DETERMINATION OF THE NONLINEAR SYSTEM OBSERVABILITY

Chun Sik Hwang, Jin Ok Park

Department of Electrinics
Korea Military Academy

ABSTRACT

A method to determine nonlinear system observability will be introduced here.

For the determination of the dererministic nonlinear system observability two conditions connectedness and univalence, are developed and used. Depending on how the conditions are satisfied, observability is classified in three categories : observability in the strict sense, wide sense, and the unobservable case. Including simple linear and nonlinear system example an underwater acoustical locaization tracking nonlinear system, the bearing - only tracking example is analyzed.

1. INTRODUCTION

Deterministic observability problem is a determination of whether every state of the system is connected to the observability mechanisn and how it is connected, if connected. So if any system is observable, then one can reconstruct a process state x(t) from measured information $\{y(t)\}$, where $t \varepsilon [t_0, t_1]$.

The observation or measurement equation of any dynamic system is modeled by

$$y(t) = h(x) \tag{1}$$

where x(t), an unknown process state (such as acoustic source location and motion etc.), is given by

$$\frac{dx}{dt} = f(x) \tag{2}$$

$x(t) \varepsilon R^n$, $y(t) \varepsilon R^m$, and $f(.)$, $h(.)$ are appropriate n and m nonlinerar functionals, respectively. Equation (2) assumes at least a certain intelligent quess as to the structure of the data source such as a submerged moving body.

For (1) and (2) linear in x(t), well - developed test criteria can be used to determine observabilty [1, 2]. But for the general nonlinear process state observability is, in general, much more complicated. In a geometric sense, a functional relationship between measurement space and state space is not generally one - to - one such that an inverse function bettween these two spaces is unique.

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For the determination of the deterministic nonlinear system observability two conditions, connectedness and univalence, are developed and used. Depending on how the conditions are satisfied, observability is classified in three categories : observability in the strict sense, wide sense, and the unobservable case. Including linear system example an underwater acoustical localization trackine nonlinear example, the bearing - only tracking example is analyzed.

2. OBSERVABILITY

In general, nonlinear observability is a very difficult property to ascertain. Sufficient data may be obseved to reconstruct the state of interest over one domain but not over another. In a geometric sense, the mapping from the measurement to the state space may not be in one - to - one correspondence globally.

Various authors have studied nonlinear state observability in different ways. Extension of linear observability criteria to the nonlinear case can be found for control processes in [3] and [4]. Here, an observability matrix rank condition [5, 6, 7] or Taylor series expansion [8] is used. Since observability really involves an inverse function, a well - known inverse function theorem from analysis is used here. In this approach, the Jacobian matrix of the observation related function plays a central role. From this view [9, 10, 11, 12] may be considered in the same category. Despite the many results, some are insufficient, too complicated to apply in practice, applicable only to special classes, or valid for only small variations (linearized models).

2. 1 Modified Inverse Function Theorem

In analysis, an inverse function theorem is widely used to get the local inverse of the function by providing a nonzero determinant of the Jacobian matrix J.

Consider an n real - valued continuous function, $F : x \to Y$, $x \varepsilon R^n$, $Y \varepsilon R^n$ such that

$$F(x) = Y \qquad (3)$$

where F(x) is a C^1 - map of R^n onto itself. The global inverse function theorem says that the necessary and sufficient conditions for F(x) to be a C^1- diffeomorphism (i. e., an inverse F^{-1} exists and is also differentiable) are given as follows [13] :

$$1)\ \det J\ F(x) \neq 0, \text{ and} \qquad (4)$$

$$2)\ \lim_{\|x\| \to \infty} \|F(x)\| = \infty, \text{ for all } x, \qquad (5)$$

where $\|x(.)\|$ is an Euclidean norm.

But the above conditions only guarantee the existence of an inverse function of (3). To be unique for all x and have flexibility in the application, the theorem can be modified further. First, consider the following :

Definition

Any individual function of (3), $F_i(x)$, i = 1, 2, . . . , n is called an absolutely independent function if it consists of only one component of $x \varepsilon R^n$.

Definitions

A cover for a ser A is collection v of sets such that $A \subset \bigcup_{V \varepsilon v} V$. Let X and Y each be connected spaces. If f maps X onto Y with the property that for each $y \varepsilon$ Y has an open neighborhood V such that each component of $u \varepsilon U$, $U = f^{-1}(V)$ is mapped homeomorphically onto V by f, then f is called a covering map. In this case if the cardinal number is n, then f is an n - covering map. If n is finite, then it is a finite - covering map, and if n = 1, then it is a one - covering map.

Remarks

i) For special cases with F including one absolutelv independent function, then

$$\det J\ F^-(x) \neq 0 \text{ for all } x,$$

where F^- consists of n-1 function found by deleting any absolutely independent function from F. And the result can be further generalized for more absolutely independent functions. Since the n - deimensional det JF (.) $\neq$ 0 always includes n - 1 dimensional case, the weakened condition will be used only for the special case whenever it applies.

ii) Further restriction of the so called " finite - covering condition " of Palais theorem [13] to a one - covering condition of the current theorem is necessary for F globally one - to - one.

iii) Neither of the conditions nonzero Jacobian and one - covering conditions alone is enough since the nonzero Jacobian condition alone lacks globality of the inverse of F, and the one - covering condition alone lacks independence of F.

iv) Since the nonzero Jacobian guarantees the existence of only a local inverse, i. e., provides "connectedness condition" in observability analysis. The one - covering condition, on the other hand, provides the uniqueness of this connection. So, this is termed a "univalence condition" for the observability problem.

Example 2 - 1

Consider the two - dimensional function f which is given by

$$f(x) = \begin{bmatrix} x_1^2 + x_2^2 \\ 2x_1^2 + 4x_2^2 \end{bmatrix}$$

Then

$$\lim_{\|x\| \to \infty} \|f(x)\| = \lim_{x_1^2 + x_2^2 \to \infty} \left\{ \left(x_1^2 + x_2^2\right)^2 + \left(2x_1^2 + 4x_2^2\right)^2 \right\} = \infty .$$

Clearly the finite - covering condition is satisfied, but actual solution of the two equations yields

$$f_1(x) = x_1^2 + x_2^2 = y_1,$$

$$f_2(x) = 2x_1^2 + 4x_2^2 = y_2,$$

with non - unique solution

$$x_1 = \pm \sqrt{4y_1 - y_2}$$

$$x_2 = \pm \sqrt{\frac{y_2 - 2y_1}{2}}$$

Thus f is only locally homeomorphic, i. e., f is not one - to - one globally. Both x_1 and x_2 are covered by the two "sheets" of cover. However, the existence of the two independent solutions is guaranteed by a nonzero determinant of the Jacobian,

$$\det Jf(x) = x_1 x_2 \neq 0,$$

$$\text{i.e, with } x_1 \neq 0 \text{ and } x_2 \neq 0,$$

2. 2 Nonlinear Observability

For nonlinear deterministic system (1) and (2), it is assumed that f(.) satisfies the required conditions to quarantee the existence and uniqueness of the solution x(t), and y(t) is assumed differentiable up to (n-1)th order in t. Define, then, system observability as follows:

Definition

The process (1), (2) is observable at t_0 if knowledge of the output measurement y(t), $t \varepsilon [t_0, t_1]$ is sufficient to determine $x(t_0)$ uniquely for finite t_1. If every state $x(t) \varepsilon R^n$ is observable on the time interval considered, then the state is completely observable.

By differentiation of (2) with respect to t and substitution of (1) and with appropriate replacement of lower order derivatives to the higher order successively (with suppression of t in variables for convenience)

$$\begin{aligned} y &= h(x, t), \\ y' &= h_1(y, x, t), \\ y'' &= h_2(y, y', x), \\ &\vdots \\ y^{(n-1)} &= h_{n-1}(y, y' \ldots, y^{(n-2)}). \end{aligned} \tag{6}$$

Denote the teanspose of the vector $Y \varepsilon R^{mn}$ by

$$Y^T = \left[y, y' \ldots y^{(n-1)} \right] \tag{7}$$

and the set Y^-, if needed, by

$$Y^- = \left\{ y, y', \ldots, y^{(n-2)} \right\}. \tag{8}$$

Then the vector notation of (6) becomes

$$Y = H(Y^-, x). \tag{9}$$

By the successive replacement of lower - order derivatives to the higher oeders, the functional dependency between the individual functional elements h, $h_1, h_2, \ldots$, vanishes since this procedure is exactly the same as the successive elimination of unknown variables in solving (6) for x. So, after this procedures the maximum independency between functional elements is obtained.

Now the main observability result may be derived by showing that the unique determination of every state $x(t_0)$ from (9) is determined uniquely from the measurement y(t), $t \varepsilon [t_0, t_1]$ and that (9) has a unique inverse for x(t). The first point is seen by the unique sufficiently smooth Taylor - series of y(t) at $t = t_0$, so that y(t) yields $Y^{(i)}(t_0)$, $i = 1, 2, \ldots, n-1$. Then, the second point depends on the above modification of the inverse function theorem. Comsequently, observability is determined by the following test.

Observability Result

System (1), (2) is observable (in the strict sense) if (9) satisfies the following conditions for all t, $t \varepsilon [t_0, t_1]$:

1) Connectedness condition

$$\det JH_n(\cdot) \neq 0, \tag{10}$$

where $JH_n = \frac{\partial H_n}{\partial x}$; $H_n(\cdot)$ is a subset of H (.) consisting arbitrarily of n of its total mn functions.

2) Univalence condition

For H_n (.), every x_i, $i = 1, 2, \ldots, n$, can be uniquely expressed in terms of only Y in (9).

Depending on the satisfaction of the conditions, define and categorize system observability as follows :

1. Observable in the strict sense.
   Both connectedness and univalence conditions are satisfied for any one or more combinations $H_n(.)$ out of mn function which comprise elements of H(.).
2. Observable in the wide sense.
   Only the connectedness condition is satisfied, i. e., multiple covering apperas in some element of x for some time t.
3. Unobservable.
   One or more states of x cannot be expressed in terms of Y. In this case, those states are unconnected to measurement space Y.

This method is readily applicable to any linear or nonlinear time - varying as well as time invariant system.

Following examples demonstrate the application of the above results.

Example 2 - 2

A falling body in the constant gravity g with position variable x_1 and velocity x_2 can be expressed as

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -g.$$

If one measures position x_1, then

$$y = x_1, \text{ and}$$
$$y' = \dot{x}_1 = x_2$$

So, both states are uniquely determined from $Y = (y,\ y')^T$, and hence the system is observable. On the other hand if velocity x_2 is measured, then

$$y = x_2,$$
$$y' = \dot{x}_2 = -g$$

Only x_2 is connected uniquely to Y. x_1 is disconnected and unobservable ; hence the system is unoservable. Classic rank test can be used to verify this.

Example 2 - 3 [9], [13]

$$\dot{x}_1 = x_2 x_3,$$
$$\dot{x}_2 = -x_1 x_3,$$
$$\dot{x}_3 = 0.$$

$$y = x_1, \text{ then}$$
$$y' = x_2 x_3,$$
$$y'' = -x_1 x_3^2 = -y x_3^2 .$$

So, det $J = 2x_1 x_3^2 \neq 0$ implies that the initial state of the form $\{x_{10} \neq 0, x_{30} \neq 0\}$ satisfies the connectedness conditions. But from the above last three equations, one obtains

$$x_1 = y$$
$$x_2 = \pm y' / \sqrt{\frac{y''}{-y}}$$
$$x_3 = \pm \sqrt{\frac{y''}{-y}}$$

and have multiple expressions or two covers for x_2 and x_3. So, the univalence condition is not satisfied. The system is only observable in the wide sense if $\{x_{10} \neq 0, x_{30} \neq 0\}$.

3. Bearing - Only - Target (BOT) tracking example

An important system observability determination example in underwater tracking is demonstrated here. The example is a bearing - only - target tracking problem where only bearing information of the target is extracted from the measurement device and used to determine the other state variables as well as whole system observability.

Consider an acoustical source or target (T) and receiver or observer (0) with velocity components V_{Tx}, V_{Ty}, V_{Ox}, V_{Oy}, the relative coordinate x(t) and y(t) can be generated as (refer to figure 1)

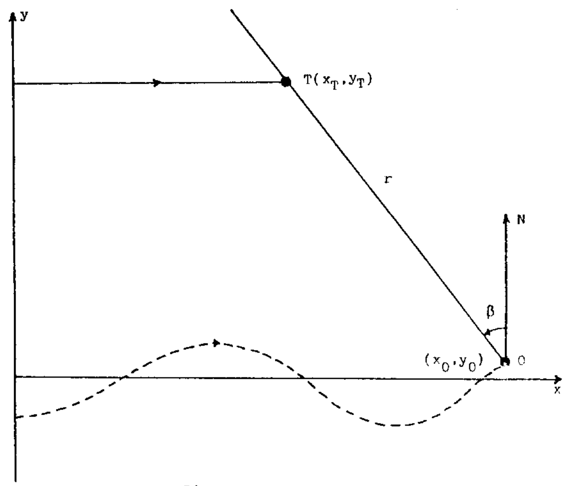

Figure 1, BOT configuration

$$x(t) = x_T(t) - x_0(t),$$
$$y(t) = y_T(t) - y_0(t).$$

Define state variables in mixed coordinates (i. e., a combination of polar and rectangular) as

$$x_1(t) = \beta(t),\ x_2(t) = r(t),$$
$$x_3(t) = V_{Tx}(t) - V_{Ox}(t) = V_{x(t)}$$
$$x_4(t) = V_{Ty}(t) - V_{Oy}(t) = V_y(t),$$

where $\beta(t)$ is bearing from north of T from O and r is range. Then, the state equation in this specific coordinate system becomes

$$\dot{x}(t) = \begin{bmatrix} \dfrac{x_3 \cos x_1 - x_4 \sin x_1}{x_2} \\ x_3 \sin x_1 + x_4 \cos x_1 \\ a_x \\ a_y \end{bmatrix} \qquad (11)$$

where $a_x(t)$, $a_y(t)$ are acceleration in each direction.

Measurement of $\beta(t)$ leads to

$$y(t) = [\,1\ 0\ 0\ 0\,]\ x(t). \quad (12)$$

Observability is checked next for the two cases where maneuvering exists, i. e., $a_x(t) \neq 0$ and / or $a_y(t) \neq 0$, and nonmaneuvering, i. e., both are zero. From (11) and (12) with $a_x(t) = 0$ and $a_y(t) = a(t) \neq 0$, (i. e., maneuvering exists only in y direction), and by successive replacment

$$y = x_1, \quad (13)$$

$$y' = \frac{x_3 \cos y - x_4 \sin y}{x_2}, \quad (14)$$

$$y'' = \frac{-(a \sin y + 2y' x_4 \cos y + 2y' x_3 \sin y)}{x_2}, \quad (15)$$

$$y''' = \frac{3ay' \cos y + x_3 (3y'' \sin y + 2(y')^2 \cos y) + x_4 (3y'' \cos y - 2(y')^2 \sin y) + a' \sin y}{x_2}, \quad (16)$$

Then, from (13) - (16)

$$x_1 = y, \quad (17)$$

$$x_2 = \frac{-2y' x_4 - a \cos y \cdot \sin y}{y'' \cos y + 2(y')^2 \sin y}, \quad (18)$$

$$x_3 = \frac{(y'' \sin y - 2(y')^2 \cos y)\, x_4 - y' a \sin y}{y'' \cos y + 2(y')^2 \sin y}, \quad (19)$$

$$x_4 = \frac{a[4(y')^3 \cos y \sin y + 6y' y'' \cos^2 y - 3y' y'' - y''' \cos y \sin y] + a' \sin y\,[y'' \cos y' + 2(y')^2 \sin y]}{2y' y''' - 3(y'')^2 + 4(y')^4}, \quad (20)$$

From (20) it is clear that if a(t) and / or a' (t) ≠ 0, (i. e., maneuvering exists), x_4 is connected to the measurement vector Y, it is unique, and thus it is observable. This imples from (18) and (19) that x_2 and x_3 are also uniquely connected. So, the system satisfies the connected condition in this case. But when a(t) = 0 and a' (t) = 0, i. e. nonmaneuvering, (20) suggests that x_4 is not connected to Y and is unoservable. This causes, again from (18) and (19), that x_2 and x_3 are disconnected from Y and thus unoservable also. Only x_1 is observale which is itself a measurement variable. After lengthy computation, the determinat of the Jacobian becomes

$$\det J = \frac{-2y' a' \sin y + 3a[2(y')^2 \cos y + y'' \sin y]}{x_2^4} \cdot \frac{-[12y' y'' \sin y(1 + \cos^2 y) + 8\cos^3 y (y')^3]\, x_3 + 4y' \cos y \sin y[2(y')^2 \cos y + 3y'' \sin y]\, x_4}{} . \quad (21)$$

From (21), this system is unoservable with det J = 0 for the following cases

1) Infinite range, $x_2 = \infty$;
2) Zero heading rate and acceleration, $\beta' = \beta'' = 0$;
3) $x_3 = x_4 = 0$, i.e, $\dot{x}_2 = 0$, with $a(t) = a'(t) = 0$ (parallel stationary movement, including tail chase);
4) Constant range with special heading such that

$$\tan \beta = \frac{6a(\beta')^2}{2a'\beta' - 3a\beta''}. \quad (22)$$

As well as certain others, the system is unoservable due to the lack of rank and thus, lack of information of some states in those cases. Consequently, the known result, i. e., bearing - only target tracking system, is observable when relative maneuvering exists with several exceptional conditions such as described in 1) through 4) above.

4. CONCLUSION

Observabiltiy problem for the deterministic nonlinear system is studied here. Since nonlinear system observability is a geometric nonlinear functional structure property, modified version of the inverse function theorem is useful. Two conditions, nonzero Jacobian and one covering condition must be satisfied for the existance and the uniquness of the inverse between the system state space and it's measurement space.

Nonzero Jacobian condition provides the connectedness condition which says that every state must be connected somehow to the measurement space for the system obserbable. On the other hand, one covering condition guarantees the univalence of this connection.

Depending on the satisfaction of these two conditions, observability is classified in three categories. Observale in strict sense means every state is connected to the measurement space and it's connection is unique (nonzero Jacibian and one - covering), observable in wide sense means every state is connected but it's connection is not unique (nonzero Jacobian and multi - covering), and unoservable case means some states are not connected to the measurement space.

Application of this method is demonstrated by some linear and nonlinear examples. Bear - omly - target tracking problem is analyzed for the another practical system example. A well known fact that the BOT system is observable when relative maneuvering exists and unoservable when non - maneuvering is proven, again.

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