The Admissible Multiperiod Mean Variance Portfolio Selection Problem with Cardinality Constraints

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ABSTRACT
Uncertain factors in financial markets make the prediction of future returns and risk of asset much difficult. In this paper, a model, assuming the admissible errors on expected returns and risks of assets, assisted in the multiperiod mean variance portfolio selection problem is built. The model considers transaction costs, upper bound on borrowing risk-free asset constraints, cardinality constraints and threshold constraints. Cardinality constraints limit the number of assets to be held in an efficient portfolio. At the same time, threshold constraints limit the amount of capital to be invested in each stock and prevent very small investments in any stock. Because of these limitations, the proposed model is a mixed integer dynamic optimization problem with path dependence. The forward dynamic programming method is designed to obtain the optimal portfolio strategy. Finally, to evaluate the model, our result of a meaningful example is compared to the terminal wealth under different constraints.

Keywords: Multiperiod Portfolio Selection, Admissible Return, Admissible Variance, Cardinality Constraints, The Forward Dynamic Programming Method

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1. INTRODUCTION

The portfolio selection problem decides the choice of limited capital to a number of potential assets according to a profitable investment strategy. The pioneering work of the portfolio selection problem is the concept of efficient set developed by Markowitz (1952). In his seminal work based on modern portfolio theory, Markowitz viewed portfolio selection as a mean-variance optimization problem with regard to two criteria: to maximize the expected return of a portfolio, and to minimize the variance of return. More formally, a desirable portfolio is defined to be a tradeoff between risk and expected return. It combines probability theory with optimization techniques to model the investment behavior under uncertainty. Mean variance model makes a basic assumption that the trend of asset markets in future can be simulated by current and before asset data. The mean and covariance are the most important indicators in portfolio selection problem; thus, the assumption means that is, the mean and covariance of assets in future is similar to the past one. Although we have to say that it is very hard to ensure the assumption is always correct in the real ever-changing asset markets, the assumption is much common in lots of works about this problem (see Yu et al., 2010).

The portfolio selection model based on fuzzy probabilities has proposed by Tanaka et al. (2000). The mean vector and covariance matrix in the model designed by Markowitz are respectively replaced by the fuzzy weighted average vector and covariance matrix. It can be regarded
as a natural extension of the model given by Markowitz because of the close relationship between probability theory and fuzzy probability. Its approach permits the incorporation of expert knowledge by means of a possibility grade, to reflect the similarity between the future state of asset markets and the state of previous periods. By using fuzzy approaches, the knowledge of the experts and the subjective opinions of the investors can be better integrated into a portfolio selection model. Zhang and Nie (2004), Zhang et al. (2006), and Zhang and Wang (2008) discussed the admissible efficient portfolio selection using the assumption that the expected return and risk of assets have admissible errors to reflect the uncertainty in real investment actions. Then, an analytic derivation of admissible efficient frontier is given, when short sales were not allowed on all risky assets. Dubois and Prade (1988) defined an interval-valued expectation of fuzzy numbers which is set as consonant random sets. They also showed that this expectation remains additive in the sense of addition of fuzzy numbers. Carlsson and Fullér (2001) employed possibility distributions by introducing lower and upper possibilistic mean values of fuzzy numbers. Huang (2008) proposed mean risk curve portfolio selection models. Li et al. (2010) proposed mean-variance-skewness fuzzy portfolio which can be solved by a genetic algorithm and fuzzy simulation. Carlsson et al. (2002) introduced a possibilistic approach to select portfolios with highest utility score under the assumption that the returns of assets are trapezoidal fuzzy numbers.

Many modifications are proposed to improve the accuracy of the Markowitz’s mean variance portfolio selection model, for example, limiting the number of assets to be held in an efficient portfolio (cardinality constraints) or prescribing lower and upper bounds on the fraction of the capital invested in each asset (threshold constraints). These improvements are based on real-world practices; however, it is clearly not desirable to administer a portfolio made up of a large number of assets because of transaction costs, complexity of management, or the policies of asset management companies. The model (often called cardinality constrained Markowitz model), and its variations have been fairly intensively studied in the last decade. Some researchers proposed exact solution methods (ie., Bienstock, 1996; Bertsimas and Shioda, 2009; Li et al., 2006; Shaw et al., 2008; Murray and Shek, 2012); Cesarone et al., 2013; Cui et al., 2013; Sun et al., 2013; Le Thi et al., 2009, 2014). Since exact solution methods were able to solve only a fraction of practically useful LAM models, many heuristic algorithms had also been proposed (ie., Fernández and Gómez, 2007; Ruiz-Torrubiano and Suarez, 2010; Anagnostopoulos and Mamanis, 2011; Woodside-Oriakhi et al., 2011; Deng et al., 2012). In these studies, it appears that the computational complexity of the solution given by the LAM (Limited Asset Markowitz) model is much greater than the one spent by the classical Markowitz model. Indeed, the standard Markowitz model is a convex quadratic programming problem, while the LAM model is a mixed integer quadratic programming problem which is a NP-hard problem. Clearly, the computational complex of LAM model is much improved.

Although the case of a long-term investment is of greater importance in practice, much less has been done in that area. Although it is heavily discussed in the recent literatures (see e.g., Li and Ng, 2000; Zhu et al., 2004; Brandt et al., 2006; Gülpinar and Rustem, 2007; Çelikyurt and Özekici, 2007; Calafiore, 2008; Yan et al., 2009, 2012; Yu et al., 2010, 2012; Wu and Li, 2012; Li and Li, 2012; Zhang et al., 2012, 2014; Liu et al., 2012, 2013; Zhang and Zhang, 2014; Bodnar et al., 2015). To the best of our knowledge, a closed-form solution is not available in the general case up to now. Closed-form solutions are only presented with the assumption of independence by Li and Ng (2000), Zhu et al. (2004), Yan et al. (2009, 2012), Yu et al. (2010, 2012), Wu and Li (2012), Li and Li (2012). For more general models, the solution is frequently determined by a numerical procedure (see e.g., van Binsbergen and Brandt, 2007; Mansini et al., 2007; Gülpinar and Rustem, 2007; Zhang et al., 2012, 2014; Liu et al., 2012, 2013; Zhang and Zhang, 2014; Köksalan and Şakar, 2014).

Since the return of asset is in a fuzzy uncertain economic environment and varies all the time, its expected return and risk cannot accurately be predicted. Based on this fact, it is reasonable to assume that the expected return and risk have admissible errors. This paper deals with the portfolio selection problem based on this assumption that the expected return and risk of asset have admissible errors to show the uncertainty in real investments. The contribution of this work is as follows. We propose a new admissible multiperiod mean variance portfolio selection with upper bound on borrowing risk-free asset constraints, transaction costs, threshold constraints and cardinality constraints. Cardinality constraints limit the number of assets held in an efficient portfolio. Threshold constraints limit the amount of capital to be invested in each stock; moreover, very small investments in any stock are prevented according to its strategy. Then, a novel forward dynamic programming method is employed to find the solution. Finally, we give an important example to illustrate the idea of the model. More importantly, the results compared to other methods of the example demonstrate the effectiveness of the designed algorithm.

This paper is organized as follows. In Section 2, we present the admissible efficient multiperiod portfolio model and define the upper and lower admissible efficient multiperiod portfolio frontiers by the spreads of the portfolio expected returns and risks from the upper and lower bounds of admissible errors. Transaction costs, upper
bound on borrowing constraints, the threshold constraints and cardinality constraints are formulated into the multi-period portfolio. A new admissible multiperiod portfolio selection model is constructed. A forward dynamic programming method is proposed to solve it in Section 3. In Section 4, A numerical example is given to illustrate our proposed effective means and approaches, and the terminal wealth under different constraints are compared in Section 4, Finally, some conclusions are given in Section 5.

2. THE ADMISSIBLE MULTIPERIOD PORTFOLIO SELECTION MODEL

In this paper, we take a very interest assumption that there are $n$ risky assets and one risk-free asset in the financial market. An investor allocates his initial wealth $W_t$ among $n+1$ assets at the beginning of time period 1, and obtains the terminal wealth at the end of time period $T$. The total process of this investment last from the beginning of period 1 to the end of period $T$. The investor can modification the choice of the $n$ risky assets at the beginning of each period during the process of investment. Suppose that the returns of portfolios among different periods are independent of each other. In order to describe conveniently, we use the following notations:

- $x_{it}$ the investment proportion of risky asset $i$ at period $t$;
- $x_{it}$ the initial investment proportion of risky asset $i$ at period 1;
- $x_t$ the portfolio at period $t$,
  where $x_t=(x_{1t}, x_{2t}, \ldots, x_{nt})'$;
- $x_{ft}$ the investment proportion of risk-free asset at period $t$, where $x_{ft}=1-\sum_{i=1}^{n} x_{it}$;
- $x^b$ upper bound on borrowing risk-free asset at period $t$,
  where $x^b \geq x^b_t , x^b_t \leq 0$;
- $R_{it}$ the random return of risky asset $i$ at period $t$;
- $r_{it}$ the expected return of $R_{it}$,
  where $r_i=(r_{1i}, r_{2i}, \ldots, r_{ni})$;
- $\sigma_{ij}$ the covariance of $R_{ij}$ and $R_{ij}$;
- $r_{ft}$ the expected return rate of the portfolio $x_t$ at period $t$;
- $r_{nt}$ the net return rate of the portfolio $x_t$ at period $t$;
- $u_{it}$ the upper bound constraints of $x_{it}$;
- $l_{it}$ the lower bound constraints of $x_{it}$;
- $W_t$ the holding wealth at the beginning of period $t$;
- $r_{it}$ the borrowing/lending rate of the risk-free asset at period $t$;
- $K$ the desired number of assets in the portfolio at period $t$.

where prime (‘) denotes matrix transposition and all non-primed vectors are column vectors.

Let the borrowing rate of the risk-free asset at period $t$ be larger than the lending rate of the risk-free asset, i.e. $r_{it}\geq r_{it}$. If the borrowing is allowed on the risk-free asset, then the expected return associated with portfolio $x_t$ are given as follows:

$$ r_{it} = \sum_{i=1}^{n} r_i x_{it} + r_{ft}(1-\sum_{i=1}^{n} x_{it}), t=1, \ldots, T $$

(1)

where 

$$ r_{it} = \begin{cases} r_{it}, & 1-\sum_{i=1}^{n} x_{it} \geq 0, \\ r_{it}, & 1-\sum_{i=1}^{n} x_{it} \leq 0, \end{cases} $$

means that lending of the risk-free asset is allowed; otherwise, it represents the borrowing from the risk-free asset.

The variance of the portfolio $x_t$ can be expressed as

$$ V(x_t) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{it} \sigma_{ij} x_{jt} $$

(2)

Since the economic environment is uncertain and $r_{it}$, $I=1, \ldots, n$ varies in the process of investment, the future states of $n$ risky assets returns cannot be predicted accurately. We extend the admissible average returns and covariances of singleperiod portfolio selection in Zhang and Nie (2004), Zhang and Wang (2006), and Zhang and Wang (2008) to multiperiod portfolio selection. Thus, the admissible average returns and covariances at period $t$ are, respectively, defined as

$$ \bar{\sigma}_{ii} = \sigma_{ii} + \epsilon_{ii}, \quad \sigma_{ij} = \epsilon_{ij}, \quad i = 1, \ldots, n, \quad t = 1, \ldots, T $$

(3)

and

$$ \overline{\sigma}_{ij} = \sigma_{ij} + \epsilon_{ij}, \quad \epsilon_{ij} \leq \epsilon_{ij} \leq \epsilon_{ij}, \quad i = 1, \ldots, n, \quad t = 1, \ldots, T $$

(4)

where $\phi'$ denotes the admissible error for $r_{it}$, $\phi_i'$ expresses the lower bound of $\phi_i'$, $\phi_i'^{+}$ is the upper bound of $\phi_i'$; $\epsilon_i'$ defines the admissible error for $\sigma_{ii}'$; $\epsilon_{ii}'$ indicates the lower bound of $\sigma_{ii}'$; $\epsilon_{ii}'$ shows the upper bound of $\sigma_{ii}'$. $\phi_i'$, $\phi_i'^{+}$, $\epsilon_i'$ and $\epsilon_{ii}'$ can be estimated by combining the forecasted information of the assets return with the opinion of experts opinion. Correspondingly, the intervals $[r_{it} + \phi_i', r_{it} + \phi_i'^{+}]$ and $[\sigma_{ii}' + \epsilon_i', \sigma_{ii}' + \epsilon_{ii}']$ are determined. $\phi_i'$ and $\epsilon_i'$ can be selected by an investor based on his attitude to return and risk.
The admissible average return vector can be defined by
\[ \overline{r}_t = r_t + \phi', \phi' \leq \phi' \leq \phi'_t, \quad t = 1, \ldots, T \]
(5)
where \( r_t = (r_{t_1}, r_{t_2}, \ldots, r_{t_n}) \), \( \phi = (\phi'_1, \phi'_2, \ldots, \phi'_t) \), and \( \phi'_t = (\phi'_1, \phi'_2, \ldots, \phi'_t) \).

Similarly, the admissible covariance matrix can be denoted by
\[ \overline{V}_t = V_t + \epsilon', \epsilon' \leq \epsilon' \leq \epsilon'_t, \quad t = 1, \ldots, T \]
(6)
where \( V_t = (e'_1, e'_2, \ldots, e'_n) \), \( \epsilon' = (\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_n) \) and \( \epsilon'_t = (\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_n) \).

The admissible expected value and variance of the return associated with portfolio \( x_t \) are respectively given by
\[ \overline{r}_{x_t} = (r_t + \phi') x_t + r_t (1 - e' x_t), \quad t = 1, \ldots, T \]
(7)
and
\[ \overline{V}_{x_t}(x_t) = x_t (V_t + \epsilon') x_t \]
(8)
where \( e = (1, 1, \ldots, 1) \).

For any \( x_t \leq 0 \), it follows that
\[ (r_t + \phi') x_t + r_t (1 - e' x_t) \leq \overline{r}_{x_t} \leq (r_t + \phi') x_t + r_t (1 - e' x_t) \]
and
\[ x_t (V_t + \epsilon') x_t \leq \overline{V}_{x_t}(x_t) \leq x_t (V_t + \epsilon') x_t \]

We assume in the sequel that the transaction costs at period \( t \) is a V shape function of difference between the \( r_t \)th period portfolio \( x_t = (x_{t_1}, x_{t_2}, \ldots, x_{t_n}) \) and the \( t-1 \)th period portfolio \( x_{t-1} = (x_{t-1_1}, x_{t-1_2}, \ldots, x_{t-1_n}) \). That’s to say, the transaction cost for asset \( i \) at period \( t \) can be expressed by
\[ C_i = c_i \left| x_{ti} - x_{(t-1)i} \right| \]
(9)

Hence, the total transaction costs of the portfolio \( x_t = (x_{t_1}, x_{t_2}, \ldots, x_{t_n}) \) at period \( t \) can be represented as
\[ C_t = \sum_{i=1}^n c_i \left| x_{ti} - x_{(t-1)i} \right|, \quad t = 1, \ldots, T \]
(10)

Thus, the admissible net return rate of the portfolio \( x_t \) at period \( t \) can be denoted as
\[ \overline{r}_{x_t} = (r_t + \phi') x_t + r_t (1 - e' x_t) - \sum_{i=1}^n c_i \left| x_{ti} - x_{(t-1)i} \right| \]
(11)

Then, the admissible holding wealth at the beginning of the period \( t \) can be written as
\[ W_{t+1} = W_t (1 + \overline{r}_{x_t}) \]
(12)

To formulate cardinality constraints into the multiperiod portfolio model, zero-one decision variables are added as:
\[ z_{it} = \begin{cases} 1 & \text{if any of asset } i \text{ of period } t \\ 0 & \text{otherwise} \end{cases} \quad (i = 1, \ldots, n; \quad t = 1, \ldots, T) \]
(13)

where \( \sum_{i=1}^n z_i \leq K \).

The threshold constraints of multiperiod portfolio selection can be expressed as
\[ l_t \leq x_{ti} \leq u_t \]
(14)

where \( l_t \) and \( u_t \) are respectively the lower and upper bounds constraints of \( x_{ti} \).

A rational investor wishes to not only maximize admissible net return but also minimize the admissible variance of the rate of return on a portfolio. Thus, it is very important to get a tradeoff of these two objectives. Let \( (1 - \theta) \) and \( \theta \) be the preference coefficients associated with criteria \( \overline{r}_{x_t} \) and \( \overline{V}_{x_t}(x_t) \) respectively. Then the investor attempts to maximize
\[ F_t(\overline{r}_{x_t}, \overline{V}_{x_t}(x_t)) = (1 - \theta) \left( 1 + \sum_{i=1}^n (r_t + \phi') x_{ti} + r_t (1 - e' x_{ti}) - \sum_{i=1}^n c_i \left| x_{ti} - x_{(t-1)i} \right| \right) \]
(15)

where \( \phi'_t \) denotes the admissible error for \( r_{ti} \), \( e'_t \) denotes the admissible error for \( e' \). Here the parameter \( \theta \) can be interpreted as the risk aversion factor of a investor. The greater the factor \( w \) is, the more risk aversion the investor has. In this paper, we assume that the investor is of risk aversion, i.e., \( 0 \leq \theta \leq 1 \).

Thus, we construct the following admissible multiperiod portfolio selection model with cardinality con-
tor makes his portfolio selection neither too optimistically
that the investor pessimistically estimates the return and
portfolio must not exceed the given value
straint (c) represents the desired number of assets in the
ly estimates the return and risk. The Model (18) means
rstion constraint; constraint (b) indicates the upper bound
on borrowing risk-free asset constraints at period
r, constraint (d) states threshold constraints of \( x_p \).

where constraint (a) denotes the wealth accumulation
constraint; constraint (b) indicates the upper bound
on borrowing risk-free asset constraints at period \( t \);
straint (c) represents the desired number of assets in the
portfolio must not exceed the given value \( K \); constraint (d)
states threshold constraints of \( x_p \).

If \( (\phi, \varepsilon) = (\phi^i, \varepsilon^i) \), then (16) can be rewrittren as:

\[
\max \sum_{t=1}^{T} (1 - \theta) \left[ 1 + i \sum_{t=1}^{T} \left( r_t + \phi^i \right) x_t + r_T (1 - \sum_{t=1}^{T} x_t) - \sum_{t=1}^{T} \varepsilon^i \left( x_t - x_{-t-1} \right) \right]
\]

\[
-W_r = (1 + i \sum_{t=1}^{T} \left( r_t + \phi^i \right) x_t + r_T (1 - \sum_{t=1}^{T} x_t) - \sum_{t=1}^{T} \varepsilon^i \left( x_t - x_{-t-1} \right) \right) W_r
\]

\[
\sum_{t=1}^{T} x_t \geq x_p^i
\]

\[
\sum_{t=1}^{T} \varepsilon_t \leq K, \varepsilon_t \in [0,1]
\]

\[
l_x \varepsilon_t \leq u_x \varepsilon_t, i = 1, \ldots, n, t = 1, \ldots, T
\]

If \( (\phi, \varepsilon) = (\phi^f, \varepsilon^f) \), then (16) can be rewrittren as:

\[
\max \sum_{t=1}^{T} (1 - \theta) \left[ 1 + i \sum_{t=1}^{T} \left( r_t + \phi^f \right) x_t + r_T (1 - \sum_{t=1}^{T} x_t) - \sum_{t=1}^{T} \varepsilon^f \left( x_t - x_{-t-1} \right) \right]
\]

\[
-W_r = (1 + i \sum_{t=1}^{T} \left( r_t + \phi^f \right) x_t + r_T (1 - \sum_{t=1}^{T} x_t) - \sum_{t=1}^{T} \varepsilon^f \left( x_t - x_{-t-1} \right) \right) W_r
\]

\[
\sum_{t=1}^{T} x_t \geq x_p^f
\]

\[
\sum_{t=1}^{T} \varepsilon_t \leq K, \varepsilon_t \in [0,1]
\]

\[
l_x \varepsilon_t \leq u_x \varepsilon_t, i = 1, \ldots, n, t = 1, \ldots, T
\]

The Model (17) means that the investor optimistically
estimates the return and risk. The Model (18) means
that the investor pessimistically estimates the return and
risk. The Model (16) covers the scenario when the investor
makes his portfolio selection neither too optimistically
nor too pessimistically.

**Definition 1.** The optimal solution of (16), \( x_t (r_t + \phi, \sigma^f \varepsilon) \)
(\( d^f \varepsilon \)) is called an admissible efficient portfolio. The optimal
solution of (17), \( x_t (r_t + \phi^i, \sigma^f \varepsilon) \) is called an upper
admissible efficient portfolio. The optimal solution of
(18), \( x_t (r_t + \phi^i, \sigma^f \varepsilon) \) is called a lower admissible efficient
portfolio.

For each admissible error couple \((\phi, \varepsilon)\) given by the
investor, an admissible efficient frontier can be obtained
by (16). If \( \phi = 0 \) and \( \varepsilon = 0 \), then the admissible efficient frontier is the classical efficient frontier. It is obvious that
the admissible portfolio selection model shown in (16) is
an extensions of the conventional multiperiod portfolio
selection models.

### 3. SOLUTION ALGORITHM

In this section we formulate a numerical solution to
(16) based on an essential assumptions. Our results about
(16) will require the following assumptions to be satisfied.

**Assumption 1.** (i) \( r_t + \phi \neq k_l \), for any \( k \in R, (\sigma^f \varepsilon) \) is a
semi-positive definite matrix.

Assumption 1 (i) is essential. Assumption 1 (ii) is easi-
sily satisfied by a proper selection of \( \varepsilon^f \).

Let \( y_a = x_{-t-1} - x_{-t-1} \). Then the Model (16) can be
turned into as follows:

\[
\max \sum_{t=1}^{T} (1 - \theta) \left[ 1 + i \sum_{t=1}^{T} \left( r_t + \phi^f \right) x_t + r_T (1 - \sum_{t=1}^{T} x_t) - \sum_{t=1}^{T} \varepsilon^f \left( x_t - x_{-t-1} \right) \right]
\]

\[
-W_r = (1 + i \sum_{t=1}^{T} \left( r_t + \phi^f \right) x_t + r_T (1 - \sum_{t=1}^{T} x_t) - \sum_{t=1}^{T} \varepsilon^f \left( x_t - x_{-t-1} \right) \right) W_r
\]

\[
\sum_{t=1}^{T} x_t \geq x_p^f
\]

\[
\sum_{t=1}^{T} \varepsilon_t \leq K, \varepsilon_t \in [0,1]
\]

\[
l_x \varepsilon_t \leq u_x \varepsilon_t, i = 1, \ldots, n, t = 1, \ldots, T
\]

where the Model (19) is the admissible multiperiod
portfolio selection.

In this section, the forward dynamic programming
method is proposed to solve the Model (19).

The sub-problem of period \( t \) of the Model (19) can be transformed into

\[
\max \left[ 1 + i \sum_{t=1}^{T} \left( r_t + \phi^f \right) x_t + r_T (1 - \sum_{t=1}^{T} x_t) - \sum_{t=1}^{T} \varepsilon^f \left( x_t - x_{-t-1} \right) \right]
\]

\[
-W_r = (1 + i \sum_{t=1}^{T} \left( r_t + \phi^f \right) x_t + r_T (1 - \sum_{t=1}^{T} x_t) - \sum_{t=1}^{T} \varepsilon^f \left( x_t - x_{-t-1} \right) \right) W_r
\]

\[
\sum_{t=1}^{T} x_t \geq x_p^f
\]

\[
\sum_{t=1}^{T} \varepsilon_t \leq K, \varepsilon_t \in [0,1]
\]

\[
l_x \varepsilon_t \leq u_x \varepsilon_t, i = 1, \ldots, n, t = 1, \ldots, T
\]
In the following section, we provide the detailed procedure of the forward dynamic programming method for finding optimal solutions of the Model (19). The procedure of the algorithm can be showed as follows:

Algorithm The forward dynamic programming method:
Step 1. When $t = 1$, $W_1$ and $x_0 = (x_{10}, \ldots, x_{0n})$ have been given, the sub-problem of period 1 of the Model (19) can be transformed into

$$
\begin{align*}
W_{1i} & = (1 + \sum_{j=1}^{m} (r_i + \phi_j)x_{1j} + r_j (1 - \sum_{j=1}^{m} x_{1j}) - \sum_{j=1}^{m} c_{ij}x_{1j})W_i \\
& - \theta \sum_{j=1}^{m} x_{1j} (\sigma_{ij} + \varepsilon_j) x_{1j}, \\
& 1 - \sum_{j=1}^{m} x_{1j} \geq x_{1i}^{*} \\
y_{1i} & = x_{1i} - x_{0i} \\
y_{1i}^- & = -(x_{1i} - x_{0i}) \\
\sum_{j=1}^{m} x_{1j} & \leq K, z_{1i} \in [0, 1] \\
l_{1i}z_{1i} & \leq u_{1i}z_{1i}, i = 1, \ldots, n
\end{align*}
$$

(20)

If the matrix $(\sigma_{ij}^{(m+1)} + \varepsilon_{ij}^{(m+1)})_{i,n}$ is a semi-positive definite matrix, the Model (21) is a cardinality constrained convex quadratic programming problem. The optimal solution of Model (22) can be obtained by branch-and-bound implementation with pivoting algorithms (Bertsimas and Shioda, 2009). At the same time,

$$
\begin{align*}
W_{m+1} & = 0 + \sum_{j=1}^{m} (r_i + \phi_j)x_{m+1j} + r_j (1 - \sum_{j=1}^{m} x_{m+1j}) - \sum_{j=1}^{m} c_{ij}x_{m+1j}W_i \\
& - \theta \sum_{j=1}^{m} x_{m+1j} (\sigma_{ij}^{(m+1)} + \varepsilon_j^{(m+1)}) x_{m+1j}, \\
& 1 - \sum_{j=1}^{m} x_{m+1j} \geq x_{1i}^{*} \\
y_{m+1i} & = x_{m+1i} - x_{0i} \\
y_{m+1i}^- & = -(x_{m+1i} - x_{0i}) \\
\sum_{j=1}^{m} x_{m+1j} & \leq K, z_{m+1i} \in [0, 1] \\
l_{m+1i}z_{m+1i} & \leq u_{m+1i}z_{m+1i}, i = 1, \ldots, n
\end{align*}
$$

(22)

and $W_{m+1}^{*}$ can be obtained, respectively.

Step 2. When $t = m$ ($m \geq 1$ and $m \in Z^{+}$), $W_{m+1}^{*}$ and $x_{m}^{*} = (x_{m1}^{*}, \ldots, x_{mn}^{*})$ have been obtained, the sub-problem of period $m$ of the Model (19) can be transformed into

$$
\begin{align*}
\max(1 - \theta) & \left[ 1 + \sum_{j=1}^{m} (r_i + \phi_j)x_{m+1j} + r_j (1 - \sum_{j=1}^{m} x_{m+1j}) - \sum_{j=1}^{m} c_{ij}x_{m+1j} \right] \\
& - \theta \sum_{j=1}^{m} x_{m+1j} (\sigma_{ij}^{(m+1)} + \varepsilon_j^{(m+1)}) x_{m+1j}, \\
1 - \sum_{j=1}^{m} x_{m+1j} & \geq x_{m+1i}^{*} \\
y_{m+1i} & = x_{m+1i} - x_{0i} \\
y_{m+1i}^- & = -(x_{m+1i} - x_{0i}) \\
\sum_{j=1}^{m} x_{m+1j} & \leq K, z_{m+1i} \in [0, 1] \\
l_{m+1i}z_{m+1i} & \leq u_{m+1i}z_{m+1i}, i = 1, \ldots, n
\end{align*}
$$

(21)

If the matrix $(\sigma_{ij}^{(m)} + \varepsilon_{ij}^{(m)})_{i,n}$ is a semi-positive definite matrix, the Model (21) is a cardinality constrained convex quadratic programming problem. The optimal solution of the Model (22) is a cardinality constrained convex quadratic programming problem. The optimal solution of the Model (22) can be obtained by branch-and-bound implementation with pivoting algorithms (Bertsimas and Shioda, 2009). At the same time,

$$
\begin{align*}
(1 - \theta) & \left[ 1 + \sum_{j=1}^{m} (r_i + \phi_j)x_{m+1j} + r_j (1 - \sum_{j=1}^{m} x_{m+1j}) - \sum_{j=1}^{m} c_{ij}x_{m+1j} \right] \\
& - \theta \sum_{j=1}^{m} x_{m+1j} (\sigma_{ij}^{(m)} + \varepsilon_j^{(m)}) x_{m+1j}, \\
1 - \sum_{j=1}^{m} x_{m+1j} & \geq x_{m+1i}^{*} \\
y_{m+1i} & = x_{m+1i} - x_{0i} \\
y_{m+1i}^- & = -(x_{m+1i} - x_{0i}) \\
\sum_{j=1}^{m} x_{m+1j} & \leq K, z_{m+1i} \in [0, 1] \\
l_{m+1i}z_{m+1i} & \leq u_{m+1i}z_{m+1i}, i = 1, \ldots, n
\end{align*}
$$

and $W_{m+1}^{*}$ can be obtained, respectively. Otherwise $t = m+1$, then turn Step 2.

At period $t$, the global optimal solutions of the Model (21) and Model (22), which are the sub-problem of the Model (19). can be obtained by branch-and-bound implementation with pivoting algorithms (Bertsimas and Shioda, 2009). So the global optimal solution of the Model (19) can also be known by the forward dynamic programming method. As a result, the global optimal solution of Model (16) can also be obtained.

The optimal solutions of (17) can be obtained as $(\phi', \varepsilon') = (\phi^{(m)}, \varepsilon^{(m)})$. The optimal solutions of (18) can be obtained as $(\phi', \varepsilon') = (\phi^{(m)}, \varepsilon^{(m)})$. Correspondingly, both the upper and lower admissible optimal solutions can also be indicated by the forward dynamic programming method.

4. NUMERICAL EXAMPLE

In this section, a numerical example is given to express the idea of the proposed model. Assume that an investor chooses twenty stocks from Shanghai Stock Exchange for his investment. The stocks codes are respectively $S_1$ (600036), $S_2$ (600002), $S_3$ (600060), $S_4$ (600362),
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The desired number of assets in the portfolio at period (600315), S18 (600518), S19 (600570), S20 (600880). He intends to make five periods of investment with initial wealth \( W_1 = 1 \) and his wealth can be adjusted at the beginning of each period. It is assumed that the returns, risk and turnover rates of the twenty stocks at each period are represented as trapezoidal fuzzy numbers. We collect historical data of them from April 2006 to March 2015 and set every three months as a period to handle the historical data.

The transaction costs of assets of the two periods take the same value \( c_{t} = 0.003 \) (\( t = 1, \ldots, 20; \tau = 1, \ldots, 5 \)), the desired number of assets in the portfolio \( \tau = 0, \ldots, 9 \) at period \( t, \tau = 1, \ldots, 5 \), upper bound on borrowing risk-free asset \( \kappa_{i}^{t} = -0.5 \), preference coefficient \( \theta = 0, 0.1, \ldots, 1 \), the borrowing rate of the risk-free asset \( r_{B} = 0.017 \), the lending rate of the risk-free asset \( r_{L} = 0.009, t = 1, \ldots, 5 \), the lower \( I_{i} = 0.05 \) and upper bound constraints \( u_{w} = 0.2 (i = 1, \ldots, 20; \tau = 1, \ldots, 5 \).

In this example, \( x_{0}, \phi_{i}, \phi_{i}^{v} \) and \( e_{i}^{v} \) are given by

\[
x_{0} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),
\]

\[
\phi_{i} = (0.025, 0.015, 0.01, 0.015, 0.025, 0.015, 0.045, 0.015, 0.045, 0.015, 0.015, 0.001, 0.001, 0.001, 0.015, 0.01, 0.01, 0.01, 0.01, 0.015, 0.015, 0.015, 0.015),
\]

\[
e_{i}^{v} = 0.0, \tau = 1, \ldots, 20; j = 1, \ldots, 20.
\]

By using the forward dynamic programming method to solve the Model (16), Model (17) and Model (18), the corresponding results can be obtained as follows.

If \( \phi_{i} = 0, e_{i}^{v} = 0, \theta = 0.5, K = 6 \), the optimal solution of the admissible multiperiod portfolio selection (Model (16)) is shown in the Table 1.

When \( \phi_{i} = 0, e_{i}^{v} = 0, \theta = 0.5, K = 6 \), the optimal investment strategy at period 1 is \( x_{11} = 0.2, x_{41} = 0.2, x_{51} = 0.2, x_{61} = 0.2, x_{71} = 0.1, x_{101} = 0.2, x_{11} = -0.2 \) and the rest of variables equal to zero, which means investor should allocate his initial wealth on asset 1, asset 4, asset 5, asset 6, asset 7, asset 10, risk-free asset and otherwise assets by the proportions of 20\%, 20\%, 30\%, 20\%, 20\%, -20\% and the rest of variables equal to zero, respectively.

In Table 1, the optimal investment strategies at period 2, period 3, period 4 and period 5 can also be obtained. In this case, the available terminal wealth is 1.9236.

If \( \phi_{i} = 0, e_{i}^{v} = 0, \theta = 0.5, K = 8 \), the optimal solution of Model (16) will be obtained as the Table 2. The available terminal wealth is 2.0611.

To display the influence of \( K \) on the optimal solution of multiperiod, its value is respectively set as 6 and 8, and the Model (16) for portfolio decision-making will be

<table>
<thead>
<tr>
<th>Asset 1</th>
<th>Asset 4</th>
<th>Asset 5</th>
<th>Asset 6</th>
<th>Asset 7</th>
<th>Asset 10</th>
<th>xf1</th>
<th>otherwise</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
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<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
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<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>-0.2</td>
<td>0</td>
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<tr>
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<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
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<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>-0.2</td>
<td>0</td>
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<tr>
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<table>
<thead>
<tr>
<th>Asset 1</th>
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<th>Asset 5</th>
<th>Asset 6</th>
<th>Asset 7</th>
<th>Asset 10</th>
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<td>0.2</td>
<td>0.1886</td>
<td>0.2</td>
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<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
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<td>0.2</td>
<td>0.1362</td>
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<td>0.2</td>
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<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.1139</td>
</tr>
</tbody>
</table>

Table 1. The optimal solution of Model (16) when \( \phi_{i} = 0, e_{i}^{v} = 0, \theta = 0.5, K = 6 \)

Table 2. The optimal solution when \( \phi_{i} = 0, e_{i}^{v} = 0, \theta = 0.5, K = 8 \)
used afterwards. After using forward dynamic programming method, the corresponding optimal investment strategies can be obtained as shown in Table 1 and Table 2. According to the results shown in Table 1 and Table 2, it can be seen that most of assets of the optimal solutions of $K = 6$ and $K = 8$ are the same. There are six assets of the same values in period 1, i.e. asset 1, asset 4, asset 5, asset 6, asset 7, asset 10.

If $\phi^i = (0.025, 0.015, 0.01, 0.015, 0.025, 0.015, 0.015, 0.015, 0.015, 0.025, 0.001, 0.001, 0.001, 0.015, 0.015, 0.015)$, $\phi^j = 0$, $\theta = 0.5$, $K = 6$, the optimal solution of the upper admissible multiperiod portfolio selection (Model (17)) will be obtained as the Table 4. Its available terminal wealth is $1.8107$. When $\theta = 0.5$, $K = 6$, the optimal terminal wealth of the lower admissible multiperiod portfolio selection (Model (18)) will be obtained as the Table 5. $W_6$ is the terminal wealth of the optimistically admissible multiperiod portfolio selection (the Model (17)). $LW_6$ is the terminal wealth of the pessimistically admissible multiperiod portfolio selection (the Model (18)).

Where $W_6$ is the terminal wealth of the admissible

<table>
<thead>
<tr>
<th>Table 3. The optimal solution of Model (17) when $\theta = 0.5, K = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
</tr>
<tr>
<td>-------</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
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<td>5</td>
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<table>
<thead>
<tr>
<th>Table 4. The optimal solution of Model (18) when $\theta = 0.5, K = 6$</th>
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</thead>
<tbody>
<tr>
<td>$t$</td>
</tr>
<tr>
<td>-------</td>
</tr>
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<td>2</td>
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</tr>
<tr>
<td>5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 5. The optimal terminal wealth of the Model (16-18) when $\theta = 0.5, K = 1, …, 9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
</tr>
<tr>
<td>-----</td>
</tr>
<tr>
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</tr>
<tr>
<td>1</td>
</tr>
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<td>2</td>
</tr>
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<td>3</td>
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<td>7</td>
</tr>
<tr>
<td>8</td>
</tr>
<tr>
<td>9</td>
</tr>
</tbody>
</table>
mutiperiod portfolio selection which $\phi_i^t = 0$, $\epsilon_{ij}^t = 0$. $UW_6$ is the terminal wealth of the optimistically admissible mutiperiod portfolio selection (the Model (17)). $LW_6$ is the terminal wealth of the pessimistically admissible mutiperiod portfolio selection (the Model (18)).

In the used data sets, the same optimal solutions at $K = 8$ is suitable for $K \geq 9$. The experiments in this paper correspond to the values of $K$ is the interval $[0, 9]$. It can be seen that, elaborated in Table 5, the terminal wealth also becomes larger, when the preset the desired number of assets in the portfolio become larger, this reflects the influence of the desired number of assets on portfolio selection.

In order to address the relationship between the $K$ and the terminal wealth of the Model (16), the Model (17) and the Model (18), Figure 1 based on the data in Table 5 is pointed as follow: $K-W_6$ is the admissible terminal wealth of the Model (16) with different $K$; $K-UW_6$ is the upper admissible terminal wealth of the Model (17) with different $K$; $K-LW_6$ is the lower admissible terminal wealth of the Model (18) with different $K$.

In Figure 1, with the same value of $K$ the terminal wealth of the upper admissible mutiperiod portfolio selection is larger than the terminal wealth of the lower admissible mutiperiod portfolio selection. For the terminal wealth under the conditions of $\phi_i^t \leq \phi_i^t \leq \phi_i^h$, and $\epsilon_{ij}^t \leq \epsilon_{ij}^t \leq \epsilon_{ij}^h$, the admissible mutiperiod portfolio selection is at middle position between upper and lower admissible mutiperiod portfolio selection.

When $K = 8$, $\theta = 0, 0.1, \ldots, \theta, 1$, the optimal terminal wealth of the Model (16), the Model (17) and the Model (18) can be respectively obtained as shown in Table 6. $W_6$ is the terminal wealth of the admissible mutiperiod portfolio selection which $\phi_i^t = 0$, $\epsilon_{ij}^t = 0$. $UW_6$ is the terminal wealth of the upper admissible mutiperiod portfolio selection (the Model (17)). $LW_6$ is the terminal wealth of the lower admissible mutiperiod portfolio selection (the Model (18)).

From Table 6, the Figure 2, which reflect the relationship between the preference coefficients $\theta$ and the terminal wealth of the Model (16), the Model (17) and the Model (18).

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_6$</td>
<td>2.0776</td>
<td>2.0771</td>
<td>2.0757</td>
<td>2.0732</td>
<td>2.0680</td>
<td>2.0611</td>
<td>2.0537</td>
<td>2.0396</td>
<td>2.0132</td>
<td>1.9498</td>
<td>1.0450</td>
</tr>
<tr>
<td>$UW_6$</td>
<td>2.2843</td>
<td>2.2843</td>
<td>2.2843</td>
<td>2.2838</td>
<td>2.2773</td>
<td>2.2579</td>
<td>2.2652</td>
<td>2.2495</td>
<td>2.2202</td>
<td>2.1371</td>
<td>1.0450</td>
</tr>
<tr>
<td>$LW_6$</td>
<td>1.9398</td>
<td>1.9398</td>
<td>1.9373</td>
<td>1.9351</td>
<td>1.9335</td>
<td>1.9261</td>
<td>1.9139</td>
<td>1.8961</td>
<td>1.8657</td>
<td>1.7972</td>
<td>1.0450</td>
</tr>
</tbody>
</table>

Figure 1. The relationship between the $K$ and the terminal wealth of the Model (16-18).

Figure 2. The relationship between the $\theta$ and the terminal wealth of the Model (16-18).
el (18), can be obtained as follows. PC is preference coefficients; PC-$W_6$ is the admissible terminal wealth of the Model (16) with different $\theta$; PC-$UW_6$ is the upper admissible terminal wealth of the Model (17) with different $\theta$; PC-$LW_6$ is the lower admissible terminal wealth of the Model (18) with different $\theta$.

In Figure 2, it can be seen that, if the $\theta$ has the same value same, the terminal wealth of the upper admissible multiperiod portfolio selection is above the terminal wealth of the lower admissible multiperiod portfolio selection. For $\phi^i_t \leq \phi^i_{t^*}$, $\epsilon^i_{t^*} \leq \epsilon^i_t \leq \epsilon^i_{t^*}$, the terminal wealth of the admissible multiperiod portfolio selection is the middle of the terminal wealth of upper and lower admissible multiperiod portfolio selection. The terminal wealth becomes smaller when preference coefficient $\theta$ becomes larger, which gives the influence of preference coefficient $\theta$ on portfolio selection.

5. CONCLUSIONS

In this paper, we have discussed the admissible multiperiod portfolio selection problem with transaction cost, borrowing constraints, threshold constraints and cardinality constraints. In the proposed model, the cardinality constraints are used to control the number of assets held in an efficient portfolio. Because of the transaction cost and cardinality constraints, the multiperiod portfolio selection is a mix integer dynamic optimization problem with path dependence. The forward dynamic programming method is designed to obtain the optimal portfolio strategy. Finally, an example using real data from the Shanghai Stock Exchange is given to illustrate the behavior of the proposed model and the designed algorithm.

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