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A NUMERICAL METHOD OF PREDETERMINED OPTIMAL RESOLUTION FOR A REDUNDANT MANIPULATOR

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Abstract

This paper proposes a numerical method for redundant manipulators using predetermined optimal resolution. In order to obtain optimal joint trajectories, it is desirable to formulate redundancy resolution as an optimization problem having an integral cost criterion. We predetermine the trajectories of redundant joints in terms of the Nth partial sum of the Fourier series, which lead to the solution in the desirable homotopy class. Then optimal coefficients of the Fourier series, which yield the optimal solution within the predetermined class, are searched by the Powell's method. The proposed method is applied to a 3-link planar manipulator for cyclic tasks in Cartesian space. As the results, we can obtain the optimal solution in the desirable homotopy class without topological liftings of the solution. To show the validity of the proposed method, we analyze both optimal and extremal solutions by the Fast Fourier Transform (FFT) and discuss joint trajectories on the phase plane.

I. Introduction

The kinematics of redundant manipulators is represented by

$$x = f(\theta) \tag{1}$$

where x is an m-dimensional vector representing the position and orientation of the end-effector in Cartesian space, θ is an n-dimensional vector representing joint variables, and f is a vector function consisting of m scalar functions, with $m < n$. Even though both x and θ are functions of time, for brevity, we use x and θ instead of $x(t)$ and $\theta(t)$, respectively. From (1), the differential kinematics, which relates the rates of changes $\dot{x}$ and $\dot{\theta}$, is described as

$$\dot{x} = J\dot{\theta} \tag{2}$$

where J is a known $m \times n$ Jacobian matrix.

For a redundant manipulator, it is natural to make use of $(n - m)$ dimensional redundant degrees of freedom so that the manipulator optimizes secondary objective while performing the primary task in Cartesian space. Secondary objectives are in the form of an instantaneous cost criterion (local optimization method) or an integral cost criterion (global optimization method). If the Jacobian matrix J is rectangular with $m < n$, the general solution $\dot{\theta}$ of (2) becomes

$$\dot{\theta} = J^+\dot{x} + (I - J^+J)z \tag{3}$$

where J^+ is the Moore-Penrose generalized inverse of J. If J has full rank, then J^+ becomes $J^t(JJ^t)^{-1}$ known as pseudoinverse of J. The matrix I in (3) is an $n \times n$ identity matrix and $I - J^+J$ is the null space projection matrix. The vector z is an n-dimensional arbitrary vector. Based on (3) or modified equations from (3), several local optimization methods have been suggested by many researchers to resolve the redundancy instantaneously. By specifying z, redundant manipulators have been used to avoid joint limits, singularities, and obstacles [1]-[3].

Klein and Huang [4] show that the closed trajectories in joint space can not be generally obtained for the closed tasks in Cartesian space by local optimization methods. Baillieul [5] proves that, without further modification, the generalized inverse method can not avoid kinematic singularities, and also a globally optimal solution is not guaranteed. Meanwhile, Chang [6] suggests an inverse kinematic method for cyclic tasks instead of a generalized inverse method. As the results of the described short-comings, global optimization methods are preferable to local optimization methods.

Martin *et al.* [7] propose a global optimization method for cyclic tasks by using Euler-Lagrange equations. They discuss the necessary conditions and the periodic boundary conditions for optimal joint trajectories of a redundant manipulator, where the problem is to minimize the following performance index

$$\int_{t_0}^{t_1} G(\theta, \dot{\theta}, t)\, dt \tag{4}$$

subject to the kinematic constraints (1) and the periodic boundary conditions

$$\begin{aligned} \theta(t_0) &= \theta(t_1), \\ \dot{\theta}(t_0) &= \dot{\theta}(t_1) \end{aligned} \tag{5}$$

where t_0 and t_1 are the initial and final time, respectively.

As a reasonable integrand of the performance index, $G(\theta, \dot{\theta}, t)$, we choose the following integrand

$$G(\theta, \dot{\theta}, t) = \frac{1}{2}\dot{\theta}^t W^{-1}\dot{\theta} + g(\theta) \tag{6}$$

where W is the $n \times n$ diagonal weighting matrix chosen by a designer to reflect the relative significance of joint velocities and $g(\theta)$ is a function of configuration such as the *manipulability measure* [8] or the distance to some obstacles. In case of $W = I$ and $g(\theta) = 0$, (6) becomes the norm of joint velocity. From (4) and (6), if the performance index is

$$\int_{t_0}^{t_1} \frac{1}{2}\dot{\theta}^t\dot{\theta}\, dt, \tag{7}$$

then the necessary condition becomes

$$\ddot{\theta} = J^+(\ddot{x} - \dot{J}\dot{\theta}). \tag{8}$$

In addition, if the given task is described by a closed path in Cartesian space, then joint trajectories must be periodic, i.e., (5) becomes additional constraints. Therefore, the necessary condition (8) must be solved with the periodic boundary conditions (5) to obtain an optimal solution. Unfortunately, due to the existence of multiple nonhomotopic extremal solutions, all these extremal solutions are not optimal and the optimal solution among them is not always unique.

The remainder of this paper is organized as follows. In Section II, the problem is described and the proposed method is derived. In Section III, the proposed method is applied to two tasks and the solutions are compared with those of other methods. In Section IV, we show the validity of assumptions by the FFT and analyze both optimal and extremal solutions on the phase plane. Finally, we draw concluding remarks in Section V.

II. Predetermined Optimal Resolution

Problem Formulation

In this paper we formulate a global optimization problem with an integral cost criterion (4) for a task (1). Particularly, if the given task in Cartesian space is closed in space or cyclic in time, then periodic joint trajectories are necessarily required for practical applications. Therefore, (5) becomes additional boundary conditions. The global optimization problem for a cyclic task can be summarized as follows:

$$\min \int_{t_0}^{t_1} G(\theta, \dot{\theta}, t)\, dt$$

subject to

$$x = f(\theta)$$

and

$$\theta(t_0) = \theta(t_1), \quad \dot{\theta}(t_0) = \dot{\theta}(t_1). \tag{9}$$

Observations on the Solution for a Cyclic Task

If a task under consideration is cyclic, then the kinematic equation (1) can be represented as a periodic function with period T, where T is given by $T = t_1 - t_0$. Then we can introduce a new constant value ω_0, defined by

$$\omega_0 = \frac{2\pi}{T}. \tag{10}$$

This new constant is called as the fundamental frequency of the task. Since we already know that the solution is periodic as far as (5) is satisfied, the optimal solution θ can be described as

$$\theta(t) = \theta_0 - \sum_{k=1}^{\infty} b_k + \sum_{k=1}^{\infty} a_k \sin(k\omega_o t) + \sum_{k=1}^{\infty} b_k \cos(k\omega_o t), \tag{11}$$

where a_k and b_k are the Fourier coefficients, and θ_0 is the given initial configuration, i.e., $\theta(t_0) = \theta_0$.

Assumptions

To generate the optimal solution of a cyclic task, we make two assumptions for the solution based on the previous observations.

1) The optimal solution represented by the Fourier series has the lowest sum of the squared amplitude of harmonic components, which is known as the total power of a signal.

2) Also it has monotonically decreasing coefficients for higher harmonic components.

The first assumption is based on the fact that the smaller is the better as far as joint trajectories satisfy (1), (4), and (5). It is easy to observe that unnecessarily large values of physical variables, e.g., velocity, acceleration, torque, and etc., are not desirable by any means. Basically joint trajectories depend on the behavior of all these physical variables, so the first assumption naturally excludes extremal solutions. The second assumption can be interpreted as follows: If a task has a fundamental frequency ω_0, the higher frequency harmonics appearing in joint trajectories are not desirable. And if the coefficients of harmonics are not monotonically decreasing, then undesirable higher frequency harmonics has larger value than lower frequency harmonics.

Predetermined Trajectories of Redundant Joints

For a redundant manipulator, joint values are not determined uniquely, it has $(n - m)$ redundant degrees of freedom which can be used to optimize some performance indices. On the other hand, this redundancy makes it difficult to solve inverse kinematics. We know that redundant degrees of freedom determine the configuration of a manipulator while tracking a given task. In other words, if the trajectories of redundant degrees of freedom θ_p are predetermined in the desirable homotopy class, the remaining degrees of freedom θ_r are accordingly determined in the desirable homotopy class, too. Based on the assumptions, $(n - m)$ redundant degrees of freedom represented by the Fourier series lead to the desirable homotopy class depending on their Fourier coefficients. Even though theoretically infinite series is required to represent any periodic trajectory, it is enough to approximate by the Nth partial sum of the Fourier series. As the results of approximate predetermination, the trajectories of redundant degrees of freedom θ_p are given by

$$\theta_j(t) = \theta_{j0} - \sum_{k=1}^{N} b_{jk} + \sum_{k=1}^{N} a_{jk} \sin(k\omega_0 t) + \sum_{k=1}^{N} b_{jk} \cos(k\omega_0 t) \tag{12}$$

where $j = 1, 2, ..., n - m$, and a_{jk} and b_{jk} are the coefficients of kth harmonics of the jth joint. By differentiating (12), we can obtain joint velocity

$$\dot{\theta}_j(t) = \omega_0 \sum_{k=1}^{N} k a_{jk} \cos(k\omega_0 t) - \omega_0 \sum_{k=1}^{N} k b_{jk} \sin(k\omega_0 t). \tag{13}$$

Then, the remaining joints θ_r can be determined by solving nonredundant inverse kinematics from given x and predetermined θ_p. So, (1) can be rewritten as

$$x = f(\theta_p, \theta_r), \tag{14}$$

where we have m equations with m unknowns. We denote the above kinematics as f'. Then, resulting nonredundant inverse kinematics problem becomes

$$\theta_r = f'^{-1}(x, \theta_p). \tag{15}$$

Multidimensional Minimum Search

To obtain an optimal solution, we change the variables, θ and $\dot{\theta}$ to the Fourier coefficients of the approximated series, a_{jk} and b_{jk}. From (12) and (15), all θ can be described by the Fourier coefficients. Also all $\dot{\theta}$ can be described by the Fourier coefficients from (13) and the inverse differential kinematics which can be derived from (2). Therefore, we can rewrite the given problem (9) as follows:

$$\min \int_{t_0}^{t_1} G(a_{jk}, b_{jk}, t)\, dt$$

subject to

$$x = f'(a_{jk}, b_{jk}) \tag{16}$$

where $j = 1, \ldots, n-m$ and $k = 1, \ldots, N$. We note that the periodic boundary conditions are already implicitly included in (16).

It is obvious that the only remaining work to solve the problem is finding the optimal coefficients. Therefore, reformulated problem becomes a multidimensional minimum search problem. In this work, we adopt the Powell's method to find the optimal coefficients a_{jk} and b_{jk}, where $j = 1, \ldots, n-m$ and $k = 1, \ldots, N$ [9].

III. Tasks and Solutions

We show two major aspects of the proposed method through numerical examples. One is that the solution by the proposed method can be a good approximation to the exact optimal solution. The other is that the proposed method does not require any additional efforts on the topological liftings of the trajectories for the optimal solution. To show the validity, two cyclic tasks are applied to a 3-link planar manipulator. Consider the 3-link planar manipulator in a horizontal plane in Fig. 1. We choose the position of the end-effector in 2-D space described in Cartesian coordinates. Accordingly, $x \in R^2$, and the degree of redundancy at nonsingular points is equal to one. Link parameters of a 3-link planar manipulator are $l_1 = 3.0$, $l_2 = 2.5$, and $l_3 = 2.0$ units, respectively.

If we denote $s_1 = \sin(\theta_1)$, $c_1 = \cos(\theta_1)$, $s_{12} = \sin(\theta_1+\theta_2)$, and $c_{12} = \cos(\theta_1+\theta_2)$, the kinematic equations are

$$x = \begin{bmatrix} l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ l_1 s_1 + l_2 s_{12} + l_3 s_{123} \end{bmatrix}. \tag{17}$$

In the numerical examples, cyclic tasks are described as

$$x = \begin{bmatrix} -R\cos(2\pi t) + C \\ -R\sin(2\pi t) \end{bmatrix}, \tag{18}$$

where R is the radius of the circle and C is the x-axis position of the center of the circle. The task is to rotate the circle of R unit radius, centered at $(C, 0)$, in unit time, in a counterclockwise, thus the initial position is $(C-R, 0)$ at $t_0 = 0$.

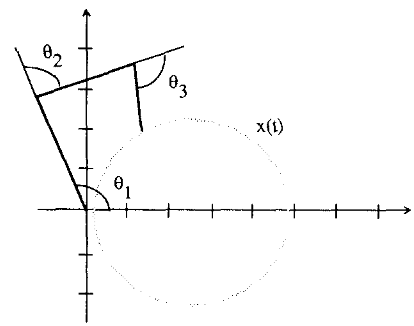

Fig. 1. Geometry of a three-link planar manipulator for Task 1

A. Exact vs. Approximate Optimal Solution

Consider Task 1 which is $R = 2.3$ and $C = 2.5$ units. For given task, we formulate the problem which has an integral cost criterion (7) subject to kinematic constraints (1) and periodic boundary conditions (5) for given initial configuration θ_0. This problem can be solved symbolically by the Euler-Lagrange equations and exact symbolic solution consisting of the necessary conditions for the optimality (8) and the periodic boundary conditions (5) can be obtained. Fig. 2 shows an exact optimal solution of Task 1 [7]. At the same time, we draw an extremal solution of Task 1 obtained from the necessary conditions (8) at different initial joint velocity $\dot{\theta}_0$.

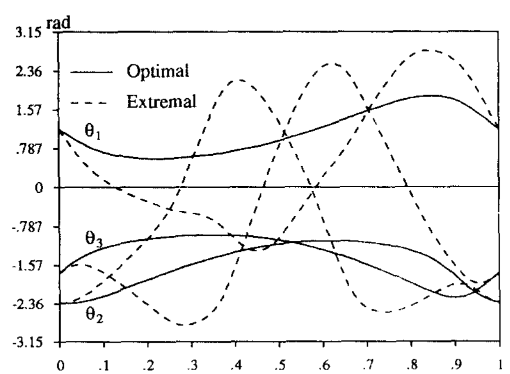

Fig. 2. Optimal and extremal solutions of Task 1

On the other hand, since the proposed method is based on the Nth partial sum of the Fourier series, the corresponding solution is inherently approximation of the exact optimal solution. To compare the approximate optimal solutions with the exact optimal solutions, we predetermine θ_1 as

$$\theta_1(t) = \theta_{10} - \sum_{k=1}^{N} b_k + \sum_{k=1}^{N} a_k \sin(k\omega_0 t) + \sum_{k=1}^{N} b_k \cos(k\omega_0 t). \quad (19)$$

Then optimal coefficients a_k and b_k are searched by the Powell's method so that we obtain an optimal solution. Fig. 3 is obtained by approximating up to the 2nd and 5th harmonics, respectively. From Figs. 2 and 3, in a practical sense, we know that there is very small difference between the exact optimal solutions and the approximate optimal solutions even though the solutions are approximated by a limited number of harmonic components.

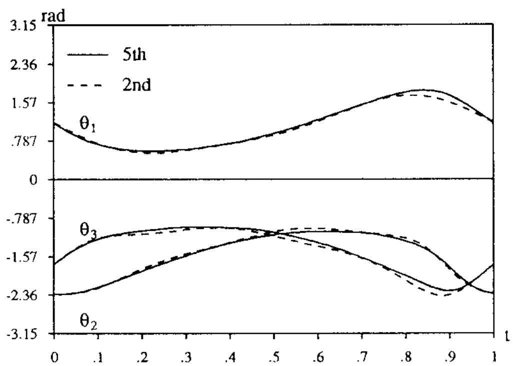

Fig. 3. Approximate optimal solutions of Task 1

B. Multiple Nonhomotopic Extremal Solutions

Multiple extremal solutions can be obtained from the necessary conditions (8) and periodic boundary conditions (5) for given initial configuration θ_0. And there are many extremal solutions depending on the initial joint velocity $\dot{\theta}_0$ as well as the initial joint angle θ_0. Unfortunately, not all these extremal solutions are optimal, nor is the optimal solution among them always unique. Therefore it is natural to investigate how to select an optimal solution among other extremal solutions for given initial configuration. For this purpose, consider Task 2 which is $R = 1$ and $C = 6$ units. Let θ_0 be (-0.47124, 1.7875, -1.8734) *radians* and we denote it as Initial-A. Fig. 4 shows initial configuration for Task 2.

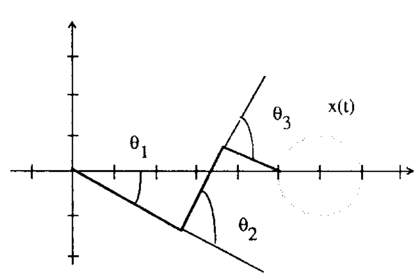

Fig. 4. Geometry of a three-link planar manipulator for Task 2

Resultant solutions based on the proposed method are shown in Fig. 5. Solid lines in Fig. 5 show the optimal solution by approximating up to 4th harmonics. For comparison, dotted lines in Fig. 5 show one solution among many extremal solutions which satisfy (8) and (5). The approximate optimal solution shown in Fig. 5 does not change the arm configuration of the manipulator. Link 3 has upper arm configuration with respect to link 2 and also link 2 has lower arm configuration with respect to link 1 through the trajectories. On the other hand, an extremal solution shown in Fig. 5 changes the arm configuration of the manipulator. For example, link 3 has upper arm configuration with respect to link 2 around the initial and final points of trajectory. But link 3 has lower arm configuration with respect to link 2 in the middle of the trajectory. Similarly, θ_2 changes the arm configuration.

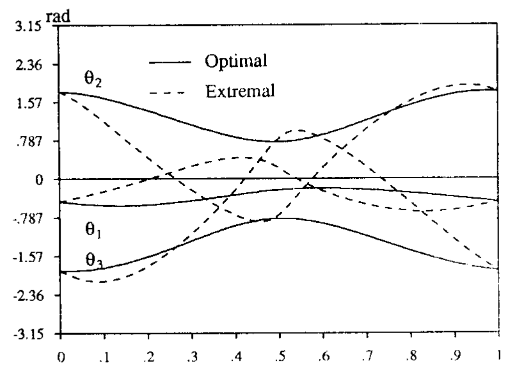

Fig. 5. Solution of Task 2 at Initial-A

IV. Analysis of Solutions

A. FFT Analysis

In the previous section, we have made two assumptions which provide the optimal solution. And it is valuable to show the validity of the assumptions in this stage. We evenly sample 4096 data points of θ_3 trajectories of both the optimal solution and the extremal solution shown in Fig. 2, respectively. Comparing these two data by using FFT (Fast Fourier Transform) as shown in Table 1, the total power of the θ_3 of the optimal solution in Fig. 2 is smaller than that of the extremal solution. The values obtained up to 2nd harmonics are 0.365 and 5.867, respectively. These data show the validity of the first assumption. We know that the power of *k*th harmonics is computed from the FFT data as $a_k^2 + b_k^2$ and the magnitude of *k*th harmonics in Complex Fourier Transform of real-valued data point is the square root of the power of *k*th harmonics. Also the magnitude is the *k*th coefficients of the Fourier series in complex form. From Table 1, we know that the optimal solution have monotonically decreasing Fourier coefficients. On the contrary, the coefficients of the extremal solution does not decrease monotonically. The coefficients in complex form are 2.185, 1.045, 0.043, 0.043, and 0.073, respectively. These values show that the extremal solution offend the second assumption.

Harmonics	Optimal Solution	Extremal Solution
1st real	-0.3580	-1.2798
1st imaginary	0.4525	-1.7713
2nd real	-0.0183	0.2585
2nd imaginary	0.1769	1.0124
3rd real	0.0132	-0.0398
3rd imaginary	0.0763	0.0156
4th real	0.0160	-0.0103
4th imaginary	0.0313	0.0418
5th real	0.0116	0.0184
5th imaginary	0.0105	0.0703

Table 1. FFT Data of θ_3 for Task 1

B. Phase Plane Analysis

Fig. 6 shows that the solid lines are the projection of the set of joint angles into the (θ_3, θ_2) phase plane that satisfies the kinematic constraints (1) for some t. Since the manipulator has one degree of redundancy, the set of such joint angles is a set of dimension one greater than the dimension of the set $x(t)$. In this case, the set can be described as a surface topologically equivalent to a deformed torus [7]. From Fig. 6, we can find two solutions which one has a short trajectory in the torus and the other has a long trajectory. In other words, it shows the existence of multiple nonhomotopic extremal solutions depending on the initial joint velocity $\dot{\theta}$ for given initial configuration. Unfortunately, nonhomotopic extremal solutions can not be continuously transformed from one homotopy class to the other, as pointed out by Martin *et al* [7]. It means that topological liftings from the undesirable class to the desirable class is necessary by any means to obtain a globally optimal solution.

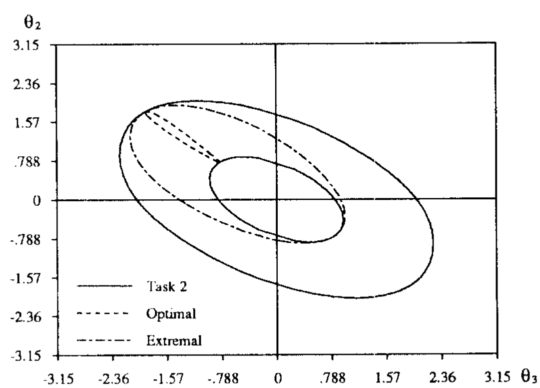

Fig. 6. (θ_3, θ_2) phase trajectories of Task 2

As shown in Fig. 6, the short trajectory does not enclose the hole of torus and remains in 2nd quadrants, this means that the arm configuration is maintained while performing Task 2. On the other hand, the long trajectory which traverses 2-3-4-1-2 quadrants in sequence encloses the hole of torus and this means that superfluous self-motions exist. Therefore, the methods based on the Euler-Lagrange equations (necessary conditions) are not sure to provide an optimal solution among nonhomotopic extremal solutions in several homotopy classes, also unable to lift trajectories in the undesirable class to the desirable class.

V. Conclusion

This paper proposed a numerical method of optimization which has an integral cost criterion for kinematically redundant manipulators. We predetermined the trajectories of redundant joints in terms of the Nth partial sum of the Fourier series, which lead to the solution in the desirable homotopy class. Then, the optimal coefficients of the Fourier series, which yield the optimal solution within the predetermined class, were searched by the Powell's method. Therefore the optimal solution can be obtained among extremal solutions. The proposed method was compared with mathematically rigorous method [7]. Simulation results showed that it was very practical and easy to implement. We analyzed several solutions by the FFT and discussed some features of solutions on the phase plane to investigate requirements of the optimal solution.

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