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NECESSARY OPTIMALITY CONDITIONS IN THE SMALL FOR DEGENERATE HYPERBOLIC DISTRIBUTED-PARAMETER CONTROL SYSTEMS

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ABSTRACT

The degenerate case of multivariable hyperbolic distributed-parameter systems (systems of hyperbolic partial differential equations) in time coordinate t and space coordinate x is characterized by a property that all the characteristic curves of the state equations are parallel to the coordinate axes of independent variables. It is a disturbing fact, although not well known, that the so-called maximum principle as applied to these systems does not exist for the control that depend on time alone.

In this paper, however, it is shown that a set of necessary conditions in the small can exist for unconstrained as well as magnitude constrained controls in a locally convex set. The necessary conditions thus derived can be used conveniently to find the optimal control for degenerate hyperbolic distributed-parameter control systems.

1. STATEMENT OF THE PROBLEM

We consider a process described by the following system of degenerate first-order hyperbolic partial differential equations [PDE's] in two independent variables, time t and position x :

$$\frac{\partial v_i}{\partial t} = f_i[t, x, v(t, x), u(t)], \quad (i = 1, 2, \cdots, n) \tag{1}$$

$$\frac{\partial v_i}{\partial x} = f_i[t, x, v(t, x), u(t)], \quad (i = n+1, n+2, \cdots, N)$$

where the N-component vector $v(t, x) \equiv [v_1(t,x), v_2(t, x), \cdots, v_N(t, x)]'$ (where superscript prime denotes transpose) describes the state of the process on a fixed domain $TXS \equiv [0, t_f] \times [0, x_f]$; $u(t) \equiv [u_1(t), u_2(t), \cdots, u_m(t)]'$ is the m-dimensional control on TXS ; $f \equiv [f_1, f_2, \cdots, f_N]'$ is a specified N-dimensional vector whose components $f_i(t, x, v, u)$ $(i = 1, 2, \cdots, N)$, are bounded piecewise continuous functions of t and x and are twice continuously differentiable with respect to v and u. The following initial data are specified:

$$v_i(0, x) = \phi_i[x], \quad (i = 1, 2, \cdots, n)$$
$$v_i(t, 0) = \psi_i[t], \quad (i = n+1, n+2, \cdots, N) \tag{2}$$

where the functions ϕ_i are piecewise continuous in x and ψ_i are piecewise continuous in t. Let the functional J given by

$$J(u) = \int_S F_1[x, \hat{v}(t_f, x)]\, dx + \int_T F_2[t, \tilde{v}(t, x_f)]\, dt + \int_S \int_T F_3[t, x, v(t, x), u(t)]\, dt\, dx \tag{3}$$

be a measure of performance of the process. In J, $\hat{v}(t, x)$ denotes the n-vector consisting of the first n-components $v_1(t, x), v_2(t, x), \cdots, v_n(t, x)$ of v and $\tilde{v}(t, x)$ is the column vector $[v_{n+1}(t,x), v_{n+2}(t, x), \cdots, v_N(t, x)]'$; the integrands F_1, F_2 and F_3 are bounded piecewise continuous functions of x, t and (t, x), respectively, and are twice continuously differentiable with respect to the remaining arguments. The optimal control problem is to find a piecewise continuous function u(t) so as to maximize the functional J subject to the state equations (1) and (2). In general, it is necessary to assume that the control u(t) belongs to a certain admissible control region U_t. We will now define this control region.

The region of admissible controls U_t is the set of all bounded and piecewise continuous functions u(t) on T such that $u(t): T \rightarrow \Omega_u$ where Ω_u is an arbitrary subset of R^m.

For this problem, it was falsely conjectured that the maximum principle would apply. That is, if u*(t) is the optimal control (v* and λ^* corresponding v and λ), then

$$\int_S H(t, x, v^*, u^*, \lambda^*)\, dx \tag{4}$$

should be maximum where H is the usual Hamiltonian function for the system. This maximum principle approach has been provn to be wrong.

The maximum principle, however, can be replaced by conditions which are locally valid and therefore termed "necessary optimality conditions in the small." We now derive these conditions.

In this derivation, it is necessary to ensure that the optimal control vector u*(t) can be enclosed in some " sufficiently rich" family of nearby trajectories which are admissible. We will ensure this by assuming that Ω_u is locally convex. Thus the following admissible control space U_t is specified.

Definition 1 : The control region U_t is the set of all bounded and piecewise continuous functions $u : T \rightarrow \Omega_u \subseteq R^m$, where Ω_u is everywhere locally convex with respect to the absolute-value metric, $\sum_{i=1}^{m} | \bar{u}_i(t) - u_i(t) |$, defined on any two points $\bar{u}$ and u in R^m.

The local convexity requirement of Ω_u implies that for every $u(t) \in U_t$, there exists a number $\rho > 0$ and $\bar{u}(t) \in U_t$ such that $\sum_{i=1}^{m} | \bar{u}_i(t) - u_i(t) | < \rho$ implies $\{u + \varepsilon(\bar{u} - u)\} \in U_t$, $0 \leq \varepsilon \leq 1$. This requirement is not satisfied only by very limited control sets which, for example, consist of isolated points of R^m. Most practical problems, however, do satisfy this requirment.

Now, in order to state and prove the optimality conditions for u(t), we require the defining Hamiltonian function to be differentiable with respect to u. Therefore we assume that the problem (1) - (3) satisfies the following conditions.

Condition 1 : For given $v \in V$, $u \in U_t$ the functions $f_i(t, x, v, u)$ $(i = 1, 2, \cdots, N)$, are piecewise continuous in t and x on TXS ; for any $(t, x) \in TXS$, f_i's are twice continuously differentiable with respect to v and u on $V \times U_t$.

Condition 2 : $F_3(t, x, v, u)$ satisfies smoothness criterion analogous to Condition 1.

2. HAMILTONIAN FUNCTION AND CO-STATE SYSTEM

We difine the Hamiltonian function of the system (1) - (3) :

$$H(t, x, v, u, \lambda) = F_3(t, x, v, u) + \sum_{i=1}^{N} \lambda_i(t, x) \cdot f_i(t, x, v, u) \tag{5}$$

where the N-dimensional $\lambda(t, x)$, known as the co-state vector, is governed by the following system of PDE's :

$$\frac{\partial \lambda_i}{\partial t} = -\frac{\partial H}{\partial v_i} = -\frac{\partial F_3}{\partial v_i} - \sum_{k=1}^{N} \lambda_k \frac{\partial f_k}{\partial v_i}, \quad (i = 1, 2, \cdots, n)$$

$$\frac{\partial \lambda_i}{\partial x} = -\frac{\partial H}{\partial v_i} = -\frac{\partial F_3}{\partial v_i} - \sum_{k=1}^{N} \lambda_k \frac{\partial f_k}{\partial v_i}, \quad (i = n+1, n+2, \cdots, N) \tag{6}$$

with final and boundary conditions given by

$$\lambda_i(t_f, x) = \frac{\partial F_1[x, \hat{v}(t_f, x)]}{\partial v_i(t_f, x)}, \quad (i = 1, 2, \cdots, n)$$

$$\tag{7}$$

$$\lambda_i(t, x_f) = \frac{\partial F_2[t, \tilde{v}(t, x_f)]}{\partial v_i(t, x_f)}, \quad (i = n+1, n+2, \cdots, N)$$

3. NECESSARY OPTIMALITY CONDITIONS

Now let v^* and $\lambda^* \in V$ be the solutions of equations (1) - (2) and (6) - (7), respectively, corresponding to an admissible control $u^*(t) \in U_t$. Then we have

Theorem 1 (The Optimality Condition in the Small)

If $u^*(t)$ maximizes J in equation (3), there exists an α-neighbourhood of u^* in U_t denoted $C_0(\alpha)$, such that for all $u \in C_0(\alpha) \subset U_t$,

$$\sum_{i=1}^{m}[u_i(t) - u^*_i(t)] \cdot \int_S \frac{\partial H}{\partial u_i}\bigg|_{(t,x,v^*,u^*,\lambda^*)} dx \leq 0 \tag{8}$$

almost everywhere on T, where

$$C_0(\alpha) = \{u : u \in U_t ; \sum_{i=1}^{m} | u_i(t) - u^*_i(t) | \leq \alpha,\ t \in T; \ \alpha > 0\} \tag{9}$$

Theorem 2 (Optimality Conditions in the Small for Magnitude Constraints)

Let the space of admissible controls be defined by

$U_t = \{[u_1(t), u_2(t), \cdots, u_m(t)] : u_{i_0} \leq u_i(t) \leq u_i^0$; $u_i(t)$ are bounded and piecewise continuous functions on T, $(i = 1, 2, \cdots, m)\}$ where u_{i_0} and u_i^0 are (2m) specified constants. Then if $u^*(t) \in U_t$ maximizes J in equation (3),

$$\int_S \frac{\partial H[t, x, v^*, u^*(t), \lambda^*]}{\partial u_k} dx = 0 \quad \text{for a.a.} \quad t \in T_1^{(k)} \tag{10}$$

$$\leq 0 \quad \text{for a.a.} \quad t \in T_2^{(k)}$$

$$\geq 0 \quad \text{for a.a.} \quad t \in T_3^{(k)}$$

$$T_1^{(k)} = \{t : t \in T,\ u_{k_0} < u^*_k(t) < u^0_k\}$$
$$T_2^{(k)} = \{t : t \in T,\ u^*_k(t) = u_{k_0}\} \tag{11}$$
$$T_3^{(k)} = \{t : t \in T,\ u^*_k(t) = u^0_k\}$$

for $k = 1, 2, \cdots, m$. Simply stated, the integrated Hamiltonian, $\int_S H[t, x, v^*, u^*(t), \lambda^*]\, dx$, must be stationary in u^*_k if $u^*_k(t)$ lies in the interior of the admissible region and must attain a local maximum in u_k at u^*_k if u^*_k is at the boundary of the admissible set.

4. PROOF OF THE THEOREMS

Proof of Theorem 1 : The theorem is proved by contradiction, that is, we assume the theorem is false. This assumption allows us to construct a control $u^1(t) \in U_t$ such that $J(u^1) > J(u^*)$, thus contradicting the optimality of u^*.

Consider the neighbourhood $C_0(\delta) \subset U_t$ of $u^*(t)$ defined by

$$C_0(\delta) = \{u : u \in U_t ; \sum_{i=1}^{m} | u_i(t) - u_i^*(t) | \leq \delta,\ t \in T\} \tag{12}$$

In view of the local convexity of U_t, there is at least one $\delta_0 > 0$ for which $C_0(\delta_0)$ is a convex set. Then it follows that for any number, $\rho \in (0, \delta_0)$, the set $C_0(\rho)$ is also convex.

If the theorem is false, for every $\rho > 0$, there exists at least one $\bar{u} \in C_0(\rho)$ and a closed subinterval $T_0(\bar{u}, \rho) \subseteq T$ such that

$$\sum_{i=1}^{m} [\bar{u}_i(t) - u^*_i(t)] \cdot \int_S \frac{\partial H}{\partial u_i}\bigg|_{(t,\, x,\, v^*,\, u^*,\, \lambda^*)} dx > \rho\,(\bar{u}, \rho) \tag{13}$$

almost everywhere on $T_0(\bar{u}, \rho)$ where $\rho(\bar{u}, \rho) > 0$ is a number dependent upon $\bar{u}$ and ρ. Let us define a new control $u^1(t)$ by :

$$u^1(t) = \begin{cases} u^*(t) + \varepsilon\,[\bar{u}(t) - u^*(t)], & t \in T_0(\bar{u}, \rho) \\ u^*(t), & t \not\in T_0(\bar{u}, \rho) \end{cases} \tag{14}$$

where $0 \leq \varepsilon \leq 1$ is a constant. Clearly from equations (12) and (14), $u^1 \in C_0(\varepsilon\rho)$ where $C_0(\varepsilon\rho)$ is the convex neighbourhood of u* of radius $\varepsilon\rho$ and is obtained from $C_0(\delta)$ by substituting $\varepsilon\rho$ in place of δ. We shall now show that for some $\varepsilon\rho > 0$, u^1 yields a value of J higher than J(u*). If we carry out the calculation, we have

$$J(u^1) - J(u^*) = \int_S\int_T [H(t, x, v^*, u^1, \lambda^*) - H(t, x, v^*, u^*, \lambda^*)] \cdot dt\, dx + \eta^* \quad (15)$$

when η^* is the remainder term that can be estimated by

$$| \eta^* | \leq L_0\int_S \{\int_T \Delta_u\hat{f}\, dt + \int_S\int_T \Delta_u f\, dt\, dx\}^2\, dx$$

$$+ L_0\int_T \{\int_S \Delta_u\tilde{f}\, dx + \int_S\int_T \Delta_u f\, dt\, dx\}^2\, dt$$

$$+ L_0\int_S\int_T \{\int_T \Delta_u\hat{f}\, dt + \int_S \Delta_u\tilde{f}\, dx + \int_S\int_T \Delta_u f\, dt\, dx\}$$

$$\cdot \{\int_T \Delta_u\hat{f}\, dt + \Delta_u\tilde{f}\, dx + \int_S\int_T \Delta_u f\, dt\, dx$$

$$+ \Delta_u g\}\, dt\, dx \quad (16)$$

In (16), $0 \leq L_0 \leq \infty$ is a constant and the quantities $\Delta_u\hat{f}$, $\Delta_u\tilde{f}$, $\Delta_u f$ and $\Delta_u g$ are defined by

$$\Delta_u\hat{f} = \sum_{i=1}^{n} | f_i(t, x, v^*, u^1) - f_i(t, x, v^*, u^*) |$$

$$\Delta_u\tilde{f} = \sum_{i=n+1}^{N} | f_i(t, x, v^*, u^1) - f_i(t, x, v^*, u^*) |$$

$$\Delta_u f = \sum_{i=1}^{N} | f_i(t, x, v^*, u^1) - f_i(t, x, v^*, u^*) |$$

$$\Delta_u g = \sum_{i=1}^{N} \left| \frac{\partial H (t, x, v^*, u^1, \lambda^*)}{\partial v_i} - \frac{\partial H (t, x, v^*, u^*, \lambda^*)}{\partial v_i} \right| \quad (17)$$

Since f_i and $\partial H / \partial v_i$ are continuously differentiable with respect to u, they satisfy the Lipschitz condition in u. Thus (17) yields

$$\Delta_u\hat{f},\ \Delta_u\tilde{f},\ \Delta_u f,\ \Delta_u g \leq L_4 \sum_{i=1}^{m} | u^1_i(t) - u^*_i(t) | \quad (18)$$

almost everywhere on T, where $0 \leq L_4 < \infty$ is a finite constant. In view of the definition of u^1 from (14), we have

$$\Delta_u\hat{f},\ \Delta_u\tilde{f},\ \Delta_u f,\ \Delta_u g \leq L_4 \cdot \varepsilon \cdot \sum^{m} | \bar{u}_i(t) - u^*_i(t) |, \quad t \in T_0(\bar{u}, \rho)$$

$$= 0 \qquad , t \notin T_0(\bar{u}, \rho) \quad (19)$$

Since $\bar{u} \in C_0(\rho)$, we have $\sum_{i=1}^{m} | \bar{u}_i(t) - u_i^*(t) | \leq \rho$. Then from (19), we have

$$\Delta_u\hat{f}, \Delta_u\tilde{f}, \Delta_u f, \Delta_u g \leq L_4\, \varepsilon\rho \quad \text{for } t \in T_0(\bar{u}, \rho)$$

$$= 0 \quad \text{for } t \notin T_0(\bar{u}, \rho) \quad (20)$$

Finally from (16) and (20), we get

$$| \eta^* | \leq L_5\, \varepsilon^2\rho^2 \{\mu(\bar{u}, \rho) + \mu^2(\bar{u}, \rho)\} \quad (21)$$

where $0 \leq L_5 < \infty$ is a constant independent of $\bar{u}$, ρ and ε; $\mu(\bar{u}, \rho)$ = Measure of $T_0(\bar{u}, \rho)$, $\mu > 0$.

We expand H in a Taylor series about (t, x, v*, u*, λ^*) and obtain:

$$H(t, x, v^*, u^1, \lambda^*) - H(t, x, v^*, u^*, \lambda^*) = \sum_{i=1}^{m} [u^1_i(t) - u^*_i(t)] \cdot \left.\frac{\partial H}{\partial u_i}\right|_{(t, x, v^*, u^*, \lambda^*)} + \eta' \quad (22)$$

where

$$\eta' = \sum_{i=1}^{m} (u^1_i - u^*_i) \cdot \left[\frac{\partial H [t, x, v^*, u^* + \sigma_0(u^1 - u^*), \lambda^*]}{\partial u_i} - \frac{\partial H [t, x, v^*, u^*, \lambda^*]}{\partial u_i}\right] \quad (23)$$

and $0 \leq \sigma_0(t) \leq 1$ is an integrable function on T. Due to the Lipschitz condition on $\partial H / \partial u_i$, equation (23) yields

$$| \eta' | \leq L_6 \{ \sum_{i=1}^{m} | u^1_i - u^*_i | \}^2 \quad (24)$$

where $0 \leq L_6 < \infty$ is a constant. In view of equation (14), and the definition of $C_0(\rho)$, the above inequality yields :

$$| \eta' | \leq L_6\, \varepsilon^2\rho^2 \quad \text{for almost all } t \in T_0(\bar{u}, \rho)$$

$$= 0 \quad \text{for} \quad t \notin T_0(\bar{u}, \rho) \quad (25)$$

Now equations (22) - (23) may be substituted into (25) to yield

$$J(u^1) - J(u^*) = \int_S\int_T \sum_{i=1}^{m} [u^1_i(t) - u^*_i(t)] \frac{\partial H}{\partial u_i}\bigg|_{(t,\, x,\, v^*,\, u^*,\, \lambda^*)} \cdot dt\, dx + \int_S\int_T \eta'\, dt\, dx + \eta^*$$

$$\geq \int_S\int_T \sum_{i=1}^{m} [u^1_i(t) - u^*_i(t)] \frac{\partial H}{\partial u_i}\bigg|_{(t,x,v^*,u^*,\lambda^*)} dt\, dx - \int_S\int_T |\eta'|\, dt\, dx - |\eta^*| \quad (26)$$

which in view of equation (14) and subsequently inequalities (13), (21) and (25) yields :

$$J(u^1) - J(u^*) \geq \int_S\int_{T_0} \varepsilon \sum_{i=1}^{m} [\bar{u}_i(t) - u^*_i(t)] \frac{\partial H}{\partial u_i}\bigg|_{(t,\, x,\, v^*,\, u^*,\, \lambda^*)} \cdot dt\, dx - \int_S\int_T |\eta'|\, dt\, dx - |\eta^*|$$

$$> \int_{T_0} \varepsilon \cdot \gamma(\bar{u}, \rho)\, dt - \int_S\int_{T_0} L_6\, \varepsilon^2 \rho^2\, dt\, dx - L_5\, \varepsilon^2 \rho^2 \{\mu(\bar{u}, \rho) + \mu^2(\bar{u}, \rho)\}$$

$$> \varepsilon\, \gamma(\bar{u}, \rho) \cdot \mu(\bar{u}, \rho) - L_7\, \varepsilon^2 \rho^2 \{\mu(\bar{u}, \rho) + \mu^2(\bar{u}, \rho)\} \quad (27)$$

where $L_7 = \mathrm{Max}\{L_6 \cdot x_f,\ L_5\}$, $0 \leq L_7 < \infty$. Here x_f is the length of the interval S. We note that L_7 is independent of ε, ρ and $\bar{u}$. Now that for all $\rho > 0$ and $\bar{u}\ \varepsilon\ C_0(\rho)$ the quantities γ and μ are bounded positive functions, we can choose some $\varepsilon > 0$ for which the right-hand side of the last inequality in (27) becomes positive. Thus for some , $\varepsilon > 0$, we obtain $J(u^1, w, z) > J(u^*, w, z)$ for all $u^1 \in C_0(\varepsilon\, \rho)$. But this is a contradiction to the fact that u* is the max-optimal control of J. Therefore by setting $\alpha = \varepsilon\, \rho$, we have proved that there exists an α-neighbourhood $C_0(\alpha) \subset U_t$ of u* for which the theorem is true. (Q.E.D.)

<u>Proof of Theorem 2</u> : Consider a small neighbourhood, $C_0(\alpha)$, of u*(t) such that for every $u(t) \in C_0(\alpha)$, the inequality (8) in Theorem 1 ia satisfied. Now since the control $\hat{u}(t)$ defined by

$$\hat{u}(t) = [u^*_1(t), u^*_2(t), \cdots, u^*_{k-1}, u_k(t), u^*_{k+1}(t), \cdots, u^*_m(t)]' \quad (28)$$

also belongs to $C_0(\alpha)$, from inequality (8) we obtain the following by replacing u by $\hat{u}$:

$$[u_k(t) - u^*_k(t)] \cdot \int_S \frac{\partial H}{\partial u_k}\bigg|_{(t,\, x,\, v^*,\, u^*,\, \lambda^*)} dx \leq 0 \quad (29)$$

almost everywhere on T.

In view of the definition of $T_1^{(k)}$, the quantity $[u_k(t) - u^*_k(t)]$ may be either positive or negative, therefore on $T_1^{(k)}$,

$\int_0 \frac{\partial H}{\partial u_k}\Big|_{(t,\, x,\, v^*,\, u^*,\, \lambda^*)} dx$ must vanish if (29) is to be satisfied.

Also on $T_2^{(k)}$, we have $u^*_k(t) = u_{k0}$; therefore $[u_k(t) - u^*_k(t)]$ can only assume positive values so that in inequality (29),

$\int_S \frac{\partial H}{\partial u_k}\Big|_{(t,\, x,\, v^*,\, u^*,\, \lambda^*)} dx$ must be nonpositive.

Similarly on $T_3^{(k)}$, the admissible control variations $[u_k(t) - u^*_k(t)]$ can only take negative values thus proving that

$\int_S \frac{\partial H}{\partial u_k} dx$ must be non-negative quantity if (29) is to be satisfied. (Q.E.D.)

5. ILLUSTRATIVE EXAMPLE

Consider a system governed by the degenerate hyperbolic equations

$$\frac{\partial v_1}{\partial t} = -v_2, \qquad v_1(0, x) = 0$$

$$\frac{\partial v_2}{\partial x} = u(t), \qquad v_2(t, 0) = 0 \quad (30)$$

It is required to obtain the scalar piecewise continuous function u(t), $1 \leq u(t) \leq 5$, $t \in [0,10]$, that gives the maximum value to

$$J = \int_0^1\int_0^{10} (-2v_1 - 4.5v_2^2 + u^2)\ dt\ dx \qquad (31)$$

The hamiltonian function and the co-state variables for this system are given by

$$H = -2v_1 - 4.5v_2^2 + u^2 - \lambda_1 v_2 + \lambda_2 u \qquad (32)$$

$$\left.\begin{array}{ll} \dfrac{\partial \lambda_1}{\partial t} = 2, & \lambda_1(10, x) = 0 \\ \dfrac{\partial \lambda_2}{\partial x} = 9v_2 + \lambda_1, & \lambda_2(t, 1) = 0 \end{array}\right] \qquad (33)$$

Now corresponding to the solution $v_1(t, x) = -x \cdot \int_0^t u(t)\ dt$, $v_2(t, x) = u(t) \cdot x$ of the state equations, the solution of (33) is given by

$$\lambda_1(t, x) = -2(10 - t) \qquad (34)$$

$$\lambda_2(t, x) = 4.5(x^2 - 1) \cdot u(t) - 2(x - 1)(10 - t)$$

Thus from equation (32) and (34), we have

$$\int_0^1 \frac{\partial H}{\partial u} dx = 2u(t) + \int_0^1 \lambda_2 dx$$

$$= 2u(t) + \{-3u(t) + (10 - t)\}$$

$$= -u(t) + (10 - t) \qquad (35)$$

In view of Theorem 2, we may argue as follows :

(i) For $t > 5$: $u(t) = 5$ gives $\int_0^1 \frac{\partial H}{\partial u} dx < 0$, therefore $u(t) = 5$ cannot be a part of the optimal control for $t > 5$.

(ii) For $t < 9$: $u = 1$ gives $\int_0^1 \frac{\partial H}{\partial u} dx > 0$, therefore $u=1$ is not part of the optimal control for $t < 9$.

(iii) The stationary condition $\int_0^1 \frac{\partial H}{\partial u} dx = 0$ yields $u(t) = 10 - t$, which for $t < 5$ and $t > 9$ calls for a control policy which is outside the admissible region, [1, 5]. Therefore, the stationary condition does not hold for $t < 5$ or $t > 9$.

Clearly the only control policy which does satisfy Theorem 2 is

$$u^*(t) = \begin{cases} 5 & t < 5 \\ 10 - t & 5 \le t \le 9 \\ 1 & t > 9 \end{cases} \qquad (36)$$

and therefore must be optimal. This is really the case.

LITERATURE

E. Buhler and D. Franke, "Topics in Identification and Distributed Parameter Systems," Friedr. Vieweg & Sohn, 1980.

N. U. Ahmed and K. L. Teo, "Optimal Control of Distributed Parameter Systems," North Holland, 1981.

P. Stravroulakis, "Distributed Parameter Systems Theory, Part I Control," Hutchinson Ross Pub, Co. 1983.

S. G. Tzafestas, " Distributed Parameter Control Systems, Theory and Application," Pergamon Press, 1982.

F. Y. Wang, I. B. Lee and K. S. Chang, " Necessary Conditions in the Optimal Control of Nonlinear Integral Equations," Proc. '89 KACC Vol 2, pp. 947-951, 1989.