A Sufficient Condition for the Feedback Quasilinearization of Control Mechanical Systems

Dong Eui Chang*, Seong-Ho Song** and Jeom Keun Kim†

Abstract – We derive a sufficient condition for feedback quasilinearizability of control mechanical systems and apply it to show the feedback quasilinearizability of the Acrobot system.

Keywords: Feedback quasilinearizability, Control mechanical system

1. Introduction

The equations of motion of a control mechanical system contain terms quadratically dependent on velocity that are usually called the Coriolis terms. Controller synthesis becomes tractable in the absence of these nonlinear terms [6], so it is useful to find a transformation that eliminates them from the equations of motion.

A control mechanical system is called quasilinearizable if there is a linear transformation of the velocity variables such that the Coriolis terms all vanish after the transformation. There has been active research on quasilinearization [2, 5, 7, 8], but there were obtained by the zero curvature condition or by some complicated PDE conditions. Then, very strong results were finally obtained in [4] where easily verifiable quasilinearizability conditions were derived.

In this paper we consider feedback transformations as well as state transformations, in order to increase possibility of removing the Coriolis terms from the dynamics. A control mechanical system is called feedback quasilinearizable if all Coriolis terms can be eliminated by a linear velocity transformation followed by a feedback transformation. We here obtain a sufficient condition for feedback quasilinearizability and apply it to prove the feedback quasilinearizability of the Acrobot system. We also derive a condition for partial quasilinearizability via a linear velocity transformation in the course of obtaining the result on feedback quasilinearizability.

2. Main Results

2.1 Review of quasilinearization theory

We review the theory of quasilinearization of mechanical systems in [4] from a slightly different viewpoint. We here use a linear bundle map from the tangent bundle $TQ$ of a given configuration space $Q$ to its cotangent bundle $T^*Q$ instead of a linear bundle map from $TQ$ to itself. This different style of presentation, however, does not affect the validity of the results in [4].

Let $Q$ be an $n$-dimensional manifold and $q = (q^i)$ a local coordinate system on $Q$; refer to [1], [3] for manifolds theory. Let $TQ$ and $T^*Q$ denote the tangent bundle and the cotangent bundle of $Q$, respectively. The natural pairing between $TQ$ and $T^*Q$ is denoted by $<,>$. The natural local coordinate bases of $TQ$ and $T^*Q$ are used.

\[ TQ = \text{span}\{\hat{e}_1, \ldots, \hat{e}_n\} \]

\[ T^*Q = \text{span}\{dq^1, \ldots, dq^n\}. \]

The symbol $\hat{e}_j$ is also used as the operator of partial differentiation with respect to $q^j$. We use the Einstein summation convention throughout this paper and the following convention for the ranges of various indices:

\[ i, j, k, \ell, r, s = 1, \ldots, n; \]

\[ a, b, c = 1, \ldots, p. \]

Consider a control mechanical system on the configuration space $Q$ with Lagrangian

\[ L(q, \dot{q}) = \frac{1}{2} m_j \dot{q}^j \dot{q}^j - V(q) \]

and $p$-dimensional control bundle $W \subset T^*Q$, where $m = (m_j)$ is the positive definite symmetric mass matrix and $V(q)$ is the potential energy of the system. Since our results are all local, we assume that $W$ is generated by $p$ independent 1-forms as follows:

\[ W = \text{span}\{W_1, \ldots, W_p\} \]

where each 1-form $W_a$, $a = 1, \ldots, p$, is written in coordinates as...
\[ W_a = W_{ia} dq^i. \]

The equations of motion of this control mechanical system are given by

\[ \ddot{q}^i + \Gamma^i_{jk} \dot{q}^j \dot{q}^k + m^{ij} \dot{\alpha}_j V = m^i W_{ja} u^a \]

for \( i = 1, \ldots, n \), where \( u = (u^a) \in \mathbb{R}^P \) is the control vector. Here, \( m^{ij} \) denotes the \((i, j)\) entry of the inverse matrix of \( m = (m_{ij}) \), and \( \Gamma^i_{jk} \) are the Christoffel symbols defined by

\[ \Gamma^i_{jk} = \frac{1}{2} m^{ki} \left( \frac{\partial m_{ij}}{\partial q^k} + \frac{\partial m_{jk}}{\partial q^i} - \frac{\partial m_{ik}}{\partial q^j} \right). \]

The quadratic terms \( \Gamma^i_{jk} \dot{q}^j \dot{q}^k \) in the equations of motion are called Coriolis terms.

Consider an invertible linear bundle map \( A: TQ \to T^* Q \) given by

\[ (q, \dot{q}) \mapsto (q, \alpha = A(q)\dot{q}). \]

In coordinates,

\[ \alpha_i = A_{ij} \dot{q}^j, \]

where \( \alpha = \alpha_i dq^i \). Let \( (B^{ij}) \) be the inverse matrix of \( (A_{ij}) \). i.e., \( B^{ik} A_{kj} = \delta^i_j \), where \( \delta^i_j \) is the Kronecker delta.

In \((x, \alpha)\) coordinates on \( T^* Q\), the equations of motion (1) become

\[ \ddot{q}^i = B^{ij} \alpha_j \]

\[ \dot{\alpha}_i = \frac{1}{2} \left( \delta_{k} A_{ij} + \delta_{j} A_{ik} - 2 A_{il} \Gamma^l_{jk} \right) B^{ij} B^{lk} \alpha_l \alpha_s - A_{ij} m^{jk} \delta_{k} V - A_{il} m^{jk} W_{ka} u^a, \]

where \( i = 1, \ldots, n \), or in vector form

\[ \dot{q} = A^{-1} \alpha \]

\[ \dot{\alpha} = f(q, \alpha) - Am^{-1} dV - Am^{-1} Wu \]

where

\[ f(q, \alpha) = \left( f_i(q, \alpha) \right) \]

\[ = \left( \frac{1}{2} \left( \delta_{k} A_{ij} + \delta_{j} A_{ik} - 2 A_{il} \Gamma^l_{jk} \right) B^{ij} B^{lk} \alpha_l \alpha_s \right) = \partial_i V \], \( W = (W_{iu}), \ u = (u^a). \)

Notice that all the Coriolis terms vanish in the \( \alpha_i \) equations in (4) if and only if

\[ \partial_k A_{ij} + \partial_j A_{ik} - 2 A_{il} \Gamma^l_{jk} = 0 \]

for all \( i, j, k \), in which case the equations of motion become

\[ \dot{s}^i = B^{ij} \alpha_j, \]

\[ \dot{\alpha}_i = -A_{ij} m^{jk} \delta_{k} V - A_{ij} m^{jk} W_{ka} u^a, \]

or in vector form

\[ \dot{q} = A^{-1} \alpha \]

\[ \dot{\alpha} = -Am^{-1} dV - Am^{-1} Wu \]

**Definition 2.1:** A control mechanical system is said to be quasi-linearizable if there is an invertible linear transformation of the form (2) that transforms the equations of motion of the system (1) to the form (6) and (7).

We can regard the configuration space \( Q \) of a mechanical system as a Riemannian manifold equipped with the metric \( m = (m_{ij}) \) that is induced from the kinetic energy of the system. A vector field \( X = X^i \partial_i \) on a Riemannian manifold \((Q, m)\) is called a Killing vector field if it satisfies

\[ X^k \delta_{k} m_{ij} + m_{ij} \delta_{k} X^k + m_{ij} \delta_{j} X^k = 0 \]

for all \( 1 \leq i \leq j \leq n \). Letting \( \alpha = mX = m_{jk} X^k dq^j \), we can write (8) as

\[ \delta_k \alpha_j + \delta_j \alpha_k - 2 \alpha_l \Gamma^l_{jk} = 0 \]

in terms of the 1-form \( \alpha \). A 1-form that satisfies (9) is called a Killing 1-form. Both (8) and (9) are called the Killing equation. Comparison of (5) and (9) implies that Eq. (5) is the Killing equations in (9) for the 1-form \( A_i := A_{ij} dq^j \) for each \( i \). Hence, a quasi-linearizing transformation consists of \( n \) pointwise independent Killing 1-forms, where each row of \( A \) is a Killing 1-form.

Let \( \text{iso}(Q, m) \) denote the set of all Killing vector fields on \((Q, m)\). It is a Lie algebra over \( \mathbb{R} \) under the usual bracket operation on vector fields. Let \( \Delta \) denote the distribution on \( Q \) that is generated by Killing vector fields, i.e.

\[ \Delta_q = \text{span}\{X(q) \in T_q Q | X \in \text{iso}(Q, m)\} \]

for each \( q \in Q \). The rank of \( \Delta_q \) is, by definition, the dimension of \( \Delta_q \) as a vector subspace of \( T_q Q \). Then the quasi-linearizability can be geometrically stated as follows.

**Theorem 2.2:** ([4]): Let \( q \) be a point in \((Q, m)\). The quasi-linearization of the system (1) is possible around \( q \)
if and only if \( \Delta_q = T_q Q \), i.e., \( \text{rank} \Delta_q = \dim Q \).

### 2.2 Partial quasilinearization and feedback quasilinearization

We now pose the following two main questions for control mechanical systems that are not quasilinearizable:

**Q1.** (Partial Quasilinearization) How many of the \( \alpha_i \) equations in (4) can be made free of the Coriolis terms via a transformation of the form (2)?

**Q2.** (Feedback Quasilinearization) If an affine feedback transformation of the form

\[
u = h(q) + \tilde{u}
\]

with \( h : Q \to \mathbb{R}^p \) and \( u \in \mathbb{R}^p \), is allowed in addition to the linear transformation of the form (2), when can a given system be transformed to the form (6) and (7), i.e., to the following form

\[
\dot{q} = A^{-1}\mathbf{a}
\]

\[
\mathbf{a} = -Am^{-3}dV - Am^{-3}Wu,
\]

which is free of the Coriolis terms?

**Definition 2.3:** A control mechanical system is called feedback quasilinearizable if its equations of motion can be transformed to the form (12) and (13) via a transformation of the form (2) followed by a feedback transformation of the form (11).

**Definition 2.4:** A point \( q \) in \( (Q,m) \) is called regular if the rank of the distribution \( \Delta \) defined in (10) is constant in a neighborhood of \( q \).

We now provide an answer to the first question we posed in the beginning of this section.

**Theorem 2.5 (Partial Quasilinearization):** Let \( q_0 \) be a regular point in \( (Q,m) \). Then, at least \( k \) \( \alpha_i \)-equations can be made free of the Coriolis terms via an invertible transformation of the form (2) around \( q_0 \) if and only if rank \( \Delta \geq k \) in a neighborhood of \( q_0 \).

**Proof:** (\( \Rightarrow \)) By hypothesis there is a linear transformation \( \mathbf{a} = A(q)\dot{q} \) such that the first \( k \) \( \alpha_i \)-equations can be written as

\[
\dot{\alpha}_i = -A_{ij}m^{jk}\partial_k V - A_{ij}m^{jk}W_{ka}u^a
\]

for \( i = 1, \ldots, k \) in a neighborhood of \( q_0 \). In other words Eq. (5) holds for \( i = 1, \ldots, k \). Hence, the first \( k \) row vectors of \( A \) are pointwise independent Killing 1-forms, which implies that rank \( \Delta \geq k \) in a neighborhood of \( q_0 \).

(\( \Leftarrow \)) This direction can be proven similarly.

The above theorem can be also interpreted as follows:

\( k \) is the maximum number of the \( \alpha_i \) equations that can be made free of Coriolis terms via a transformation of the form (2) around a regular point \( q_0 \) if and only if rank \( \Delta = k \) in a neighborhood of \( q_0 \).

We now answer the second question posed in the beginning of this section.

**Theorem 2.6 (Feedback Quasilinearization):** A control mechanical system is feedback-quasilinearizable around a regular point \( q_0 \) if

\[
\Delta^0_q \subset W_q
\]

for each \( q \) in a neighborhood of \( q_0 \), where \( \Delta^0_q \) is the codistribution on \( Q \) that annihilates \( \Delta \), i.e., pointwise

\[
\Delta^0_q = \{ \beta \in T^*_q Q \mid \beta \cdot X_q = 0, \forall X_q \in \Delta_q \}.
\]

**Proof:** Let \( k \) be the constant rank of \( \Delta \) around \( q_0 \). Then there exist \( k \) Killing vector fields \( X_1, \ldots, X_k \) that span \( \Delta \) pointwise around \( q_0 \). Choose \( (n-k) \) more vector fields \( X_{k+1}, \ldots, X_n \) such that the set of vector fields \( \{X_1, \ldots, X_n\} \) span \( TQ \) around \( q_0 \). One can find \((n-k)\) 1-forms \( \beta_{k+1}, \ldots, \beta_n \) in \( \Delta^0 \) around \( q_0 \) such that

\[
\langle \beta_i, X_j \rangle = \begin{cases} 
0, & \text{if } 1 \leq j \leq k \\
\delta_{ij}, & \text{if } k + 1 \leq j \leq n
\end{cases}
\]

for \( k + 1 \leq i \leq n \). Since \( \Delta^0 \subset W \) by hypothesis, there exist vectors \( u_{k+1}, \ldots, u_n \) in \( \mathbb{R}^p \) such that

\[
\beta_i = W_{u_i}
\]

for \( k + 1 \leq i \leq n \), where \( \beta_i \) and \( u_i \) are assumed to be in column vector form. Let

\[
A = \begin{bmatrix} mX_1 & \cdots & mX_n \end{bmatrix}^T = \begin{bmatrix} X_1^T \mathbf{m} \\
\vdots \\
X_n^T \mathbf{m} \end{bmatrix}
\]

the first \( k \) rows of which are Killing 1-forms since \( X_1, \ldots, X_k \) are Killing vector fields. Let

\[
\alpha = A\dot{q}
\]

or in coordinates \( \alpha_i = A_{ij}\dot{q}^j \). Change coordinates from \( \dot{q} \) to \( \alpha \) to transform (1) to (3) and (4), where the first \( k \) \( \alpha_i \)-equations in (4) become free of the Coriolis terms. Apply the following control \( u \in \mathbb{R}^p \)

\[
u = \begin{bmatrix} u_{k+1} & \cdots & u_n \end{bmatrix} \begin{bmatrix} f_{k+1} \\
\vdots \\
f_n \end{bmatrix} + \tilde{u},
\]

http://www.jeet.or.kr | 743
where
\[ f_i = \frac{1}{2} \left( \ddot{x}_k A_{ij} + \ddot{y}_j A_{ik} - 2 A_{ij} \dot{x}_k \right) B_{ij} B_{jk} a_i a_s \]
for \( k + 1 \leq i \leq n \). It is then easy to see that the system (3) and (4) is transformed via this feedback control to the system (12) and (13). Therefore, the system is feedback quasilinearizable around \( q_0 \).

### 3. Example

Consider the Acrobat system in Fig. 1, where there is an actuation \( u \) on the outer joint. Let \( M_1 \) and \( M_2 \) be the masses of the bobs and \( \ell_1 \) and \( \ell_2 \) the lengths of the massless rods. The gravitational acceleration is denoted by \( g \). Let \( \theta_1 \) denote the angle of the first rod measured counter-clockwise from the upward vertical, and \( \theta_2 \) the angle measured counterclockwise from the ray containing the first rod to the second rod.

The Lagrangian of the system is given by
\[
L = \frac{1}{2} m_1 \dot{\theta}_1^2 + m_2 \dot{\theta}_2^2 + \frac{1}{2} m_2 \dot{\theta}_2^2 \\
- (M_1 + M_2) g \ell_1 \cos \theta_1 - M_2 g \ell_2 \cos(\theta_1 + \theta_2)
\]
where
\[
m_{1} = M_{1} \ell_{1}^{2} + M_{2}(\ell_{1}^{2} + \ell_{2}^{2} + 2 \ell_{1} \ell_{2} \cos \theta_{2}), \\
m_{2} = M_{2}(\ell_{2}^{2} + \ell_{1} \ell_{2} \cos \theta_{2}), \\
m_{22} = M_{2} \ell_{2}^{2}.
\]

The scalar curvature \( R_S \) of the metric \( m = (m_{ij}) \) is computed as
\[
R_S = \frac{2m_1 \cos \theta_2}{\ell_1 \ell_2 (M_1 + M_2 - M_2 \cos^2 \theta_2)^2},
\]
which is not constant. Hence, the system is not quasilinearizable by Theorem III.1 in [4].

Let us now investigate feedback quasilinearizability of this system. The Acrobat has only one Killing vector field up to a scalar factor and it is given by \( X = \ell_1 \), which can be easily obtained using software Maple. Hence,
\[
\Delta = \text{span}\{\ell_1\}, \Delta^0 = \text{span}\{d\theta_2\}.
\]

The control bundle of the Acrobat is given by
\[
W = \text{span}\{d\theta_2\}.
\]

Since \( \Delta^0 \subset W \), the Acrobat is feedback quasilinearizable by Theorem 2.6.

### Acknowledgements

This research was supported by Hallym University Research Fund 2015 (HRF-201506-006).

### References

Dong Eui Chang  He received the B.S degree in control and Instrumentation engineering and the M.S. degree from electrical engineering, both, from Seoul National University and the Ph.D. in control & dynamical systems from the California Institute of Technology. He is currently associate professor in applied mathematics at the University of Waterloo, Canada. His research interests lie in control, mechanics and various engineering applications.

Seong-Ho Song  He received the B.S, M.S, and Ph. D degree in measurement and control engineering from Seoul National University. He is currently professor in electronics engineering at Hallym University, Korea. His research interests are nonlinear control, aerospace engineering, mechatronics and vision systems.

Jeom Keun Kim  He received the B.S, M.S, and Ph. D degree in measurement and control engineering from Seoul National University, Korea. He is currently professor in electronics engineering at Hallym University, Korea. His research interests are mechatronics, and control applications for medical equipments.