A NEW COMBINATION THEOREM FOR RELATIVELY HYPERBOLIC GROUPS

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ABSTRACT. We prove a new combination theorem for relatively hyperbolic groups by analyzing diagrams over HNN-extensions of relatively hyperbolic groups.

1. Introduction

We recall Osin’s definition [3] of relatively hyperbolic groups among many equivalent definitions of relatively hyperbolic groups. Let $G$ be a group, $\mathbb{H} = \{H_\lambda\}_{\lambda \in \Lambda}$ a collection of subgroups of $G$, $X$ a subset of $G$. Suppose that $X$ is a relative generating set for $(G, \mathbb{H})$, namely, $G$ is generated by the set $(\bigcup_{\lambda \in \Lambda} H_\lambda) \cup X$ (for convenience, we assume that $X = X^{-1}$). Then $G$ can be regarded as the quotient group of the free product

$$F = (\ast_{\lambda \in \Lambda} \tilde{H}_\lambda) * F(X),$$

where the groups $\tilde{H}_\lambda$ are isomorphic copies of $H_\lambda$, and $F(X)$ is the free group generated by $X$. Let $\mathcal{H}$ be the disjoint union

$$\mathcal{H} = \bigsqcup_{\lambda \in \Lambda} (\tilde{H}_\lambda \setminus \{1\}).$$

(1)

For every $\lambda \in \Lambda$, we denote by $S_\lambda$ the set of all words over the alphabet $\tilde{H}_\lambda \setminus \{1\}$ that represent the identity in $F$. Then we may describe $G$ as a relative presentation

$$\langle X, \mathcal{H} | S_\lambda, \lambda \in \Lambda, \mathcal{R} \rangle$$

(2)

with respect to the collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$, where $\mathcal{R} \subseteq F$. For brevity, we often use the following shorthand for presentation (2):

$$\langle X, \mathbb{H} | \mathcal{R} \rangle.$$

(3)

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If both the sets $\mathcal{R}$ and $X$ are finite, relative presentation (2) or (3) is said to be finite and the group $G$ is said to be \textit{finitely presented relative to the collection of subgroups} $\mathbb{H}$.

For every word $w$ in the alphabet $X \cup \mathcal{H}$ representing the identity in the group $G$, there exists an expression
\[
w = F \prod_{i=1}^{k} f_i^{-1} R_i f_i \]
with the equality in the group $F$, where $R_i \in \mathcal{R}$ and $f_i \in F$ for $i = 1, \ldots, k$. The smallest possible number $k$ in a presentation of the form (4) is called the \textit{relative area} of $w$ and is denoted by $\text{Area}^{\text{rel}}(w)$.

\textbf{Definition 1.} A group $G$ is said to be \textit{hyperbolic relative to a collection of subgroups} $\mathbb{H}$ if $G$ admits a relatively finite presentation (2) with respect to $\mathbb{H}$ satisfying a \textit{linear relative isoperimetric inequality}. That is, there is a constant $C > 0$ such that for any cyclically reduced word $w$ in the alphabet $X \cup \mathcal{H}$ representing the identity in $G$, we have
\[
\text{Area}^{\text{rel}}(w) \leq C|w|_{X \cup \mathcal{H}},
\]
where the symbol $|w|_{X \cup \mathcal{H}}$ means the word length of $w$ over $X \cup \mathcal{H}$. The constant $C$ is called an \textit{isoperimetric constant} of relative presentation (2).

Let $G$ be a group that is hyperbolic relative to a collection of subgroups $\mathbb{H} = \{H_{\lambda}\}_{\lambda \in \Lambda}$. Suppose that there exists a monomorphism $\iota : H_{\mu} \rightarrow H_{\nu}$ for some $\mu, \nu \in \Lambda$. Osin [4] proved that if $\mu \neq \nu$ and $H_{\mu}$ is finitely generated, then the HNN-extension
\[
G^* = \langle G, t \mid t^{-1} h t = \iota(h), \ h \in H_{\mu} \rangle
\]
is hyperbolic relative to $\mathbb{H} \setminus \{H_{\mu}\}$. Our new combination theorem covers the case when $\mu = \nu$, and is stated as follows. (cf. For finitely generated groups, a similar result was obtained by Dahmani [1].)

\textbf{Theorem 1.1.} Suppose that a group $G$ is hyperbolic relative to a collection of subgroups $\mathbb{H} = \{H_{\lambda}\}_{\lambda \in \Lambda}$. Assume in addition that there exists a monomorphism $\iota : K_{\mu} \rightarrow H_{\mu}$ for some $\mu \in \Lambda$, where $K_{\mu}$ is a subgroup of $H_{\mu}$ and it need not be finitely generated. Then the HNN-extension
\[
G^* = \langle G, t \mid t^{-1} k t = \iota(k), \ k \in K_{\mu} \rangle
\]
is hyperbolic relative to the collection $\mathbb{H} \setminus \{H_{\mu}\} \cup \{H_{\mu}^*\}$, where $H_{\mu}^* = \langle H_{\mu}, t \rangle \leq G^*$.

In particular, if $G = \langle X, \mathbb{H} \mid \mathcal{R} \rangle$ is a relative presentation of $G$ with respect to the collection of subgroups $\mathbb{H}$, then $G^* = \langle X, \mathbb{H} \setminus \{H_{\mu}\} \cup \{H_{\mu}^*\} \mid \mathcal{R} \rangle$, and these two relative presentations have the same isoperimetric constant.

As an immediate corollary, we obtain
Corollary 1.2. Suppose that a group $G = \langle X, H_\lambda, \lambda \in \Lambda \mid R \rangle$ is hyperbolic relative to $\{ H_\lambda \}_{\lambda \in \Lambda}$. Then the group $G^*$ defined by a relative presentation $G^* = \langle X, H^*_\lambda, \lambda \in \Lambda \mid R \rangle$ is hyperbolic relative to $\{ H^*_\lambda \}_{\lambda \in \Lambda}$, where $H^*_\lambda \cong H_\lambda \times A_\lambda$ for some (finitely or infinitely generated) free abelian group $A_\lambda$ for each $\lambda \in \Lambda$.

2. Proof of Theorem 1.1

A word in an alphabet is called cyclically reduced if all its cyclic permutations are reduced. A cyclic word is defined to be the set of all cyclic permutations of a cyclically reduced word. By $(w)$, we denote the cyclic word associated with a cyclically reduced word $w$. Also, by $(u) \equiv (w)$, we mean the visual equality of two cyclic words $(u)$ and $(w)$. For other terminology and notation used throughout this section, we refer the reader to [4, Sections 2 and 3].

Let us fix a finite relative presentation

$$G = \langle X, H_\lambda, \lambda \in \Lambda \mid R \rangle$$

of $G$ with respect to $\{ H_\lambda \}_{\lambda \in \Lambda}$. Clearly HNN-extension (5) has a finite relative presentation

$$G^* = \langle X, H^*_\mu, H_\lambda, \lambda \in \Lambda \setminus \{ \mu \} \mid R \rangle$$

in view of shorthand (3).

For $\mathcal{H}$ defined as in (1), let

$$\mathcal{H}^* = \mathcal{H} \setminus (\tilde{H}_\mu \setminus \{1\}) \cup (\tilde{H}^*_\mu \setminus \{1\}),$$

where $\tilde{H}^*_\mu$ is an isomorphic copy of $H^*_\mu$. Also let $w$ be a cyclically reduced word in the alphabet $X \cup \mathcal{H}^*$ such that $w$ represents the identity in $G^*$. We use the symbol $\|w\|$ to mean the word length of $w$ over $X \cup \mathcal{H}^*$. Not only for $w$ but also for any element in $G^*$, the symbol $\|\cdot\|$ will be used to mean its word length over $X \cup \mathcal{H}^*$. Let $C$ be an isoperimetric constant of relative presentation (6). Then we will show that

$$\text{Area}^\text{rel}(w) \leq C \|w\|. \quad (8)$$

By van Kampen’s Lemma, there is a reduced van Kampen diagram $\Delta$ over presentation (7) such that a boundary label of $\Delta$ is visually equal to $w$ (cf. [2]). In particular, we can take $\Delta$ so that $\Delta$ has the least number of $R$-cells among all van Kampen diagrams over (7) with a boundary label $w$. If $\text{Area}^\text{rel}(\Delta)$ denotes the number of $R$-cells in $\Delta$, this implies that $\text{Area}^\text{rel}(w) = \text{Area}^\text{rel}(\Delta)$. So in order to show (8), it suffices to show

$$\text{Area}^\text{rel}(\Delta) \leq C \|w\|. \quad (9)$$

A cell in a diagram over presentation (7) is called a $t$-cell if it corresponds to a relation of the form $t^{-1}kt = \iota(k)$, where $k \in K_\mu$. They are shown on Figure 1(a). A configuration of $t$-cells, as shown on Figure 1(b), we call a $t$-annulus.

Claim. We may assume that $\Delta$ does not contain a $t$-annulus.
Proof of Claim. Suppose that \( \Delta \) contains a \( t \)-annulus. Take an innermost \( t \)-annulus \( T \) in \( \Delta \), meaning that there is not another \( t \)-annulus inside \( T \). Then the label of the internal contour \( p \) of \( T \) represents the identity in \( H_\mu \). Since \( \iota \) is a monomorphism, the label of the external contour \( q \) of \( T \) also represents the identity in \( H_\mu \). This implies that the circular subdiagram, say \( D \), bounded by the contour \( p \) consists of only \( H_\mu \)-cells and that we may replace \( D \sqcup T \) with \( D' \) consisting of only \( H_\mu \)-cells with the contour \( q \). By repeating this process to remove all \( t \)-annuli from \( \Delta \), we obtain a new van Kampen diagram \( \Delta' \) such that \( \text{Lab}(\partial \Delta') \equiv \text{Lab}(\partial \Delta) \) and \( \text{Area}_{\rel}(\Delta') = \text{Area}_{\rel}(\Delta) \) (see Figure 2), where \( \text{Lab} \) is a labeling function. Hence we may rename \( \Delta' \) as \( \Delta \). \( \square \)

![Figure 1.](image1.png)

**Figure 1.**

By Claim, \( t \)-cells can only form \( t \)-strips, and these \( t \)-strips must end on the boundary of \( \Delta \). To show inequality (9), we proceed by induction on the number of \( t \)-strips in \( \Delta \). If there is no \( t \)-strip in \( \Delta \), then \( \Delta \) is a van Kampen diagram over (6), and hence (9) holds.

Now assume that \( \Delta \) contains at least one \( t \)-strip. Take any \( t \)-strip, say \( T \), in \( \Delta \). Let \( \Delta_1 \) and \( \Delta_2 \) be the subdiagrams lying in the left and right of \( T \), respectively, so that \( \Delta = \Delta_1 \sqcup T \sqcup \Delta_2 \) (see Figure 3).

Clearly \( t \)-cells belong to \( H_\mu^* \)-cells, so they are not counted in \( \text{Area}_{\rel}(\Delta) \). Hence
\[
\text{Area}_{\rel}(\Delta) = \text{Area}^\text{rel}(\Delta_1) + \text{Area}^\text{rel}(\Delta_2).
\] (10)

Moreover, for each \( i = 1, 2 \), note that \( \Delta_i \) is a van Kampen diagram over (7) which has the smallest relative area among all van Kampen diagrams over (7)
with the same boundary label as $\Delta_i$. Then by the induction hypothesis, we have

$$\text{Area}^{rel}(\Delta_i) \leq C||\text{Lab}(\partial\Delta_i)||$$

for all $i = 1, 2$.

Let $(\text{Lab}(\partial\Delta_i)) \equiv (w_ik_i)$, where $w_i$ is a reduced word over $X \cup \mathcal{H}^*$ and $k_i$ is a reduced word over $\tilde{H}_\mu^*$ for all $i = 1, 2$, so that $(w) \equiv (\text{Lab}(\partial\Delta)) \equiv (w_1t^{\pm 1}w_2t^{\mp 1})$ (see Figure 4).

Put $w_i \equiv w_i\bar{w}_iw_{ie}$, where $||w_{ib}|| = ||w_{ie}|| = 1$ for all $i = 1, 2$. Note that for each $i = 1, 2$,

$$\begin{align*}
||\text{Lab}(\partial\Delta_i)|| &= ||w_i|| - 1 & \text{if both } w_{ib} \text{ and } w_{ie} \text{ belong to } \tilde{H}_\mu^*; \\
||\text{Lab}(\partial\Delta_i)|| &= ||w_i|| & \text{if either } w_{ib} \text{ or } w_{ie} \text{ but not both belongs to } \tilde{H}_\mu^*; \\
||\text{Lab}(\partial\Delta_i)|| &= ||w_i|| + 1 & \text{if neither } w_{ib} \text{ nor } w_{ie} \text{ belongs to } \tilde{H}_\mu^*.
\end{align*}$$
Let \( Y = \{w_{1b}, w_{1e}, w_{2b}, w_{2e}\} \cap \tilde{H}_\mu^* \). It then follows that
\[
\begin{align*}
\|\text{Lab}(\partial \Delta_1)\| + \|\text{Lab}(\partial \Delta_2)\| &= \|w_1\| + \|w_2\| - 2 \quad \text{if } |Y| = 4; \\
\|\text{Lab}(\partial \Delta_1)\| + \|\text{Lab}(\partial \Delta_2)\| &= \|w_1\| + \|w_2\| - 1 \quad \text{if } |Y| = 3; \\
\|\text{Lab}(\partial \Delta_1)\| + \|\text{Lab}(\partial \Delta_2)\| &= \|w_1\| + \|w_2\| \quad \text{if } |Y| = 2; \\
\|\text{Lab}(\partial \Delta_1)\| + \|\text{Lab}(\partial \Delta_2)\| &= \|w_1\| + \|w_2\| + 1 \quad \text{if } |Y| = 1; \\
\|\text{Lab}(\partial \Delta_1)\| + \|\text{Lab}(\partial \Delta_2)\| &= \|w_1\| + \|w_2\| + 2 \quad \text{if } |Y| = 0.
\end{align*}
\]
In view of \((w) \equiv (\text{Lab}(\partial \Delta)) \equiv (w_1 t^{\pm 1} w_2 t^{\mp 1})\), note also that
\[
\begin{align*}
\|w\| &= \|w_1\| + \|w_2\| - 2 \quad \text{if } |Y| = 4; \\
\|w\| &= \|w_1\| + \|w_2\| - 1 \quad \text{if } |Y| = 3; \\
\|w\| &= \|w_1\| + \|w_2\| \quad \text{if } |Y| = 2; \\
\|w\| &= \|w_1\| + \|w_2\| + 1 \quad \text{if } |Y| = 1; \\
\|w\| &= \|w_1\| + \|w_2\| + 2 \quad \text{if } |Y| = 0.
\end{align*}
\]
Therefore, in any of five cases, we have
\[
\|\text{Lab}(\partial \Delta_1)\| + \|\text{Lab}(\partial \Delta_2)\| = \|w\|.
\]
This together with (10) and (11) finally yields (9), which completes the proof of Theorem 1.1.

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References


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