RICCI SOLITONS AND RICCI ALMOST SOLITONS ON PARA-KENMOTSU MANIFOLD

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Abstract. The purpose of this article is to study the Ricci solitons and Ricci almost solitons on para-Kenmotsu manifold. First, we prove that if a para-Kenmotsu metric represents a Ricci soliton with the soliton vector field $V$ is contact, then it is Einstein and the soliton is shrinking. Next, we prove that if a $\eta$-Einstein para-Kenmotsu metric represents a Ricci soliton, then it is Einstein with constant scalar curvature and the soliton is shrinking. Further, we prove that if a para-Kenmotsu metric represents a gradient Ricci almost soliton, then it is $\eta$-Einstein. This result is also hold for Ricci almost soliton if the potential vector field $V$ is pointwise collinear with the Reeb vector field $\xi$. 

1. Introduction

A pseudo-Riemannian metric $g$, defined on a manifold $M^n$, is called a Ricci soliton metric, or in short a Ricci soliton if there exist a constant $\lambda \in \mathbb{R}$ and a vector field $V \in \chi(M)$ such that

$$\frac{1}{2} \mathcal{L}_V g + \text{Ric} = \lambda g, \quad (1.1)$$

where $\mathcal{L}_V$ denotes the Lie-derivative in the direction of $V$ and $\text{Ric}$ is the Ricci tensor of $g$. A Ricci soliton is said to be trivial if $V$ is either zero or Killing on $M$. Ricci soliton is considered as a generalization of Einstein metric and often arises as a fixed point of Hamilton’s Ricci flow. In [19], Pigoli-Rigoli-Rinoldi-Setti generalized the notion of Ricci soliton to Ricci almost soliton by allowing the soliton constant $\lambda$ to be a smooth function. We denote it by $(M^n, g, V, \lambda)$. The Ricci almost soliton is said to be shrinking, steady, and expanding accordingly as $\lambda$ is positive, zero, and negative respectively. Moreover, if the potential vector field $V$ is the gradient of some smooth function $u$ on $M^n$, i.e., $V = Du$, where $D$ is the gradient operator of $g$ on $M^n$, then the Ricci soliton is called a gradient Ricci soliton and the soliton Eq. (1.1) becomes

$$\text{Hess } u + \text{Ric} = \lambda g, \quad (1.2)$$

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where \( \text{Hess} \, u \) denotes the Hessian of \( u \). The function \( u \) is known as the potential function.

As a generalization of Einstein metric, Ricci solitons grow interest on a new class of pseudo-Riemannian geometry called paracontact geometry which is introduced by Kaneyuki and Williams [15]. The importance of paracontact manifolds comes from the theory of para-Kähler manifolds. Since then many authors studied the paracontact geometry (see [1, 3, 5, 6, 9, 15, 16, 18, 25, 26]). Specially, Calvaruso-Perrone [4] explicitly studied the Ricci solitons on almost paracontact metric three-manifolds and describe more examples and Bejancrasmareanu [1] studied Ricci solitons on 3-dimensional normal paracontact manifolds. Further, Blaga [2] studied the \( \eta \)-Ricci soliton on para-Kenmotsu manifolds. On the other hand, studies on Ricci solitons in the frame work of contact geometry are very interesting and therefore many authors have been developed (see [7, 8, 10–14, 17, 20] and references therein). Among these many contexts: on Kenmotsu manifolds [10, 11], on \( K \)-contact and \((\kappa, \mu)\)-contact manifolds [20], on Sasakian manifolds [14], on Kähler manifolds [8] etc. Recently, the present author and Ghosh explicitly studied the Ricci solitons and \( \ast \)-Ricci solitons in the frame-work of Sasakian and \((\kappa, \mu)\)-contact manifolds (see [12, 13]). Further, the study of Ricci solitons on almost Kenmotsu manifolds was started by the Wang and Liu [23] and explicitly studied by Wang (see [21, 22]). Motivated by the above results we study the Ricci solitons and Ricci almost solitons on para-Kenmotsu manifolds.

This paper is organized as follows. In Section 2, the basic information about paracontact metric manifolds and para-Kenmotsu manifolds are given. In Section 3, we consider Ricci solitons on para-Kenmotsu manifold and prove that if a para-Kenmotsu metric \( g \) represents a Ricci soliton where the soliton vector field \( V \) is contact, then it becomes a shrinking soliton which is Einstein. In Section 4, we prove that if a para-Kenmotsu metric \( g \) represents a gradient Ricci almost soliton, then it is \( \eta \)-Einstein. Also we prove this result for Ricci almost soliton with the potential vector field \( V \) is pointwise collinear with the Reeb vector field \( \xi \).

### 2. Notes on paracontact metric manifolds

In this section, we recall some information about paracontact metric manifolds. We refer to [3, 5, 6, 9, 15, 16, 25, 26] for more details as well as some examples. A \((2n + 1)\)-dimensional smooth manifold \( M^{2n+1} \) has an almost paracontact structure \((\varphi, \xi, \eta)\) if it admits a \((1, 1)\)-tensor field \( \varphi \), a vector field \( \xi \) and a 1-form \( \eta \) satisfying the following conditions:

\[
\varphi^2 = I - \eta \circ \xi, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1,
\]

and there exists a distribution \( \mathcal{D} : p \in M \to \mathcal{D}_p \subset T_pM : \mathcal{D}_p = \text{Ker}(\eta) = \{ x \in T_pM : \eta(x) = 0 \} \), called paracontact distribution generated by \( \eta \). If an
almost paracontact manifold \( M^{2n+1} \) with a structure \((\varphi, \xi, \eta)\) admits a pseudo-Riemannian metric \( g \) such that

\[
g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y)
\]

for all vector fields \( X, Y \) on \( M \), then we say that \( M \) has an almost paracontact metric structure and \( g \) is called a compatible metric. The fundamental 2-form \( \Phi \) of an almost paracontact metric structure \((\varphi, \xi, \eta, g)\) defined by \( \Phi(X, Y) = g(X, \varphi Y) \) for all vector fields \( X, Y \) on \( M \). If \( \Phi = d\eta \), then the manifold \( M^{2n+1}(\varphi, \xi, \eta, g) \) is called a paracontact metric manifold. In this case, \( \eta \) is a contact form, i.e., \( \eta \wedge (d\eta)^n \neq 0 \), \( \xi \) is its Reeb vector field and \( M \) is a contact manifold (see [6, 16, 18]). An almost paracontact metric manifold is said to be para-Kenmotsu manifold if

\[
(\nabla_X \varphi)Y = \eta(Y)\varphi X + g(X, \varphi Y)\xi
\]

for all vector fields \( X, Y \) on \( M \). On para-Kenmotsu manifold [25]:

\[
\nabla X \xi = -X + \eta(X)\xi,
\]

\[
R(X, Y)\xi = \eta(X)Y - \eta(Y)X,
\]

\[
Q \xi = -2n\xi,
\]

for all vector fields \( X, Y \) on \( M \), where \( \nabla \) is the operator of covariant differentiation of \( g \) and \( Q \) denotes the Ricci operator associated with the Ricci tensor given by \( Rie(X, Y) = g(QX, Y) \) for all vector fields \( X, Y \) on \( M \).

### 3. Para-Kenmotsu metric as a Ricci soliton

In this section, we study the Ricci Solitons on para-Kenmotsu manifold. First we recall the following.

**Lemma 3.1.** Let \( M^{2n+1}(\varphi, \xi, \eta, g) \) be a para-Kenmotsu manifold. Then we have

\[
R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X,
\]

\[
(\nabla_X \eta)Y = -g(X, Y) + \eta(X)\eta(Y),
\]

\[
(\ell_\xi g)(Y, Z) = -2\{g(Y, Z) - \eta(Y)\eta(Z)\},
\]

for all vector fields \( Y, Z \) on \( M \).

We can prove Lemma 3.1 by simple routine calculation. Using these results now we prove the following lemma for later use.

**Lemma 3.2.** Let \( M^{2n+1}(\varphi, \xi, \eta, g) \) be a para-Kenmotsu manifold. Then we have

\[
(\ell_\xi Q)Y = 2QY + 4nY = (\nabla_\xi Q)Y
\]

for all vector fields \( Y \) on \( M \).
Proof. First taking the covariant derivative of (3.3) along an arbitrary vector field $X$ on $M$ and using (3.2) we obtain

$$(3.5) \quad (\nabla_X \xi) g(Y, Z) = -2\{g(X, Y)\eta(Z) + g(X, Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)\}$$

for all vector fields $X, Y, Z$ on $M$. Now, we recalling the following commutation formula (see Yano [24], p. 23):

$$(3.6) \quad (\mathcal{L}_Y \nabla g - \nabla Z \xi g - \nabla [V, Z] g)(X, Y) = -g((\mathcal{L}_Y \nabla)(Z, X), Y)$$

for all vector fields $X, Y, Z$ on $M$. By virtue of parallelism of the pseudo-Riemannian metric $g$, this formula reduces to

$$(3.7) \quad (\mathcal{L}_Y \nabla)(Y, Z) = 2\{\eta(Y)\eta(Z)\xi - g(Y, Z)\xi\}$$

for all vector fields $Y, Z$ on $M$. Taking covariant differentiation of (3.7) along $X$ and using (2.2), we find

$$(\nabla_X \mathcal{L}_Y)(Y, Z) = -2\{g(X, Y)\eta(Z)\xi + g(X, Z)\eta(Y)\xi + g(Z, X)\eta(Y)\xi - g(Y, Z)\eta(X)\eta(Y)\eta(Z)\xi$$

for all vector fields $Y, Z$ on $M$. Using this in the following commutation formula (see Yano [24], p. 23)

$$(3.8) \quad (\mathcal{L}_Y R)(X, Y) Z = (\nabla_X \mathcal{L}_Y \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_Y \nabla)(X, Z),$$

we can compute

$$(\mathcal{L}_Y R)(X, Y) Z = -2\{g(X, Z)Y - g(Y, Z)X + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}$$

for all vector fields $X, Y, Z$ on $M$. Now, contracting the foregoing Eq. over $X$, we find

$$(3.9) \quad (\mathcal{L}_Y \text{Ric})(Y, Z) = -4n\{\eta(Y)\eta(Z) - g(Y, Z)\}$$

for all vector fields $Y, Z$ on $M$. On the other hand, taking Lie derivative of $\text{Ric}(Y, Z) = g(QY, Z)$ with respect to $\xi$, we get

$$(3.10) \quad (\mathcal{L}_Y \text{Ric})(Y, Z) = (\mathcal{L}_Y g)(QY, Z) + g((\mathcal{L}_Y Q)Y, Z)$$

for all vector fields $Y, Z$ on $M$. Now, replacing $Y$ by $QY$ in (3.3) and using (2.6), we find

$$(3.11) \quad (\mathcal{L}_Y g)(QY, Z) = -2\{g(QY, Z) + 2n\eta(Y)\eta(Z)\}$$
for all vector fields $Y$, $Z$ on $M$. By virtue of (3.11) and (3.10), Eq. (3.9) reduces to $(\mathcal{L}_\xi Q)Y = 2QY + 4nY$ for all vector fields $Y$ on $M$. Further, it is well known that
\begin{align*}
(\mathcal{L}_\xi Q)Y & = \nabla_\xi QY - \nabla_Q Y \xi - Q(\nabla_\xi Y) + Q\nabla_Y \xi \\
& = (\nabla_\xi Q)Y - \nabla_Q Y \xi + Q\nabla_Y \xi
\end{align*}
for all vector fields $Y$ on $M$. Thus, by virtue of (2.4) and (2.6) we see that $(\mathcal{L}_\xi Q)Y = (\nabla_\xi Q)Y$ for all vector fields $Y$ on $M$. This completes the proof. □

Now we consider a para-Kenmotsu metric as a Ricci soliton where the soliton vector field $V$ is contact and proof the following.

**Theorem 3.1.** Let $M^{2n+1}(\varphi, \xi, \eta, g)$, $n > 1$, be a para-Kenmotsu manifold. If $g$ represents a Ricci soliton, then the soliton is shrinking. Further, if the soliton vector field $V$ is contact, then $V$ is strict and $g$ is Einstein with Einstein constant $-2n$.

**Proof.** First, from (2.4) we get $R(X, \xi) \xi = -X + \eta(X) \xi$ and the Lie derivative of this along $V$ provides
\begin{equation}
(\mathcal{L}_V R)(X, \xi) \xi + R(X, \mathcal{L}_V \xi) \xi + R(X, \xi) \mathcal{L}_V \xi
\end{equation}
for all vector fields $X$ on $M$. Now, taking covariant derivative of (1.1) along an arbitrary vector field $Z$ on $M$ and using (3.6) we have
\begin{equation}
g((\mathcal{L}_V \nabla)(Z, X), Y) + g((\mathcal{L}_V \nabla)(Z, Y), X) = -2(\nabla_Z \text{Ric})(X, Y)
\end{equation}
for all vector fields $X$, $Y$, $Z$ on $M$. By a straightforward combinatorial combination of the last equation one can deduce
\begin{equation}
g((\mathcal{L}_V \nabla)(X, Y), Z) = (\nabla_Z \text{Ric})(X, Y) - (\nabla_X \text{Ric})(Y, Z)
\end{equation}
(3.13)
for all vector fields $X$, $Y$, $Z$ on $M$. Next, differentiating (2.6) along an arbitrary vector field $X$ on $M$ and recalling (2.2) we get
\begin{equation}
(\nabla_X Q) \xi = QX + 2nX
\end{equation}
(3.14)
for all vector fields $X$ on $M$. Taking into account of this, (3.4) and replacing $Y$ by $\xi$ in (3.13) we deduce
\begin{equation}
(\mathcal{L}_V \nabla)(X, \xi) = -2QX - 4nX
\end{equation}
(3.15)
for all vector fields $X$ on $M$. Taking covariant derivative of (3.15) along $Y$ and using (2.2), (3.15) we obtain
\begin{equation}
(\nabla_Y \mathcal{L}_V \nabla)(X, \xi) - (\mathcal{L}_V \nabla)(X, Y) - 2\eta(Y)(QX + 2nX) = -2(\nabla_Y Q)X
\end{equation}
for all vector fields $X$ on $M$. Making use of this in (3.8) yields
\begin{equation}
(\mathcal{L}_V R)(X, Y) \xi = 2[\eta(X)QY - \eta(Y)QX + 2n\{\eta(X)Y - \eta(Y)X\}]
\end{equation}
\[(3.16) \quad - \{[\nabla_X Q] Y - (\nabla_Y Q) X]\]

for all vector fields \(X, Y\) on \(M\). Now, replacing \(Y\) by \(\xi\) in (3.16) and using (3.14) and (3.4) we have \((\mathcal{L}_V R)(X, \xi) \xi = 0\). Making use of this along with (2.4), (3.1) in (3.12), one can deduce
\[
(3.17) \quad g(X, \mathcal{L}_V \xi) - 2\eta(\mathcal{L}_V \xi) X = \{(\mathcal{L}_V \eta) X\} \xi
\]
for all vector fields \(X\) on \(M\). Next, taking into account (1.1), (2.6) in the Lie differentiation \(g(\xi, \xi) = 1\) along \(V\) leads to
\[
(3.18) \quad \eta(\mathcal{L}_V \xi) = \lambda - 2n.
\]
Further, by virtue of (2.6), the soliton equation (1.1) reduces to
\[
(3.19) \quad (\mathcal{L}_V \eta) X = g(X, \mathcal{L}_V \xi) - 2(\lambda - 2n) \eta(X)
\]
for all vector fields \(X\) on \(M\). By the help of (3.19) and (3.18), Eq. (3.17) provides \(\lambda = 2n\) and therefore the soliton is shrinking. Further, Eq. (3.18) together with (3.19) yields
\[
(3.20) \quad \mathcal{L}_V \xi = 0 = \mathcal{L}_V \eta.
\]
Also by our assumption, \(V\) is a contact vector field, i.e., \(\mathcal{L}_V \xi = f \xi\). Making use of this in (3.18) gives \(f = \lambda - 2n\) and therefore \(f = 0\). Thus, \(\mathcal{L}_V \xi = 0\), and hence \(V\) is strict. Now, recall the well known formula (see [24, p. 23]):
\[
(3.21) \quad \mathcal{L}_V \nabla_X Y - \nabla_X \mathcal{L}_V Y - \nabla_{[X,Y]} Y = (\mathcal{L}_V \nabla)(X, Y)
\]
for all vector fields \(X, Y, V\) on \(M\). Next, taking \(\xi\) instead of \(Y\) in the preceding equation and using (3.20) we get
\[
(\mathcal{L}_V \nabla)(X, \xi) = \mathcal{L}_V \nabla_X \xi - \mathcal{L}_V X + \eta(\mathcal{L}_V X)
\]
for all vector fields \(X, V\) on \(M\). Taking into account (2.2) and (3.20), the last equation provides \((\mathcal{L}_V \nabla)(X, \xi) = 0\) for all vector fields \(X, V\) on \(M\). By virtue of this, Eq. (3.15) proves that \(g\) is Einstein. This completes the proof. \(\square\)

A pseudo-Riemannian manifold is called \(\eta\)-Einstein, if the Ricci tensor \(\text{Ric}\) is of the form
\[
(3.22) \quad \text{Ric} = a g + b \eta \otimes \eta,
\]
where \(a, b\) are smooth functions on \(M\). For a para-Sasakian manifold of dimension > 3, the functions \(a, b\) are constant (see [25]).

Lemma 3.3. Let \(M^{2n+1}(\varphi, \xi, \eta, g)\) be a para-Kenmotsu manifold. If \(M\) is an \(\eta\)-Einstein manifold, we have
\[
(3.23) \quad \text{Ric}(Y, Z) = (1 + \frac{r}{2n}) g(Y, Z) - \{(2n + 1) + \frac{r}{2n}\} \eta(Y) \eta(Z)
\]
for all vector fields \(Y, Z\) on \(M\).

Proof. Equations (3.22) and (2.6) gives \(r = (2n + 1)a + b\) and \(a + b = -2n\). Thus, we have \(a = 1 + \frac{r}{2n}\) and \(b = -(2n + 1) + \frac{r}{2n}\). Then the Eq. (3.22) can be written as the required form. This completes the proof. \(\square\)
Theorem 3.2. Let $M^{2n+1}(\varphi, \xi, \eta, g)$, $n > 1$, be a $\eta$-Einstein para-Kenmotsu manifold. If $g$ represents a Ricci soliton, then the soliton is shrinking and $g$ is Einstein with constant scalar curvature $r = -2n(2n + 1)$.

Proof. By the help of (3.23), the soliton Eq. (1.1) becomes

$$\mathcal{L}_Y g(Y, Z) = \left\{2(\lambda - 1) - \frac{r}{n}\right\} g(Y, Z) + \left\{2(2n + 1) + \frac{r}{n}\right\} \eta(Y) \eta(Z)$$

for all vector fields $Y, Z$ on $M$. Differentiating this along an arbitrary vector field $X$ on $M$ and using (3.24), (3.26) we have

$$\mathcal{L}_Y \mathcal{L}_Y (Z, X), Y) + g((\mathcal{L}_Y \mathcal{L}_Y)(Z, Y), X)$$

for all vector fields $Y, Z$ on $M$. By straightforward combinatorial computation of the last equation provides

$$2ng((\mathcal{L}_Y \mathcal{L}_Y)(X, Y), Z) = -\left\{\eta(Y)\eta(Z) - \{2(2n + 1) + \frac{r}{n}\}\right\} g(X, Y)\eta(Z)$$

for all vector fields $X, Y, Z$ on $M$. Consider a local orthonormal basis $\{e_i : i = 1, 2, \ldots, 2n + 1\}$ of tangent space at each point of $M$. Next, setting $X = Z = e_i$ in (3.32) and summing over $i : 1 \leq i \leq 2n + 1$, we have $(\mathcal{L}_Y \mathcal{L}_Y)(e_i, e_i) = 0$. Now, putting $X = Y = e_i$ in (3.26) gives

$$(\xi_r)\eta(Z) + (n - 1)(Zr) = 2n\{2n(2n + 1) + r\}\eta(Z)$$

for all vector fields $Z$ on $M$. Taking $Z = \xi$ in the last equation we get $(\xi_r) = 2\{2n(2n + 1) + r\}$. By virtue of this, Eq. (3.27) yields $Dr = (\xi_r)\xi$. Next, substituting $X$ by $\xi$ in (3.26) we obtain

$$2n(\mathcal{L}_Y \mathcal{L}_Y)(\xi, Y) = -((\xi_r)\{Y - \eta(Y)\})$$

for all vector fields $Y$ on $M$. Taking covariant derivative of this along $X$ and using (2.6) and (3.15) we get

$$2n(\nabla_X \mathcal{L}_Y \mathcal{L}_Y)(\xi, Y) = 2n(\mathcal{L}_Y \mathcal{L}_Y)(X, Y) - \xi_r\{Y - \eta(Y)\}\xi$$

for all vector fields $Y$ on $M$. Next, interchanging $X, Y$ in (3.29) and using the well known formula (see [24, p. 23]):

$$(\mathcal{L}_Y R)(X, Y, Z) = (\nabla_X \mathcal{L}_Y \mathcal{L}_Y)(Y, Z) - (\nabla_Y \mathcal{L}_Y \mathcal{L}_Y)(X, Z),$$

it follows that

$$2n(\mathcal{L}_Y R)(X, Y)\xi = Y(\xi_r)\{X - \eta(X)\} - X(\xi_r)\{Y - \eta(Y)\}$$
−2(ξr){η(Y)X − η(X)Y}

for all vector fields $X, Y$ on $M$. Contracting this over $X$ we have $(£_V \text{Ric})(Y, \xi) = 0$, where we use $Dr = (ξr)ξ$. Further, using (3.23), (3.30) in the Lie derivative of $\text{Ric}(Y, \xi) = −2nη(Y)$ along $V$ yields

\[(1 + \frac{r}{2n})g(Y, £_V \xi) − \{(2n + 1) + \frac{r}{2n}\}η(Y)η(£_V \xi)\]

(3.31)

\[= −4n(λ − 2n)η(Y) − 2ng(Y, £_V \xi)\]

for all vector fields $Y$ on $M$. Taking $Y = \xi$ in the last equation we get $λ = 2n$ and therefore the soliton is shrinking. Again, setting $Y = Z = \xi$ in (3.24) we obtain $η(£_V \xi) = 0$. Using this in (3.31) yields

\[\{r + 2n(2n + 1)\}£_V \xi = 0.\]

(3.32)

Suppose $r \neq −2n(2n + 1)$ on some open set $O$ of $M$. Then from (3.32) it follows that $£_V \xi = 0$. Thus, from (2.4) we deduce that $∇_V V = V − η(V)ξ$. Using this, (2.4), (2.5) and (3.28) in the identity (see [24, p. 39]):

\[(£_V ∇)(X, Y) = ∇_X ∇_Y V − ∇_X η(V) − R(V, X)Y,\]

we obtain $ξr = 0$. As $Dr = (ξr)ξ$, so the scalar curvature $r$ is constant. This shows from (3.27) that $r = −2n(2n + 1)$ on $O$, which is a contradiction on $O$. Thus, Eq. (3.32) gives $r = −2n(2n + 1)$ and therefore we can conclude from (3.23) that $M$ is Einstein. This completes the proof.

4. Para-Kenmotsu metric as a Ricci almost soliton

In this section, we study the Ricci almost solitons on para-Kenmotsu manifold. First, we consider a para-Kenmotsu metric as a gradient Ricci almost soliton. Thus, the equation (1.1) and (1.2) holds for a smooth function $λ$.

Theorem 4.1. Let $M^{2n+1}(φ, ξ, η, g)$ be a Kenmotsu manifold. If $g$ represents a gradient Ricci almost soliton, then it is $η$-Einstein. Moreover, if the Reeb vector field $ξ$ leaves the scalar curvature $r$ invariant, then $g$ is Einstein with constant scalar curvature $−2n(2n + 1)$.

Proof. Making use of (1.2) in the well known expression of the curvature tensor $R(X, Y) = [∇_X, ∇_Y] − ∇_[X, Y]$, one can obtain

\[R(X, Y)Df = (∇_Y Q)X − (∇_X Q)Y + (Xλ)Y − (Yλ)X\]

(4.1)

for all vector fields $X, Y$ on $M$. Now, replacing $Y$ by $ξ$ in (4.1) and using (3.4) and (3.14) we deduce

\[R(X, ξ)Df = QX + 2nX + (Xλ)ξ − (ξλ)X\]

for all vector fields $X$ on $M$. By virtue of (3.1), the preceding equation reduces to

\[g(X, Df − Dλ)ξ = QX + 2nX + {[(ξf) − (ξλ)]}X\]

(4.2)
for all vector fields $X$ on $M$. Taking scalar product of (4.2) with $\xi$ and using (2.6) yields

\[(4.3) \quad Df - D\lambda = \{(\xi f) - (\xi \lambda)\}\xi.\]

Using this in (4.2) we have

\[(4.4) \quad \text{Ric}(X,Y) = -(2n + (\xi f) - (\xi \lambda))g(X,Y) + \{(\xi f) - (\xi \lambda)\}\eta(X)\eta(Y)\]

for all vector fields $X, Y$ on $M$. Consider a local orthonormal basis $\{e_i : i = 1, 2, \ldots, 2n + 1\}$ of tangent space at each point of $M$. Next, taking the inner product of (4.1) with $Z$ and then setting $X = Z = e_i$ and summing over $i : 1 \leq i \leq 2n + 1$, we have

\[(4.5) \quad \text{Ric}(Y, D\lambda) = \{\sum_{i=1}^{2n+1} g((\nabla_{\nabla Y}e_i, e_i) - (\nabla_{e_i} Y)e_i)\} - 2n(Y\lambda)\]

for all vector fields $Y$ on $M$. Contraction of Bianchi’s second identity gives $\text{div} Q = \frac{1}{2} D\lambda$ and therefore Eq. (4.5) yields

\[(4.6) \quad \text{Ric}(Y, D\lambda) = \frac{1}{2} Yr - 2nY\lambda\]

for all vector fields $Y$ on $M$. Thus, $M$ is $\eta$-Einstein. Moreover, if $\xi$ leaves the scalar curvature $r$ invariant, i.e., $\xi r = 0$ and therefore, $r = -2n(2n + 1)$. This transform the Eq. (4.7) into $\text{Ric} = -2ng$, i.e., $g$ is Einstein. This complete the proof. $\square$

Next, we extend the above Theorem from gradient Ricci almost soliton to Ricci almost soliton and consider para-Kenmotsu metric as a Ricci almost soliton and the potential vector field $V$ is pointwise collinear with the Reeb vector field $\xi$ and prove:

**Theorem 4.2.** Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a para-Kenmotsu manifold. If $g$ represents a non-trivial Ricci almost soliton such that the potential vector field $V$ is pointwise collinear with the Reeb vector field $\xi$, then it is $\eta$-Einstein.

**Proof.** By hypothesis: $V = \rho \xi$ for some smooth function $\rho$ on $M$. Taking covariant derivative of this along an arbitrary vector field $X$ on $M$ and using (2.2) provides

\[(4.8) \quad \nabla_X V = (X\rho)\xi - \rho(X + \eta(X))\xi.\]

Then the soliton equation (2.1) reduces to

\[(4.9) \quad 2\text{Ric}(X,Y) = 2(\rho - \lambda)g(X,Y) - (X\rho)\eta(Y) - (Y\rho)\eta(X) - 2\rho\eta(X)\eta(Y)\]
for all vector fields $X, Y$ on $M$. Now, replacing $X = Y = \xi$ in the foregoing equation and using (2.6), we have $\xi \rho = 2n - \lambda$. Taking into account of this, (2.6) and putting $Y = \xi$ in (4.9) gives $X \rho = (2n - \lambda) \eta(X)$. using this in (4.9), we have

\begin{equation}
Ric(X, Y) = (\rho - \lambda) g(X, Y) - (2n + \rho - \lambda) \eta(X) \eta(Y)
\end{equation}

for all vector fields $X, Y$ on $M$. Tracing the preceding equation gives

\begin{equation}
\rho - \lambda = \frac{r}{2n} + 1.
\end{equation}

This transform the Eq. (4.10) into

\begin{equation}
Ric(X, Y) = \left(\frac{r}{2n} + 1\right) g(X, Y) - \left(\frac{r}{2n} + 2n + 1\right) \eta(X) \eta(Y)
\end{equation}

for all vector fields $X, Y$ on $M$. This implies that $M$ is $\eta$-Einstein. This complete the proof. \hfill \Box

Next, if we take $\rho$ a constant instead of a function, then from $X \rho = (2n - \lambda) \eta(X)$, we have $\lambda = 2n$, which is constant. Thus from (4.11) follows that $\xi r = 0$. Again, tracing (3.4) gives $(\xi r) = 2(r + 2n(2n + 1))$. Hence $r = -2n(2n + 1)$. Making use of this in (4.11) we see that $\rho = 0$, and therefore from the soliton Eq. we conclude that $g$ is Einstein. Thus, we have the following.

**Corollary 4.1.** If a para-Kenmotsu metric $g$ represents a non-trivial Ricci almost soliton with $V = \rho \xi$ for some constant $\rho$, then it is Einstein with constant scalar curvature $r = -2n(2n + 1)$.

In particular, we can also say that if a para-Kennmotsu metric $g$ represents a non-trivial Ricci almost soliton where the potential vector field $V$ is $\xi$, then it is Einstein with constant scalar curvature $r = -2n(2n + 1)$.

**References**


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