. L. Gourley and T. J. Drummond, Appl. Phys. Lett., vol. 49, 489 (1986).

[2] M. Ogura, T. Hata, and T. Yao, Jpn. J. Appl. Phys., vol. 23, L512 (1984).

[3] J. L. Jewell, A. Scherer, S. L. McCall, A. C. Gossard, and J. H. English, Appl. Phys. Lett., vol. 51, 94 (1987).

[4] M. Born and E. Wolf, *Principles of Optics*, MacMillan, New York, 1964.

[5] A. Yariv and P. Yeh, *Optical Waves in Crystals*, John Wiley & Sons, New York, 1984.

[6] A. K. Ghatak, K. Thyagarajan, and M. R. Shenoy, J. Lightwave Technol., vol. LT-5, 660 (1987).

[7] W. Streifer, D. R. Scifres, and R. D. Burnham, IEEE J. Quantum Electron., vol. QE-12, 422, (1976).

[8] H. A. Haus, *Waves and Fields in Optoelectronics*, Prentice-Hall, Inc., Englewood Cliffs, 1984.

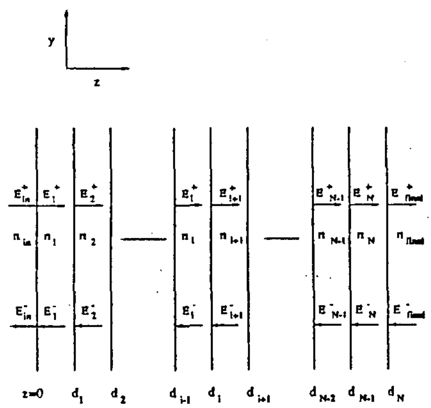

**Fig. 1:** The layered structure used in formulating the matrix method.

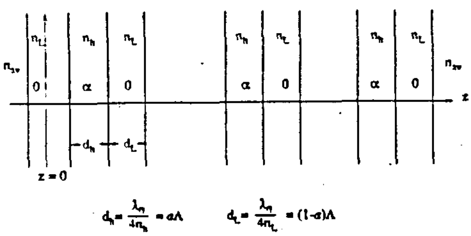

**Fig. 2:** The geometry of a Bragg reflector.

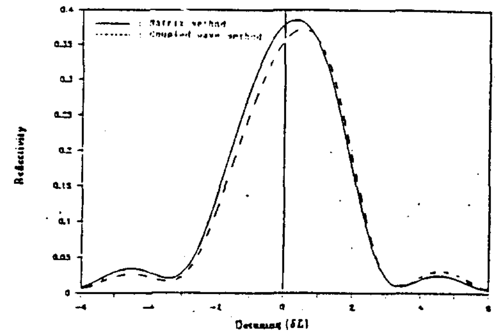

**Fig. 3:** The reflectance spectrum obtained by the two methods in the case of an absorption loss as 1 for $KL = 1.1$ and $\Delta n/n_L = 0.2$.

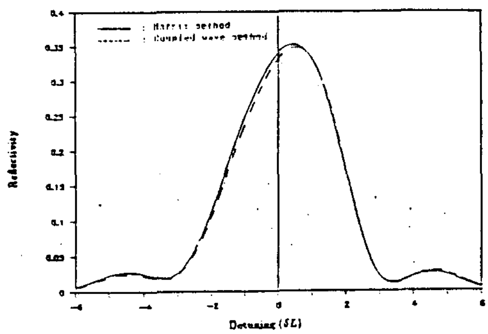

**Fig. 4:** The reflectance spectrum obtained by the two methods in the case of an absorption loss as 1 for $KL = 1.1$ and $\Delta n/n_L = 0.067$.

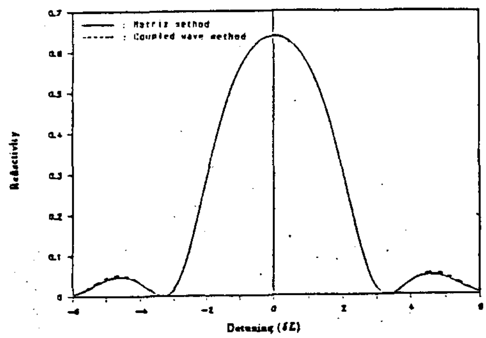

**Fig. 5:** The reflectance spectrum obtained by the two methods in a lossless case for $KL = 1.1$ and $\Delta n/n_L = 0.2$.