Stability and Robust H_∞ Control for Time-Delayed Systems with Parameter Uncertainties and Stochastic Disturbances

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Abstract – This paper investigates the problem of stability analysis and robust H_∞ controller for time-delayed systems with parameter uncertainties and stochastic disturbances. It is assumed parameter uncertainties are norm bounded and mean and variance for disturbances of them are known. Firstly, by constructing a newly augmented Lyapunov-Krasovskii functional, a stability criterion for nominal systems with time-varying delays is derived in terms of linear matrix inequalities (LMIs). Secondly, based on the result of stability analysis, a new controller design method is proposed for the nominal form of the systems. Finally, the proposed method is extended to the problem of robust H_∞ controller design for a time-delayed system with parameter uncertainties and stochastic disturbances. To show the validity and effectiveness of the presented criteria, three examples are included.

Keywords: Stochastic disturbances, Parameter uncertainties, Time-varying delays, Lyapunov method

1. Introduction

During the past several decades, the topic of stability analysis and stabilization for time-delayed systems has been tackled by many researchers since time delays occur in many practical systems. For examples, most sensors in systems usually have time-delays since they take some times to measure. Also, it is unavoidable that communications have time-delays since processes of modulation and demodulation require some times. Moreover, in digital systems, time-delays occur in some processes like a sampling. It is well known that occurrence of time delays cause poor performance or even instability. Therefore, a great deal of efforts has been done in stability analysis and stabilization for time-delayed systems [1-26] based on delay-dependent approach since delay-dependent criteria are less conservative than delay-independent ones especially when the sizes of time-delays are small.

One of the main issue in the field of delay-dependent stability analysis and stabilization is to find larger delay bounds for guaranteeing the asymptotic stability of the concerned dynamic system. Therefore, the size of maximum delay bound obtained by stability or stabilization criteria has been the main index for comparing the superiority of stability or stabilization criteria. With this regard, to enhance the feasible region of stability or stabilization, some new Lyapunov-Krasovskii functional such as triple or quadruple integral form were introduced

in [3-7, 16, 18, 20], and [22]. The application of free-weighting matrix techniques to reduce the conservatism of stability criteria for various systems can be found in [1, 2, 5, 13, 15, 17, 18, 21, 23], and [24]. In [5], delay-partitioning techniques with a newly augmented Lyapunov-Krasovskii functional were utilized in stability analysis for uncertain neutral systems with time-varying delays. Zhu et al. [12] tried to reduce the decision number of LMI variable by not introducing slack variables in delay-dependent stability analysis of time-delay systems. The model transformation such as a descriptor system was utilized in [14, 25], and [26]. To obtain more tighter upper bound of time-derivative of Lyapunov-Krasovskii functional, Jensen's inequality [10], reciprocally convex optimization approach [8], and Wirtinger-based integral inequality [9] are well recognized in reducing the conservatism of delay-dependent stability analysis. The application of reciprocally convex optimization approach [8] in discrete-time systems with time-varying delays can be found in [11] and [19]. From the analysis of delay-dependent stability and stabilization for dynamic systems with time-delays mentioned before, one can see that how to construct Lyapunov-Krasovskii functional and estimate its time-derivative value with some techniques play key roles to increase maximum delay bounds.

In practical systems, there exist some uncertainties because it is very difficult to be obtain an exact mathematical model due to environment noise, system complexities, friction, uncertain or slowly varying parameters, and so on. Hence, considerable efforts in [20-24], [27-29] have been devoted to the stability and stabilization for uncertain dynamics with norm-bounded parameter uncertainties.

On the other hand, disturbances can have an adverse effect on the stability of systems. Thus, it is important to design a controller for the systems with disturbances. For

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an example, when disturbances like earthquake occurred, building and structure systems require controllers that minimize the effect of external disturbances as well as stabilize the system. Therefore, an H_∞ control [30] has been used to minimize the effect of the disturbances because the goal of the H_∞ control is to design the controllers such that the closed-loop systems are stable and their H_∞ -norm of the transfer functions between the controlled output and the disturbances will not exceed prescribed level of performances. With this regard, a number of research results on H_∞ control has been addressed for various systems such as time-delayed systems in [25-28], singular systems [29], and so on.

Recently, a variety of stochastic systems have been researched such as systems with stochastic sampling and stochastic systems with missing measurement in [31, 32]. Moreover, systems with randomly occurring uncertainties have been introduced in [33, 34]. In [31], it is assumed that the time-delay has a stochastic characteristic to describe the systems which has sampling period varying by stochastic characteristic. In [32], it was assumed that the system output has the stochastic variables for probabilistic missing data. Also, the parameter uncertainties are multiplied by the stochastic variables in [33, 34].

However, it is natural to assume that stochastic characteristic exist not only in the mentioned systems in [31-34] but also in disturbances since the disturbances are affected by random change of environment. For an example, aircraft is affected by disturbances when the wind is strong like a storm. Other existing literatures in [31-34], even though stochastic variables are used, variance is not considered. Thus, it is desirable and realistic that the stochastic data of the disturbances such as mean and variance is utilized in stability analysis and stabilization for systems with stochastic disturbances. However, the systems with stochastic disturbances have not been fully investigated yet.

With motivations mentioned in above discussions, this paper focuses on the problem of the H_∞ controller design for linear time-delayed systems with stochastic disturbances. The main contribution of this research lies in two aspects.

- Some new augmented Lyapunov-Krasovskii functional are introduced in stability and stabilization problem for time-delayed systems with parameter uncertainties.
- The problem of designing a robust H_∞ control for the systems with both stochastic disturbance and parameter uncertainties is investigated for the first time.

First, by construction of a newly Lyapunov-Krasovskii functional and utilization of reciprocally convex approach [8] with some new zero equalities, a new stability criterion for the nominal form of systems with time-varying delays is derived in Theorem 1 with the LMI framework, which can be formulated as convex optimization algorithms which are amenable to computer solution [35]. Secondly,

based on the results of Theorem 1, a new controller design method for the nominal form of systems with time-varying delays will be proposed in Theorem 2. Finally, Theorem 2 will be extended to Theorem 3 which deals with the robust H_∞ controller design methods for the systems with both stochastic disturbance and parameter uncertainties. Through three numerical examples, the advantage and effectiveness of the proposed theorems will be shown.

Notation: $\mathbf{R}^n$ is the n -dimensional Euclidean space, $\mathbf{R}^{m \times n}$ denotes the set of $m \times n$ real matrix. For symmetric matrices X and Y , $X > Y$ (respectively, $X \geq Y$) means that the matrix $X - Y$ is positive definite (respectively, nonnegative). X^T denotes the transposition of X . I_n denotes the $n \times n$ identity matrix. 0_n and $0_{n \times m}$ denote the $n \times n$ zero matrix and the $n \times m$ zero matrix, respectively. $\|\cdot\|$ refers to the Euclidean vector norm and the induced matrix norm. $\text{diag}\{\dots\}$ denotes the block diagonal matrix, respectively. $*$ represents the elements below the main diagonal of a symmetric matrix. $L_2[0, \infty)$ is the space of square integrable vector. $\mathbf{E}\{x\}$ and $\mathbf{E}\{x|y\}$ denote the expectation of x and the expectation of x conditional on y , respectively. $X_{[f(t)]}$ means that the elements of the matrix $X_{[f(t)]}$ includes the value of $f(t)$; e.g., $X_{[a]} = X_{[f(t)=a]}$. $\mathbf{Pr}\{x\}$ means the varying probability of the event x .

2. Problem statements

Consider the linear system with uncertain parameters and a time-varying delay:

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A(t))x(t) + (A_d + \Delta A_d(t))x(t - h(t)) \\ &\quad + Bu(t) + B_w w(t), \\ z(t) &= Cx(t), \end{aligned} \quad (1)$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^m$ is the control input, $w(t) \in \mathbf{R}^p$ is the disturbance input which belongs to $L_2[0, \infty)$, $z(t) \in \mathbf{R}^q$ is the vector of controlled output, $A, A_d \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$, $B_w \in \mathbf{R}^{n \times p}$ and $C \in \mathbf{R}^{q \times n}$ are known real constant matrices, $\Delta A(t)$ and $\Delta A_d(t)$ are the parameter uncertainties of system matrices of the form

$$[\Delta A(t), \Delta A_d(t)] = DF(t)[E_a, E_d] \quad (2)$$

in which $D \in \mathbf{R}^{n \times s}$, $E_a \in \mathbf{R}^{r \times n}$ and $E_d \in \mathbf{R}^{r \times n}$ are known constant matrices and $F(t) \in \mathbf{R}^{s \times r}$ is norm bounded with $F^T(t)F(t) \leq I_r$. Also, $h(t)$ is a time-delay satisfying time-varying continuous function as follows:

$$0 \leq h(t) \leq h_M, \dot{h}(t) \leq h_d, \quad (3)$$

where h_M is a known positive scalar and h_d is any constant one.

In this paper, it is assumed that the disturbance has stochastic properties. Let us define the stochastic variable

$\rho(t)$ which satisfies the following conditions:

$$\mathbf{E}\{\rho(t)\} = \rho_0, \quad \mathbf{E}\{(\rho(t) - \rho_0)^2\} = \sigma^2, \quad (4)$$

where ρ_0 and σ^2 are mean and variance of $\rho(t)$, respectively. From (4), the expectation of $\rho^2(t)$ can be obtained as

$$\mathbf{E}\{\rho^2(t)\} = \sigma^2 + \rho_0^2. \quad (5)$$

For the assumption about the stochastic disturbances, in this paper, the term $B_w w(t)$ in Eq. (1) is multiplied by $\rho(t)$. Thus, in this paper, the following model is considered:

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A(t))x(t) + (A_d + \Delta A_d(t))x(t-h(t)) \\ &\quad + Bu(t) + \rho(t)B_w w(t), \\ z(t) &= Cx(t). \end{aligned} \quad (6)$$

This model has a form of $\rho(t)B_w w(t)$ which obtain the stochastic variable $\rho(t)$ and reflects a stochastic character of disturbances. For the various disturbances, even if means of disturbances are same, the disturbances can be different each other because of variances. If the variance of $\rho(t)$ is 0, $\rho(t)$ can be assumed by the constant value ρ_0 .

Remark 1. It is supposed that the $\rho(t)$ has the stochastic characteristic and its mean and variance are known. Thus, the disturbances considered in Eq. (6) can be characterized by stochastic properties. Since the disturbances are affected by random change of environment, considering mean and variance of $\rho(t)$ is reasonable. For the aircraft as an example, if wind is strong like storm, then the flight performance of the aircraft will be deteriorated since the strength of the disturbances is intense. That is, mean of $\rho(t)$ will be increased. Also, if wind is fluctuating sharply, variance of $\rho(t)$ will be increased. Thus, if the stochastic information of $\rho(t)$ when a controller is designed is utilized, then it may be expected that more practical controllers can be provided to the system (6) than other ones in existing literatures.

Let us consider a memoryless state feedback controller:

$$u(t) = Kx(t), \quad (7)$$

where $K \in \mathbf{R}^{m \times n}$ is a gain matrix of the feedback controller. In order to develop a delay-dependent H_∞ controller for the system (6), the following conditions are satisfied:

- (i) With $w(t) = 0$, the closed loop system (6) with control input is asymptotically stable.
- (ii) Under zero initial condition, the closed-loop system satisfies

$$J = \mathbf{E} \left\{ \int_0^\infty (z^T(s)z(s) - \gamma^2 w^T(s)w(s))ds \right\} < 0 \quad (8)$$

where $\gamma > 0$ is a prescribed scalar. Under these condition, the system (6) is said to be stabilizable with an H_∞ disturbance attenuation level γ . Then, the obtained controller $u(t)$ is said to be an H_∞ stabilization controller.

The objective of this paper is to design a state feedback controller (7) such that system (6) is asymptotically stable and an H_∞ disturbance attenuation level γ is minimized. To derive main results, the following lemmas are utilized in deriving the main results.

Lemma 1 [4]. For a positive matrix Z , scalars $h_2 > h_1 > 0$ such that the integrations are well defined, then

$$\begin{aligned} (h_2 - h_1) \int_{t-h_2}^{t-h_1} x^T(s)Zx(s)ds \\ \geq \left(\int_{t-h_2}^{t-h_1} x(s)ds \right)^T Z \left(\int_{t-h_2}^{t-h_1} x(s)ds \right), \end{aligned} \quad (9)$$

$$\begin{aligned} \frac{h_2^2 - h_1^2}{2} \int_{t-h_2}^{t-h_1} \int_u^t x^T(u)Zx(u)duds \\ \geq \left(\int_{t-h_2}^{t-h_1} \int_u^t x(u)duds \right)^T Z \left(\int_{t-h_2}^{t-h_1} \int_u^t x(u)duds \right), \end{aligned} \quad (10)$$

$$\begin{aligned} \frac{h_2^3 - h_1^3}{6} \int_{t-h_2}^{t-h_1} \int_u^t \int_v^t x^T(v)Zx(v)dvdu ds \\ \geq \left(\int_{t-h_2}^{t-h_1} \int_u^t \int_v^t x(v)dvdu ds \right)^T Z \left(\int_{t-h_2}^{t-h_1} \int_u^t \int_v^t x(v)dvdu ds \right). \end{aligned} \quad (11)$$

Lemma 2 [8]. For a scalar α ($0 < \alpha < 1$), a given matrix $R \in \mathbf{R}^{n \times n} > 0$, two matrices $W_1 \in \mathbf{R}^{n \times m}$ and $W_2 \in \mathbf{R}^{n \times m}$, any vector $\xi \in \mathbf{R}^m$, let us define the function $\Theta(\alpha, R)$ given by:

$$\Theta(\alpha, R) = \frac{1}{\alpha} \xi^T W_1^T R W_1 \xi + \frac{1}{1-\alpha} \xi^T W_2^T R W_2 \xi. \quad (12)$$

Then, if there exists a matrix $X \in \mathbf{R}^{n \times n}$ such that $\begin{bmatrix} R & X \\ * & R \end{bmatrix} > 0$, then the following inequality holds

$$\min_{\alpha \in (0,1)} \Theta(\alpha, R) \geq \begin{bmatrix} W_1 \xi \\ W_2 \xi \end{bmatrix}^T \begin{bmatrix} R & X \\ * & R \end{bmatrix} \begin{bmatrix} W_1 \xi \\ W_2 \xi \end{bmatrix}. \quad (13)$$

Lemma 3 (Finsler's lemma) [36]. Let $\zeta \in \mathbf{R}^n$, $\Phi = \Phi^T \in \mathbf{R}^{n \times n}$ and $B \in \mathbf{R}^{m \times n}$ such that $\text{rank}(B) < n$. The following statements are equivalent:

$$\zeta^T \Phi \zeta < 0, \quad \forall B\zeta = 0, \quad \zeta \neq 0,$$

$(B^\perp)^T \Phi B^\perp < 0$, where $B^\perp$ is right orthogonal complement of B .

Lemma 4 [11]. For a positive matrix Λ , a symmetric matrix Ξ , and a matrix Γ , two following statements are equivalent:

$$(1) \quad \Xi - \Gamma^T \Lambda \Gamma < 0,$$

- (2) There exists a matrix U of appropriate dimension such that

$$\begin{bmatrix} \Xi + U^T \Gamma + \Gamma^T U & U^T \\ U & -\Lambda \end{bmatrix} < 0. \quad (14)$$

Lemma 5 [37]. Let E , H , and $F(t)$ be real matrices of appropriate dimensions, and let $F(t)$ satisfy $F^T(t)F(t) \leq I$ for all t . Then, for a positive scalar ε , the following inequality holds:

$$EF(t)H + H^T F^T(t)E^T \leq \varepsilon^{-1}EE^T + \varepsilon H^T H. \quad (15)$$

3. Main results

This section consists of three subsections. The goal of first subsection is stability analysis of a nominal system. The second subsection will be extended to a stabilization. And the final section will introduce a design method of a robust H_∞ controller for uncertain linear systems with time-varying delays and stochastic disturbances.

3.1 Stability analysis

Firstly, in this subsection, a delay-dependent stability criterion for the nominal system of (6) with $w(t) = 0$, $u(t) = 0$ and without parameter uncertainties is derived. Here, the following nominal system of (6) is given by

$$\dot{x} = Ax(t) + A_d x(t-h(t)). \quad (16)$$

Now, for simplicity of matrix and vector representation, $e_i \in \mathbf{R}^{9 \times n}$ ($i=1, \dots, 9$) are defined as block entry matrices which will be used in Theorem 1. The other notations are defined as

$$\begin{aligned} \zeta^T(t) = & \left[x^T(t), x^T(t-h(t)), x^T(t-h_M), \right. \\ & \dot{x}^T(t), \dot{x}^T(t-h_M), \int_{t-h(t)}^t x^T(s) ds, \int_{t-h_M}^{t-h(t)} x^T(s) ds, \\ & \left. \int_{t-h(t)}^t \int_s x^T(t) duds, \int_{t-h_M}^{t-h(t)} \int_s x^T(t) duds \right], \end{aligned}$$

$$\begin{aligned} \Psi_{11} = & \left[e_1, e_3, e_6 + e_7, e_8 + e_9, \left(h_M^2/2 \right) e_1 - e_8 - e_9 \right], \\ \Psi_{12} = & \left[e_4, e_5, e_1 - e_3, h_M e_1 - e_6 - e_7, \left(h_M^2/2 \right) e_4 - h e_1 + e_6 + e_7 \right], \\ \Psi_{21} = & \left[e_1, e_4, 0_{9 \times n}, 0_{9 \times n} \right], \Psi_{22} = \left[e_3, e_4, 0_{9 \times n}, 0_{9 \times n} \right], \\ \Psi_{23} = & \left[e_6 + e_7, e_1 - e_3, h_M e_1 - e_6 - e_7, e_8 + e_9 \right], \\ \Psi_{24} = & \left[0_{9 \times n}, 0_{9 \times n}, e_4, e_1 \right], \Psi_{31} = \left[e_1, 0_{9 \times n}, 0_{9 \times n} \right], \\ \Psi_{32} = & \left[e_2, e_2 - e_1, e_6 \right], \Psi_{33} = \left[e_6, -e_6, e_8 \right], \\ \Psi_{34} = & \left[0_{9 \times n}, e_4, e_1 \right], \Psi_{41} = \left[e_1, e_4 \right], \Psi_{42} = \left[e_1, e_2, e_3 \right], \\ \Psi_{43} = & \left[e_6, e_1 - e_2, e_7, e_2 - e_3 \right], \Psi_5 = \left[e_1, e_3 \right], \end{aligned}$$

$$\begin{aligned} \Psi_6 = & \left[\left(h_M^2/2 \right) e_1 - e_8 - e_9 \right], \\ \Lambda_1 = & \begin{bmatrix} Q_1 + \begin{bmatrix} 0_n & P_1 \\ P_1 & 0_n \end{bmatrix} & S_1 \\ S_1^T & Q_1 + \begin{bmatrix} 0_n & P_2 \\ P_2 & 0_n \end{bmatrix} \end{bmatrix}, \\ \Lambda_2 = & \begin{bmatrix} Q_2 & S_2 \\ S_2^T & Q_2 \end{bmatrix}, \\ \Xi_1 = & \Psi_{11} R \Psi_{12}^T + \Psi_{12} R \Psi_{11}^T, \\ \Xi_2 = & \Psi_{21} N \Psi_{21}^T - \Psi_{22} N \Psi_{22}^T + \Psi_{23} N \Psi_{24}^T + \Psi_{24} N \Psi_{23}^T, \\ \Xi_3 = & \Psi_{31} G \Psi_{31}^T - (1-h_d) \Psi_{32} G \Psi_{32}^T + \Psi_{33} G \Psi_{34}^T + \Psi_{34} G \Psi_{33}^T, \\ \Xi_4 = & h_M^2 \Psi_{41} Q_1 \Psi_{41}^T + h_M \Psi_{42} \text{diag}\{P_1, P_2 - P_1, -P_2\} \Psi_{42}^T - \Psi_{43} \Lambda_1 \Psi_{43}^T, \\ \Xi_5 = & \left(h_M^2/2 \right)^2 \Psi_5 Q_2 \Psi_5^T, \\ \Xi_6 = & \left(h_M^3/6 \right)^2 e_4 Q_3 e_4^T - \Psi_6 Q_3 \Psi_6^T, \\ \Gamma_{[h(t)]} = & \left[e_8, h(t) e_1 - e_6, e_9, (h_M - h(t)) e_1 - e_7 \right]^T, \\ \Theta = & A e_1^T + A_d e_2^T - e_4^T, \\ \Omega_{[h(t)]} = & \begin{bmatrix} 0_{n \times 9n} \\ h(t) e_1^T \\ 0_{n \times 9n} \end{bmatrix}^T G \begin{bmatrix} 0_{n \times 9n} \\ e_4^T \\ e_1^T \end{bmatrix} + \begin{bmatrix} 0_{n \times 9n} \\ e_4^T \\ e_1^T \end{bmatrix}^T G \begin{bmatrix} 0_{n \times 9n} \\ h(t) e_1^T \\ 0_{n \times 9n} \end{bmatrix}, \\ \Upsilon_{[h(t)]} = & \begin{bmatrix} \left(\Theta^\perp \right)^T \left(\sum_{i=1}^6 \Xi_i + \Omega_{[h(t)]} \right) \Theta^\perp \\ \left(\Theta^\perp \right)^T \Gamma_{[h(t)]} U + U^T \Gamma_{[h(t)]} \Theta^\perp \\ U \\ -\Lambda_2 \end{bmatrix}. \quad (17) \end{aligned}$$

Now, the following theorem is given as a delay-dependent stability criterion for the system (16).

Theorem 1. For given scalars $0 \leq h_M$ and h_d , the system (16) is asymptotically stable for $0 \leq h(t) \leq h_M$, $\dot{h}(t) \leq h_d$, if there exist positive definite matrices $R \in \mathbf{R}^{5n \times 5n}$, $N \in \mathbf{R}^{4n \times 4n}$, $G \in \mathbf{R}^{3n \times 3n}$, $Q_1 \in \mathbf{R}^{2n \times 2n}$, $Q_2 \in \mathbf{R}^{2n \times 2n}$, $Q_3 \in \mathbf{R}^{n \times n}$, symmetric matrices $P_1 \in \mathbf{R}^{n \times n}$, $P_2 \in \mathbf{R}^{n \times n}$, any matrices $S_1 \in \mathbf{R}^{2n \times 2n}$, $S_2 \in \mathbf{R}^{2n \times 2n}$ and $U \in \mathbf{R}^{4n \times 8n}$ satisfying the following LMIs:

$$\Upsilon_{[h(t)=0]} < 0, \quad (18)$$

$$\Upsilon_{[h(t)=h_M]} < 0, \quad (19)$$

$$\Lambda_i \geq 0 \ (i=1, 2). \quad (20)$$

Proof. Let us consider the following Lyapunov-Krasovskii functional candidate as

$$V(t) = \sum_{i=1}^6 V_i(t), \quad (21)$$

where

$$V_1(t) = \begin{bmatrix} x(t) \\ x(t-h_M) \\ \int_{t-h_M}^t x(s)ds \\ \int_{t-h_M}^t \int_s^t x(u)duds \\ \int_{t-h_M}^t \int_s^t \int_u^t \dot{x}(v)dvduds \end{bmatrix}^T R \begin{bmatrix} x(t) \\ x(t-h_M) \\ \int_{t-h_M}^t x(s)ds \\ \int_{t-h_M}^t \int_s^t x(u)duds \\ \int_{t-h_M}^t \int_s^t \int_u^t \dot{x}(v)dvduds \end{bmatrix},$$

$$V_2(t) = \int_{t-h_M}^t \begin{bmatrix} x(s) \\ \dot{x}(s) \\ \int_s^t \dot{x}(u)du \\ \int_s^t x(u)du \end{bmatrix}^T N \begin{bmatrix} x(s) \\ \dot{x}(s) \\ \int_s^t \dot{x}(u)du \\ \int_s^t x(u)du \end{bmatrix} ds,$$

$$V_3(t) = \int_{t-h(t)}^t \begin{bmatrix} x(s) \\ \int_s^t \dot{x}(u)du \\ \int_s^t x(u)du \end{bmatrix}^T G \begin{bmatrix} x(s) \\ \int_s^t \dot{x}(u)du \\ \int_s^t x(u)du \end{bmatrix} ds,$$

$$V_4(t) = h_M \int_{t-h_M}^t \int_s^t \begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix}^T Q_1 \begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix} duds,$$

$$V_5(t) = (h_M^2/2) \int_{t-h_M}^t \int_s^t \int_u^t \begin{bmatrix} x(v) \\ \dot{x}(v) \end{bmatrix}^T Q_2 \begin{bmatrix} x(v) \\ \dot{x}(v) \end{bmatrix} dvduds,$$

$$V_6(t) = (h_M^3/6) \int_{t-h_M}^t \int_s^t \int_u^t \int_v^t \dot{x}^T(\lambda) Q_3 \dot{x}(\lambda) d\lambda dvduds.$$

Now, the time-derivative of $V_1(t)$ is

$$\dot{V}_1(t) = 2 \begin{bmatrix} x(t) \\ x(t-h_M) \\ \int_{t-h(t)}^t x(s)ds + \int_{t-h_M}^{t-h(t)} x(s)ds \\ \int_{t-h_M}^t \int_s^t x(u)duds \\ \int_{t-h_M}^t \int_s^t \int_u^t \dot{x}(v)dvduds \end{bmatrix}^T R \begin{bmatrix} \dot{x}(t) \\ \dot{x}(t-h_M) \\ x(t) - x(t-h_M) \\ \left(h_M x(t) - \int_{t-h(t)}^t x(s)ds \right) \\ \left(-\int_{t-h_M}^{t-h(t)} x(s)ds \right) \\ \left(\left(h_M^2/2 \right) \dot{x}(t) - h_M x(t) \right) \\ \left(+ \int_{t-h(t)}^t x(s)ds + \int_{t-h_M}^{t-h(t)} x(s)ds \right) \end{bmatrix} = \zeta^T(t) \Xi_1 \zeta(t). \quad (22)$$

Also, $\dot{V}_2(t)$ is calculated as

$$\begin{aligned} \dot{V}_2(t) &= \begin{bmatrix} x(t) \\ \dot{x}(t) \\ 0_{n \times 1} \\ 0_{n \times 1} \end{bmatrix}^T N \begin{bmatrix} x(t) \\ \dot{x}(t) \\ 0_{n \times 1} \\ 0_{n \times 1} \end{bmatrix} - \begin{bmatrix} x(t-h_M) \\ \dot{x}(t-h_M) \\ x(t) - x(t-h_M) \\ \int_{t-h_M}^t x(u)du \end{bmatrix}^T \\ &\quad N \begin{bmatrix} x(t-h_M) \\ \dot{x}(t-h_M) \\ x(t) - x(t-h_M) \\ \int_{t-h_M}^t x(u)du \end{bmatrix} + 2 \int_{t-h_M}^t \begin{bmatrix} x(s) \\ \dot{x}(s) \\ x(t) - x(s) \\ \int_s^t x(u)du \end{bmatrix}^T ds N \begin{bmatrix} 0_{n \times 1} \\ 0_{n \times 1} \\ \dot{x}(t) \\ \dot{x}(t) \end{bmatrix} \\ &= \zeta^T(t) \Xi_2 \zeta(t). \end{aligned} \quad (23)$$

The upper-bound of $\dot{V}_3(t)$ can be given as follows:

$$\begin{aligned} V_3(t) &= \begin{bmatrix} x(t) \\ 0_{n \times 1} \\ 0_{n \times 1} \end{bmatrix}^T G \begin{bmatrix} x(t) \\ 0_{n \times 1} \\ 0_{n \times 1} \end{bmatrix} - (1-h(t)) \begin{bmatrix} x(t-h(t)) \\ \int_{t-h(t)}^t \dot{x}(u)du \\ \int_{t-h(t)}^t x(u)du \end{bmatrix}^T \\ &\quad G \begin{bmatrix} x(s) \\ \int_{t-h(t)}^t \dot{x}(u)du \\ \int_{t-h(t)}^t x(u)du \end{bmatrix} + 2 \begin{bmatrix} \int_{t-h(t)}^t x(s)ds \\ h(t)x(t) - \int_{t-h(t)}^t x(s)ds \\ \int_{t-h(t)}^t \int_s^t x(u)duds \end{bmatrix}^T \\ &\quad G \begin{bmatrix} 0_{n \times 1} \\ \dot{x}(t) \\ x(t)du \end{bmatrix} \leq \begin{bmatrix} x(t) \\ 0_{n \times 1} \\ 0_{n \times 1} \end{bmatrix}^T G \begin{bmatrix} x(t) \\ 0_{n \times 1} \\ 0_{n \times 1} \end{bmatrix} \\ &\quad - (1-h_d) \begin{bmatrix} x(t-h(t)) \\ \int_{t-h(t)}^t \dot{x}(u)du \\ \int_{t-h(t)}^t x(u)du \end{bmatrix}^T G \begin{bmatrix} x(s) \\ \int_{t-h(t)}^t \dot{x}(u)du \\ \int_{t-h(t)}^t x(u)du \end{bmatrix} \\ &\quad + 2 \begin{bmatrix} \int_{t-h(t)}^t x(s)ds \\ h(t)x(t) - \int_{t-h(t)}^t x(s)ds \\ \int_{t-h(t)}^t \int_s^t x(u)duds \end{bmatrix}^T G \begin{bmatrix} 0_{n \times 1} \\ \dot{x}(t) \\ x(t)du \end{bmatrix} \\ &= \zeta^T(t) (\Xi_3 + \Omega_{[h(t)]}) \zeta(t). \end{aligned} \quad (24)$$

Moreover, $\dot{V}_4(t)$ can be derived as:

$$\dot{V}_4(t) = h_M^2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T Q_1 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + h_M \int_{t-h_M}^t \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix}^T Q_1 \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} ds. \quad (25)$$

Inspired by the work in [13], for symmetric matrices P_1 and P_2 , let us consider the following zero equalities:

$$\begin{aligned} 0 &= x^T(t)(h_M P_1)x(t) - x^T(t-h(t))(h_M P_1)x(t-h(t)) \\ &\quad - 2h_M \int_{t-h(t)}^t x^T(s)P_1 \dot{x}(s)ds, \\ 0 &= x^T(t-h(t))(h_M P_2)x(t-h(t)) \\ &\quad - x^T(t-h_M)(h_M P_2)x(t-h_M) \\ &\quad - 2h_M \int_{t-h_M}^{t-h(t)} x^T(s)P_2 \dot{x}(s)ds. \end{aligned} \quad (26)$$

By adding (26) into (25) and using Lemma 1, an upper bound of $\dot{V}_4(t)$ is still same as

$$\begin{aligned} \dot{V}_4(t) &\leq h_M^2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T Q_1 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\ &\quad - \frac{1}{\phi(t)} \begin{bmatrix} \int_{t-h(t)}^t x(s)ds \\ \int_{t-h(t)}^t \dot{x}(s)ds \end{bmatrix}^T \left(Q_1 + \begin{bmatrix} 0_n & P_1 \\ P_1 & 0_n \end{bmatrix} \right) \begin{bmatrix} \int_{t-h(t)}^t x(s)ds \\ \int_{t-h(t)}^t \dot{x}(s)ds \end{bmatrix} \\ &\quad - \frac{1}{1-\phi(t)} \begin{bmatrix} \int_{t-h_M}^{t-h(t)} x(s)ds \\ \int_{t-h_M}^{t-h(t)} \dot{x}(s)ds \end{bmatrix}^T \left(Q_1 + \begin{bmatrix} 0_n & P_2 \\ P_2 & 0_n \end{bmatrix} \right) \begin{bmatrix} \int_{t-h_M}^{t-h(t)} x(s)ds \\ \int_{t-h_M}^{t-h(t)} \dot{x}(s)ds \end{bmatrix} \\ &\quad + h_M \begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t-h_M) \end{bmatrix}^T \begin{bmatrix} P_1 & 0_n & 0_n \\ 0_n & -P_1+P_2 & 0_n \\ 0_n & 0_n & -P_2 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t-h_M) \end{bmatrix}, \end{aligned} \quad (27)$$

where $\phi(t) = h(t)/h_M$ which satisfies $0 < \phi(t) < 1$ when $0 < h(t) < h_M$. It should be pointed that when $h(t)=0$, equalities $\int_{t-h(t)}^t x(s)ds = \int_{t-h(t)}^t \dot{x}(s)ds = 0$ is obtained, and when $h(t)=h_M$, equalities $\int_{t-h_M}^{t-h(t)} x(s)ds = \int_{t-h_M}^{t-h(t)} \dot{x}(s)ds = 0$ is obtained. Thus, equality (27) still holds when $0 \leq h(t) \leq h_M$.

From Lemma 2 with $\Lambda_1 \geq 0$ which is defined in (17), a new upper-bound of $\dot{V}_4(t)$ calculated by

$$\begin{aligned} \dot{V}_4(t) &\leq h_M^2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T Q_1 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\ &\quad - \begin{bmatrix} \int_{t-h(t)}^t x(s)ds \\ x(t) - x(t-h(t)) \\ \int_{t-h_M}^{t-h(t)} x(s)ds \\ x(t-h(t)) - x(t-h_M) \end{bmatrix}^T \Lambda_1 \begin{bmatrix} \int_{t-h(t)}^t x(s)ds \\ x(t) - x(t-h(t)) \\ \int_{t-h_M}^{t-h(t)} x(s)ds \\ x(t-h(t)) - x(t-h_M) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} &+ h_M \begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t-h_M) \end{bmatrix}^T \begin{bmatrix} P_1 & 0_n & 0_n \\ 0_n & -P_1+P_2 & 0_n \\ 0_n & 0_n & -P_2 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t-h_M) \end{bmatrix} \\ &= \zeta^T(t) \Xi_4 \zeta(t). \end{aligned} \quad (28)$$

In succession, through Lemma 1, $\dot{V}_5(t)$ can be obtained as

$$\begin{aligned} \dot{V}_5(t) &= \left(\frac{h_M^2}{2} \right)^2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T Q_2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\ &\quad - \frac{h_M^2}{2} \int_{t-h(t)}^t \int_s^t \begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix}^T Q_2 \begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix} du ds \\ &\quad - \frac{h_M^2}{2} \int_{t-h_M}^{t-h(t)} \int_s^t \begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix}^T Q_2 \begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix} du ds \\ &\leq \left(\frac{h_M^2}{2} \right)^2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T Q_2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\ &\quad - \nu_1^T(t) \begin{bmatrix} \frac{1}{\phi_2(t)} Q_2 & 0_{2n} \\ 0_{2n} & \frac{1}{1-\phi_2(t)} Q_2 \end{bmatrix} \nu_1(t), \end{aligned} \quad (29)$$

where $\nu_1(t) = \begin{bmatrix} \int_{t-h(t)}^t \int_s^t x(u) du ds, & \int_{t-h(t)}^t \int_s^t \dot{x}(u) du ds, \\ \int_{t-h_M}^{t-h(t)} \int_s^t x(u) du ds, & \int_{t-h_M}^{t-h(t)} \int_s^t \dot{x}(u) du ds \end{bmatrix}$, $\phi_2(t) = (h(t)/h_M)^2$ which satisfies $0 < \phi_2(t) < 1$ when $0 < h(t) < h_M$.

Note that when $h(t)=0$, equalities $\int_{t-h(t)}^t \int_s^t x(u) du ds = \int_{t-h(t)}^t \int_s^t \dot{x}(u) du ds = 0$ is obtained, and when $h(t)=h_M$, equalities $\int_{t-h_M}^{t-h(t)} \int_s^t x(u) du ds = \int_{t-h_M}^{t-h(t)} \int_s^t \dot{x}(u) du ds = 0$ is obtained. Thus, equality (29) still holds when $0 \leq h(t) \leq h_M$.

Using Lemma 2 with $\Lambda_2 \geq 0$ defined in (17) yields the following inequality:

$$\begin{aligned} \dot{V}_5(t) &\leq \left(\frac{h_M^2}{2} \right)^2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T Q_2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} - \nu_2^T(t) \Lambda_2 \nu_2(t) \\ &= \zeta^T(t) \left(\Xi_5 - \Gamma_{[h(t)]}^T \Lambda_2 \Gamma_{[h(t)]} \right) \zeta(t), \end{aligned} \quad (30)$$

where $\nu_2(t) = \begin{bmatrix} \int_{t-h(t)}^t \int_s^t x(u) du ds, & h(t)x(t) - \int_{t-h(t)}^t x(s)ds, \\ \int_{t-h_M}^{t-h(t)} \int_s^t x(u) du ds, & (h(t)-h_M)x(t) - \int_{t-h_M}^{t-h(t)} x(s)ds \end{bmatrix}$.

Moreover, from Lemma 1, $\dot{V}_6(t)$ has a new upper-bound given by

$$\begin{aligned} \dot{V}_6(t) &\leq \left(\frac{h_M^3}{6} \right)^2 \dot{x}^T(t) Q_3 \dot{x}(t) - \left(\frac{h_M^2}{2} x(t) - \int_{t-h_M}^t \int_s^t x(u) du ds \right)^T \\ &\quad \times Q_3 \left(\frac{h_M^2}{2} x(t) - \int_{t-h_M}^t \int_s^t x(u) du ds \right) \\ &= \zeta^T(t) \Xi_6 \zeta(t), \end{aligned} \quad (31)$$

By combining (22)-(31), an upper-bound of $\dot{V}(t)$ is obtained as follows:

$$\dot{V}(t) \leq \zeta^T(t) \left(\sum_{i=1}^6 \Xi_i + \Omega_{[h(t)]} - \Gamma_{[h(t)]}^T \Lambda_2 \Gamma_{[h(t)]} \right) \zeta(t). \quad (32)$$

From the dynamic equation of the system (16), it is true that $0 = \Theta \zeta(t)$. Therefore, by using Lemma 3, a stability criterion for system (16) can be obtained as

$$(\Theta^\perp)^T \left(\sum_{i=1}^6 \Xi_i + \Omega_{[h(t)]} - \Gamma_{[h(t)]}^T \Lambda_2 \Gamma_{[h(t)]} \right) (\Theta^\perp) < 0. \quad (33)$$

For any matrix U and by using Lemma 4, a new upper-bound of (33) is given by

$$\Upsilon_{[h(t)]} = \begin{bmatrix} \left((\Theta^\perp)^T \left(\sum_{i=1}^6 \Xi_i + \Omega_{[h(t)]} \right) \Theta^\perp \right) & U^T \\ + (\Theta^\perp)^T \Gamma_{[h(t)]} U + U^T \Gamma_{[h(t)]} \Theta^\perp & \\ U & -\Lambda_2 \end{bmatrix} < 0. \quad (34)$$

Note that inequality (34) is affinely dependent on $h(t)$ where $0 \leq h(t) \leq h_M$. So, $\Upsilon_{[h(t)]} < 0$ is satisfied if and only if LMIs (18) and (19). This completes our proof. ■

Remark 2. Compared with the existing works, the main differences of Lyapunov-Krasovskii functionals are $V_1(t)$, $V_2(t)$ and $V_3(t)$. They are used to reduce the conservatism of stability and stabilization criteria for the time-delayed system since the utilized augmented vector in $V_1(t)$ includes not only single and double integral terms of states but also triple integral term $\int_{t-h_M}^t \int_s^t \int_u^t \dot{x}(v) dv du ds$. In the proposed methods, more information on states were utilized by using $V_1(t)$ with this vector as one term of the Lyapunov-Krasovskii functional in (21). By calculating the time-derivative of $V_2(t)$ and $V_3(t)$, some cross terms such as

$$2 \int_{t-h_M}^t \begin{bmatrix} x(s) \\ \dot{x}(s) \\ x(t) - x(s) \\ \int_s^t x(u) du \end{bmatrix} ds N \begin{bmatrix} 0_{n \times 1} \\ 0_{n \times 1} \\ \dot{x}(t) \\ \dot{x}(t) \end{bmatrix}$$

and

$$2 \begin{bmatrix} \int_{t-h(t)}^t x(s) ds \\ h(t)x(t) - \int_{t-h(t)}^t x(s) ds \\ \int_{t-h(t)}^t \int_s^t x(u) du ds \end{bmatrix}^T G \begin{bmatrix} 0_{n \times 1} \\ \dot{x}(t) \\ x(t) du \end{bmatrix}$$

are obtained and utilized in estimating the time-derivative of $V(t)$ by including the integral terms of derivative of state as the integrands of $V_2(t)$ and $V_3(t)$. In next section, it will be shown Theorem 1 can provide larger delay bounds than the works in other literature by comparing maximum delay bounds for a system which has been utilized in many literature to check the conservatism of delay-dependent stability criteria.

3.2 Controller design

Let us consider the following system

$$\dot{x}(t) = Ax(t) + A_d x(t-h(t)) + Bu(t), \quad (35)$$

where $h(t)$ is satisfied with (3) and $u(t)$ is the control input which is defined in (7). In this section, a stabilization method for the system (35) will be introduced based on the result of Theorem 1. For simplicity of matrix representation, some notations are defined as

$$\begin{aligned} \hat{\Lambda}_1 &= \begin{bmatrix} \hat{Q}_1 + \begin{bmatrix} 0_n & \hat{P}_1 \\ \hat{P}_1 & 0_n \end{bmatrix} & \hat{S}_1 \\ \hat{S}_1^T & \hat{Q}_1 + \begin{bmatrix} 0_n & \hat{P}_2 \\ \hat{P}_2 & 0_n \end{bmatrix} \end{bmatrix}, \quad \hat{\Lambda}_2 = \begin{bmatrix} \hat{Q}_2 & \hat{S}_2 \\ \hat{S}_2^T & \hat{Q}_2 \end{bmatrix}, \\ \hat{\Xi}_1 &= \Psi_{11} \hat{R} \Psi_{12}^T + \Psi_{12} \hat{R} \Psi_{11}^T, \\ \hat{\Xi}_2 &= \Psi_{21} \hat{N} \Psi_{21}^T - \Psi_{22} \hat{N} \Psi_{22}^T + \Psi_{23} \hat{N} \Psi_{24}^T + \Psi_{24} \hat{N} \Psi_{23}^T, \\ \hat{\Xi}_3 &= \Psi_{31} \hat{G} \Psi_{31}^T - (1-h_d) \Psi_{32} \hat{G} \Psi_{32}^T + \Psi_{33} \hat{G} \Psi_{34}^T + \Psi_{34} \hat{G} \Psi_{33}^T, \\ \hat{\Xi}_4 &= h_M^2 \Psi_{41} \hat{Q}_1 \Psi_{41}^T + h_M \Psi_{42} \text{diag} \{ \hat{P}_1, \hat{P}_2, -\hat{P}_1, -\hat{P}_2 \} \Psi_{42}^T \\ &\quad - \Psi_{43} \hat{\Lambda}_1 \Psi_{43}^T, \\ \hat{\Xi}_5 &= (h_M^2/2)^2 \Psi_{52} \hat{Q}_2 \Psi_{52}^T, \quad \hat{\Xi}_6 = (h_M^3/6)^2 e_4 \hat{Q}_3 e_4^T - \Psi_{62} \hat{Q}_3 \Psi_{62}^T, \\ \hat{\Xi}_7 &= (\alpha e_1 + e_4) (AXe_1^T + A_d X e_2^T + B Y e_1^T - X e_4^T) \\ &\quad + (AXe_1^T + A_d X e_2^T + B Y e_1^T - X e_4^T)^T (\alpha e_1 + e_4)^T, \\ \hat{\Omega}_{[h(t)]} &= \begin{bmatrix} 0_{n \times 9n} \\ h(t) e_1^T \\ 0_{n \times 9n} \end{bmatrix}^T \hat{G} \begin{bmatrix} 0_{n \times 9n} \\ e_4^T \\ e_1^T \end{bmatrix} + \begin{bmatrix} 0_{n \times 9n} \\ e_4^T \\ e_1^T \end{bmatrix}^T \hat{G} \begin{bmatrix} 0_{n \times 9n} \\ h(t) e_1^T \\ 0_{n \times 9n} \end{bmatrix}, \\ \hat{\Upsilon}_{[h(t)]} &= \begin{bmatrix} \left(\sum_{i=1}^7 \hat{\Xi}_i + \hat{\Omega}_{[h(t)]} \right) & \hat{U}^T \\ + \Gamma_{[h(t)]} \hat{U} + \hat{U}^T \Gamma_{[h(t)]} & \\ \hat{U} & -\hat{\Lambda}_2 \end{bmatrix}, \end{aligned} \quad (36)$$

where other terms were defined in (17).

Now, the following theorem is given by the second result.

Theorem 2. For given scalars $0 \leq h_M, h_d$ and $\alpha > 0$ the system (35) is asymptotically stable for $0 \leq h(t) \leq h_M$ and $\dot{h}(t) \leq h_d$ if there exist positive definite matrices $\hat{R} = [\hat{R}_{ij}]_{5 \times 5} \in \mathbf{R}^{5n \times 5n}$, $\hat{N} \in \mathbf{R}^{4n \times 4n}$, $\hat{G} \in \mathbf{R}^{3n \times 3n}$, $\hat{Q}_1 \in \mathbf{R}^{2n \times 2n}$, $\hat{Q}_2 \in \mathbf{R}^{2n \times 2n}$, $\hat{Q}_3 \in \mathbf{R}^{n \times n}$, symmetric matrices $\hat{P}_1 \in \mathbf{R}^{n \times n}$, $\hat{P}_2 \in \mathbf{R}^{n \times n}$, any matrices $\hat{S}_1 \in \mathbf{R}^{2n \times 2n}$, $\hat{S}_2 \in \mathbf{R}^{2n \times 2n}$, $\hat{U} \in \mathbf{R}^{4n \times 9n}$, $X \in \mathbf{R}^{n \times n}$ and $Y \in \mathbf{R}^{m \times n}$ satisfying the following LMIs:

$$\hat{Y}_{[h(t)=0]} < 0, \quad (37)$$

$$\hat{Y}_{[h(t)=h_M]} < 0, \quad (38)$$

$$\hat{\Lambda}_i \geq 0 (i=1, 2). \quad (39)$$

Then, if the above conditions are feasible, a desired controller gain matrix is obtained by $K = YX^{-1}$.

Proof. For the same Lyapunov-Krasovskii candidate functional in (21) and by (32), the upper-bound of $\dot{V}(t)$ is given as follows:

$$\dot{V}(t) \leq \zeta^T(t) \left(\sum_{i=1}^6 \Xi_i + \Omega_{[h(t)]} - \Gamma_{[h(t)]}^T \Lambda_2 \Gamma_{[h(t)]} \right) \zeta(t) \quad (40)$$

with the inequalities (20).

Then, for any matrices Z_1 and Z_2 , the following zero equality is considered:

$$0 = 2\zeta^T(t) (Z_1 e_1^T + Z_2 e_4^T)^T \hat{\Theta} \zeta(t), \quad (41)$$

where $\hat{\Theta} = (A + BK)e_1^T + A_d e_2^T - e_4^T$.

For $R = [R_{ij}]_{5 \times 5} \in \mathbf{R}^{5n \times 5n}$ and a given scalar $\alpha > 0$, let us define $Z_1 = \alpha R_{11}$ and $Z_2 = R_{11}$. By adding (41) to (40), a new upper-bound of $\dot{V}(t)$ is given by

$$\begin{aligned} \dot{V}(t) &\leq \zeta^T(t) \left(\sum_{i=1}^6 \Xi_i + \Omega_{[h(t)]} - \Gamma_{[h(t)]}^T \Lambda_2 \Gamma_{[h(t)]} \right) \zeta(t) \\ &\quad + 2\zeta^T(t) (\alpha e_1 + e_4) R_{11} (A e_1^T + A_d e_2^T + BK e_1^T - e_4^T) \zeta(t) \\ &< 0. \end{aligned} \quad (42)$$

Let us define $X = R_{11}^{-1} > 0$ and $\Phi_i = \text{diag}\{X, \dots, X\}$. For examples, $\Phi_1 = X$ and $\Phi_2 = \text{diag}\{X, X\}$. Then, to derive more simply, let us define following matrices as $\hat{R} = \Phi_5^T R \Phi_5$, $\hat{N} = \Phi_4^T N \Phi_4$, $\hat{G} = \Phi_3^T G \Phi_3$, $\hat{Q}_1 = \Phi_2^T Q_1 \Phi_2$, $\hat{Q}_2 = \Phi_2^T Q_2 \Phi_2$, $\hat{Q}_3 = \Phi_1^T Q_3 \Phi_1$, $\hat{P}_1 = \Phi_1^T P_1 \Phi_1$, $\hat{P}_2 = \Phi_1^T P_2 \Phi_1$,

$\hat{S}_1 = \Phi_2^T S_1 \Phi_2$, $\hat{S}_2 = \Phi_2^T S_2 \Phi_2$ and $Y = KX$. Then, the following inequalities can be obtained by multiplying Φ_i to pre and post of (42)

$$\begin{aligned} 0 &> \sum_{i=1}^6 \hat{\Xi}_i + \hat{\Omega}_{[h(t)]} - \Gamma_{[h(t)]}^T \hat{\Lambda}_2 \Gamma_{[h(t)]} \\ &\quad + (\alpha e_1 + e_4) (A X e_1^T + A_d X e_2^T + B Y e_1^T - X e_4^T) \\ &\quad + (A X e_1^T + A_d X e_2^T + B Y e_1^T - X e_4^T)^T (\alpha e_1 + e_4) \\ &= \sum_{i=1}^7 \hat{\Xi}_i + \hat{\Omega}_{[h(t)]} - \Gamma_{[h(t)]}^T \hat{\Lambda}_2 \Gamma_{[h(t)]}. \end{aligned} \quad (43)$$

At this time, the conditions are attended as follows: $\Phi_4^T \Lambda_i \Phi_4 = \hat{\Lambda}_i \geq 0 (i=1, 2)$.

Also, by Lemma 4 with any matrix $\hat{U}$, an inequality (43) is equivalent to

$$\hat{Y}_{[h(t)]} = \begin{bmatrix} \left(\sum_{i=1}^7 \hat{\Xi}_i + \hat{\Omega}_{[h(t)]} \right) & \hat{U}^T \\ + \Gamma_{[h(t)]} \hat{U} + \hat{U}^T \Gamma_{[h(t)]} & \\ \hat{U} & \\ - \hat{\Lambda}_2 & \end{bmatrix} < 0. \quad (44)$$

Note that inequality (44) is affinely dependent on $0 \leq h(t) \leq h_M$. So, $\hat{Y}_{[h(t)]} < 0$ is satisfied if and only if (37) and (38). This completes our proof. ■

3.3 Robust H_∞ controller design for stochastic disturbances

In this subsection, the robust H_∞ controller design for the system (6) will be derived based on Theorem 2. The notations are defined as

$$\begin{aligned} \hat{\Xi}_8 &= \lambda \rho_0^2 (\alpha e_1 + e_4) D^T D (\alpha e_1^T + e_4^T), \\ \bar{L}_{[h(t)]} &= \sum_{i=1}^6 \hat{\Xi}_i + \rho_0 \hat{\Xi}_7 + \hat{\Xi}_8 + \hat{\Omega}_{[h(t)]} + \Gamma_{[h(t)]} \hat{U} + \hat{U}^T \Gamma_{[h(t)]}, \\ \bar{Y}_{[h(t)]} &= \begin{bmatrix} \bar{L}_{[h(t)]} & \bar{Y}_{12} & \bar{Y}_{13} & \hat{U}^T & e_1 X C^T \\ * & -\gamma^2 I_p & 0_{p \times n} & 0_{p \times 4n} & 0_{p \times q} \\ * & * & -\lambda I_n & 0_{p \times 4n} & 0_{n \times q} \\ * & * & * & -\hat{\Lambda}_2 & 0_{4n \times q} \\ * & * & * & * & -I_q \end{bmatrix}, \\ \bar{Y}_{12} &= (\rho_0^2 + \sigma^2) (\alpha e_1 + e_4) B_w^T, \\ \bar{Y}_{13} &= (E_d X e_1^T + E_d X e_2^T)^T, \end{aligned} \quad (45)$$

where other terms were defined in (17) and (36).

Theorem 3. For given scalars $0 \leq h_M, h_d$ and $\alpha > 0$ the system (35) is asymptotically stable for $0 \leq h(t) \leq h_M$ and $\dot{h}(t) \leq h_d$ if there exist positive definite matrices $\hat{R} = [\hat{R}_{ij}]_{5 \times 5} \in \mathbf{R}^{5n \times 5n}$, $\hat{N} \in \mathbf{R}^{4n \times 4n}$, $\hat{G} \in \mathbf{R}^{3n \times 3n}$,

$\hat{Q}_1 \in \mathbf{R}^{2n \times 2n}$, $\hat{Q}_2 \in \mathbf{R}^{2n \times 2n}$, $\hat{Q}_3 \in \mathbf{R}^{n \times n}$, symmetric matrices
 $\hat{P}_1 \in \mathbf{R}^{n \times n}$, $\hat{P}_2 \in \mathbf{R}^{n \times n}$, any matrices $\hat{S}_1 \in \mathbf{R}^{2n \times 2n}$,
 $\hat{S}_2 \in \mathbf{R}^{2n \times 2n}$, $\hat{U} \in \mathbf{R}^{4n \times 9n}$, $X \in \mathbf{R}^{n \times n}$, $Y \in \mathbf{R}^{m \times n}$ and a
 positive scalar λ satisfying the following LMIs:

$$\bar{\Upsilon}_{[h(t)=0]} < 0, \quad (46)$$

$$\bar{\Upsilon}_{[h(t)=h_M]} < 0, \quad (47)$$

$$\hat{\Lambda}_i \geq 0 (i=1, 2). \quad (48)$$

Then, if the above conditions are feasible, a desired controller gain matrix is obtained by $K = YX^{-1}$.

Proof. Let us consider the same Lyapunov-Krasovskii candidate functional in (21). By infinitesimal operator $\mathbf{L}$ in [31], a new upper-bound of $\mathbf{L}V(t)$ is obtained by

$$\mathbf{L}V(t) \leq \zeta^T(t) \left(\sum_{i=1}^6 \Xi_i + \Omega_{[h(t)]} - \Gamma_{[h(t)]}^T \Lambda_2 \Gamma_{[h(t)]} \right) \zeta(t), \quad (49)$$

with the inequalities (20).

Then, the following equation is obtained for any matrix Z_1 and Z_2 :

$$0 = 2\zeta^T(t) \rho(t) (Z_1 e_1^T + Z_2 e_4^T)^T \bar{\Theta} [\zeta(t), w^T(t)]^T, \quad (50)$$

where

$$\bar{\Theta} = [(A + BK + DF(t)E_a)e_1^T + (A_d + DF(t)E_d)e_2^T - e_4^T, \rho(t)B_w].$$

Let us define $Z_1 = \alpha R_{11}$ and $Z_2 = R_{11}$. By adding (50) into (49), a new upper-bound of $\mathbf{L}V(t)$ can be obtained as follows:

$$\begin{aligned} \mathbf{L}V(t) &\leq \zeta^T(t) \left(\sum_{i=1}^6 \Xi_i + \Omega_{[h(t)]} - \Gamma_{[h(t)]}^T \Lambda_2 \Gamma_{[h(t)]} \right) \zeta(t) \\ &\quad + 2\zeta^T(t) \rho(t) (Z_1 e_1^T + Z_2 e_4^T)^T \bar{\Theta} [\zeta(t), w^T(t)]^T \\ &= \zeta^T(t) \left(\sum_{i=1}^6 \Xi_i + \Omega_{[h(t)]} - \Gamma_{[h(t)]}^T \Lambda_2 \Gamma_{[h(t)]} \right) \zeta(t) \quad (51) \\ &\quad + 2\rho(t) \zeta^T(t) (\alpha e_1 + e_4) R_{11} \left(\begin{matrix} (A + BK) e_1^T \\ + A_d e_2^T - e_4^T \end{matrix} \right) \zeta(t) \\ &\quad + 2\rho^2(t) \zeta^T(t) (\alpha e_1 + e_4) R_{11} B_w w(t) \\ &\quad + 2\rho(t) \zeta^T(t) (\alpha e_1 + e_4) R_{11} \left(\begin{matrix} DF(t) E_a e_1^T \\ + DF(t) E_d e_2^T \end{matrix} \right) \zeta(t). \end{aligned}$$

The expectation of (51) can be obtained with (4) and (5) as follows:

$$\mathbf{E}\{\mathbf{L}V(t)\} \leq \zeta^T(t) \left(\sum_{i=1}^6 \Xi_i + \Omega_{[h(t)]} - \Gamma_{[h(t)]}^T \Lambda_2 \Gamma_{[h(t)]} \right) \zeta(t)$$

$$\begin{aligned} &+ 2\rho_0 \zeta^T(t) (\alpha e_1 + e_4) R_{11} \left(\begin{matrix} (A + BK) e_1^T \\ + A_d e_2^T - e_4^T \end{matrix} \right) \zeta(t) \\ &+ 2(\rho_0^2 + \sigma^2) \zeta^T(t) (\alpha e_1 + e_4) R_{11} B_w w(t) \quad (52) \\ &+ 2\rho_0 \zeta^T(t) (\alpha e_1 + e_4) R_{11} \left(\begin{matrix} DF(t) E_a e_1^T \\ + DF(t) E_d e_2^T \end{matrix} \right) \zeta(t). \end{aligned}$$

Since $V(0)=0$ and $V(\infty) \geq 0$ are satisfied under the zero initial condition, the following inequality can be obtained from H_∞ performance index in (8) as

$$J \leq \mathbf{E} \left\{ \int_0^\infty [z^T(s)z(s) - \gamma^2 w^T(s)w(s) + \mathbf{L}V(s)] ds \right\} < 0. \quad (53)$$

If the inequality (53) is satisfied, the system (6) is stable with H_∞ performance level γ under the obtained controller (7). With $z^T(t)z(t) = x^T(t)C^T Cx(t)$, (53) is equivalent to the following inequality

$$\begin{aligned} 0 &> \zeta^T(t) \left(\sum_{i=1}^6 \Xi_i + \Omega_{[h(t)]} - \Gamma_{[h(t)]}^T \Lambda_2 \Gamma_{[h(t)]} \right) \zeta(t) \\ &\quad + 2\rho_0 \zeta^T(t) (\alpha e_1 + e_4) R_{11} \left(\begin{matrix} (A + BK) e_1^T \\ + A_d e_2^T - e_4^T \end{matrix} \right) \zeta(t) \\ &\quad + 2(\rho_0^2 + \sigma^2) \zeta^T(t) (\alpha e_1 + e_4) R_{11} B_w w(t) \quad (54) \\ &\quad + 2\rho_0 \zeta^T(t) (\alpha e_1 + e_4) R_{11} \left(\begin{matrix} DF(t) E_a e_1^T \\ + DF(t) E_d e_2^T \end{matrix} \right) \zeta(t) \\ &\quad + x^T(t) C^T Cx(t) - \gamma^2 w^T(t)w(t). \end{aligned}$$

Using Lemma 5, an upper-bound of (54) can be obtained for $\lambda > 0$ as follows:

$$0 > \begin{bmatrix} \zeta(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} \Pi_{[h(t)]} & (\rho_0^2 + \sigma^2)(\alpha e_1 + e_4) R_{11} B_w \\ * & -\gamma^2 I_p \end{bmatrix} \begin{bmatrix} \zeta(t) \\ w(t) \end{bmatrix}, \quad (55)$$

where

$$\begin{aligned} \Pi_{[h(t)]} &= \sum_{i=1}^6 \Xi_i + \Omega_{[h(t)]} - \Gamma_{[h(t)]}^T \Lambda_2 \Gamma_{[h(t)]} \\ &\quad + \rho_0 (\alpha e_1 + e_4) R_{11} ((A + BK) e_1^T + A_d e_2^T - e_4^T) \\ &\quad + \rho_0 ((A + BK) e_1^T + A_d e_2^T - e_4^T)^T R_{11}^T (\alpha e_1 + e_4) \\ &\quad + \lambda \rho_0^2 (\alpha e_1 + e_4) R_{11}^T D^T D R_{11} (\alpha e_1 + e_4) \\ &\quad + \lambda^{-1} (E_a e_1^T + E_d e_2^T)^T (E_a e_1^T + E_d e_2^T) \\ &\quad + x^T(t) C^T Cx(t). \end{aligned}$$

Therefore, inequality (55) is equivalent to the following condition

$$\begin{bmatrix} \Pi_{[h(t)]} & (\rho_0^2 + \sigma^2)(\alpha e_1 + e_4)R_{11}B_w \\ * & -\gamma^2 I_p \end{bmatrix} < 0. \quad (56)$$

Let us define $X = R_{11}^{-1}$, $\Phi_i = \text{diag}\{X, \dots, X\}$, $\hat{R} = \Phi_5^T R \Phi_5$, $\hat{N} = \Phi_4^T N \Phi_4$, $\hat{G} = \Phi_3^T G \Phi_3$, $\hat{Q}_1 = \Phi_2^T Q_1 \Phi_2$, $\hat{Q}_2 = \Phi_2^T Q_2 \Phi_2$, $\hat{Q}_3 = \Phi_1^T Q_3 \Phi_1$, $\hat{P}_1 = \Phi_1^T P_1 \Phi_1$, $\hat{P}_2 = \Phi_1^T P_2 \Phi_1$, $\hat{S}_1 = \Phi_2^T S_1 \Phi_2$, $\hat{S}_2 = \Phi_2^T S_2 \Phi_2$ and $Y = KX$. Then, the following inequalities can be obtained by pre- and post-multiplying (56) by $\text{diag}\{\Phi_9, I_p\}$

$$\begin{bmatrix} \hat{\Pi}_{[h(t)]} & (\rho_0^2 + \sigma^2)(\alpha e_1 + e_4)B_w \\ * & -\gamma^2 I_p \end{bmatrix} < 0. \quad (57)$$

where

$$\begin{aligned} \hat{\Pi}_{[h(t)]} = & \sum_{i=1}^6 \hat{\Xi}_i + \hat{\Xi}_7 + \hat{\Xi}_8 + \hat{\Omega}_{[h(t)]} - \Gamma^T_{[h(t)]} \Lambda_2 \Gamma_{[h(t)]} \\ & + \lambda^{-1} (E_a e_1^T + E_d e_2^T)^T (E_a e_1^T + E_d e_2^T) \\ & + x^T(t) C^T C x(t). \end{aligned}$$

At this time, the conditions are attended as follows: $\hat{\Lambda}_i \geq 0 (i=1, 2)$.

Using Lemma 4 with any matrix $\hat{U}$, (57) is equivalent to

$$\bar{\Upsilon}_{[h(t)]} < 0. \quad (58)$$

Note that inequality (58) is affinely dependent on $h(t)$ where $0 \leq h(t) \leq h_M$. So, $\bar{\Upsilon}_{[h(t)]} < 0$ is satisfied if and only if (46) and (47). This completes our proof. ■

4. Numerical examples

In this section, three numerical examples demonstrate the effectiveness of the proposed criteria.

Example 1. Consider the system (16) with following parameters:

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}. \quad (59)$$

For different h_d , the maximum delay bounds for guaranteeing the asymptotic of system (16) with above parameters are listed in Table 1 which conducts the comparison of the obtained results by Theorem 1 with the previous results. Table 1 shows that when h_d is bigger, the maximum upper-bound of $h(t)$ is smaller. It means that h_d

Table 1. The maximum h_M in Example 1.

h_d	0	0.05	0.1	0.5	3.0
[21]	4.47	3.98	3.60	2.0	-
[1]	4.47	3.98	3.60	2.4	-
[2]	4.47	4.01	3.66	2.33	1.86
[3]	4.60	4.12	3.70	2.33	1.86
[8]	4.66	4.17	3.76	2.12	-
[6]	4.97	4.35	3.86	2.33	1.86
Thm. 1	5.1240	4.4290	3.9523	2.3605	2.2331

affects the stability region of the system. Moreover, it can be seen that Theorem 1 is less conservative than those of the existing results in [1-3], [6], [8] and [12]. Thus, it can be confirm that the proposed Layponove-Krasovskii functional and some utilized techniques in Theorem 1 are effective in reducing the conservatism of stability criterion. It should be noted that when time-delay is constant, the maximum delay bound for guaranteeing the asymptotic stability of system (16) with above parameters is 6.1725.

Example 2. Consider the system (35) with following parameters:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_d = \begin{bmatrix} -1 & -1 \\ 0 & 0.9 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (60)$$

When $h_d=0$, by applying Theorem 2 to the system (35) with the above parameters, the obtained maximum delay bounds and the corresponding controller gain are listed and comparison with the previous results are conducted in Table 2. The obtained maximum allowable delay bounds of $h(t)$ is 50.01 and the controller gain is $K = 10^4 \times [-1.7155, -1.7331]$. At this time, the tuning parameter α is 0.0334 when $h_M=50.01$. In order to confirm one of the results, the simulation result with the time delay $h(t)=50.01$ and the designed controller gain K is illustrated in Fig. 1 which shows the state trajectories goes to zero as time increases. To compare the obtained results by Theorem 2 with some other ones, the various h_M and corresponding controller gain are listed in Table 2 when h_M are 1, 2, 5, 10 and 11. From Table 2, one can confirm that the feasible region of Theorem 2 is much larger than the previous literature [14-16], [24-26].

Table 2. The maximum h_M with $h_d=0$ in Example 2.

Methods	h_M	α	K
[14]	1.40	-	-
[25]	1.51	-	[58.31, -294.9]
[26]	3.20	-	[-7.964, -14.77]
[15]	6.00	-	[-70.18, -77.67]
[24]	7.00	-	[-86.92, -98.21]
[16]	11	-	[-153.1753, -164.7362]
Thm. 2	1	0.97	[-0.6399, -3.4970]
	2	1.10	[-3.3392, -7.1933]
	5	0.49	[-107.3444, -129.7523]
	10	0.27	[-344.1206, -367.8740]
	11	0.20	[-477.8811, -507.9710]
	50.01	0.0334	$10^4 \times [-1.7155, -1.7331]$

Fig. 1. Simulation for Example 2 with the controller gain K when $h_M=50.01$ and $h_d=0$.

Example 3. Consider the system (6) with following parameters:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.3 & 0.3 & -0.0004 & 0.0004 \\ 0.3 & -0.3 & 0.0004 & -0.0004 \end{bmatrix},$$

$$A_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.03 & 0.03 & -0.00004 & 0.00004 \\ 0.03 & -0.03 & 0.00004 & -0.00004 \end{bmatrix},$$

$$B = [0, 0, 1, 0]^T, B_w = 0.5I_4, C = [1, 1, 0, 0], D = I_4,$$

$$F(t) = \sin(5t)[1, 1, 1, 1]^T, E_a = E_d = [0, 0, 0.01, 0]. \quad (61)$$

Also, let us define the disturbances as

$$w(t) = [w_1^T(t), w_2^T(t), w_3^T(t), w_4^T(t)]^T, \quad (62)$$

where

$$w_1(t) = \begin{cases} 0.9, & 1 \leq t \leq 6, \\ 0, & \text{otherwise}, \end{cases}$$

$$w_2(t) = \begin{cases} 0.8(\sin(2\pi 10t) + 1), & 3 \leq t \leq 8, \\ 0, & \text{otherwise}, \end{cases}$$

$$w_3(t) = \begin{cases} 0.6(\cos(2\pi 10t) + 1), & 5 \leq t \leq 9, \\ 0, & \text{otherwise}, \end{cases}$$

$$w_4(t) = \begin{cases} 5(1 - e^{-0.9t}), & 3 \leq t \leq 7, \\ 0, & \text{otherwise}. \end{cases}$$

Moreover, $\rho(t)$ is a stochastic variable of which mean and variance are ρ_0 and σ^2 , respectively.

By applying Theorem 3, the feedback controller gains K can be obtained with minimum γ when $h_M=2$, $h_d=1$, $\alpha=4$, $\rho_0=1$ and various σ^2 in Table 3. More specifically, in order to show the effectiveness of the variance for the

(a) Simulation for $\rho_0 = 1$ and $\sigma^2 = 0$

(b) Simulation for $\rho_0 = 1$ and $\sigma^2 = 1$

(c) Simulation for $\rho_0 = 1$ and $\sigma^2 = 2$

(d) Simulation for $\rho_0 = 1$ and $\sigma^2 = 3$

Fig. 2. Simulation for Example 3 with parameters in Table 3.

system, the results are obtained when σ^2 are 0, 1, 2 and 3. Even though the means of the disturbances are same, minimum values of γ can be obtained differently which

are increased as σ^2 is increased. The results are dependent on variance and are obtained by Theorem 3 which is derived for our new system model. Also, Fig. 2

(a) Simulation for $\rho_0 = 1$ and $\sigma^2 = 1$

(b) Simulation for $\rho_0 = 2$ and $\sigma^2 = 1$

(c) Simulation for $\rho_0 = 3$ and $\sigma^2 = 1$

(d) Simulation for $\rho_0 = 4$ and $\sigma^2 = 1$

Fig. 3. Simulation for Example 3 with parameters in Table 4.

Table 3. Minimum value of γ with $\rho_0=1$, $h_M=2$, $h_d=1$, $\alpha=4$ and various σ^2 in Example 3.

σ^2	γ_{min}	K
0	2.4812	$10^5 \times [-0.6630, -0.6125, -0.0889, -2.8387]$
1	4.9624	$10^5 \times [-0.7346, -0.6787, -0.0985, -3.1455]$
2	7.4437	$10^5 \times [-0.7803, -0.7210, -0.1046, -3.3415]$
3	9.9248	$10^5 \times [-0.9646, -0.8914, -0.1293, -4.1307]$

Table 4. Minimum value of γ with $\sigma^2=1$, $h_M=2$, $h_d=1$, $\alpha=4$ and various ρ_0 in Example 3.

ρ_0	γ_{min}	K
1	4.9624	$10^5 \times [-0.7346, -0.6787, -0.0985, -3.1455]$
2	6.2031	$10^5 \times [-0.7315, -0.6758, -0.0981, -3.1323]$
3	8.2708	$10^5 \times [-0.6317, -0.5835, -0.0847, -2.7048]$
4	10.5453	$10^5 \times [-0.5497, -0.5077, -0.0738, -2.3535]$

is illustrated for state response of the system (6) with stochastic disturbances and the time-delay $h(t) = 0.5h_M(\cos(2h_d t/h_M) + 1)$ by simulation with parameters listed in Table 3. It shows that the closed-loop system (6) is asymptotically stable with H_∞ disturbance attenuation level γ for any time-varying delay $h(t)$ satisfying (3). Even if the mean of disturbances are same, the effect of disturbances increases when the variance σ^2 becomes large.

In addition, in Table 4, minimum values of γ and controller gains K which dependent on mean of $\rho(t)$ are listed when $\sigma^2=1$, $h_M=2$, $h_d=1$, $\alpha=4$ and various ρ_0 . It can be shown that the minimum of γ is increased as ρ_0 is increased which means that the system is influenced more by the disturbances. It is natural that the results are worse when the mean of disturbances increases because disturbances cause poor performances. Fig. 3 shows the simulation results with parameters listed in Table 4 and the time-delay $h(t) = 0.5h_M(\cos(2h_d t/h_M) + 1)$. It shows the state responses of the system (6) when the stochastic disturbances occurred with various ρ_0 . From Fig. 3, the closed-loop system (6) with the controller gain in Table 4 is asymptotically stable with H_∞ disturbance attenuation level γ for any time-varying delay $h(t)$ satisfying (3). Also, when ρ_0 increases, the signal of disturbance become large. This means that system is more affected by disturbance and thus the performance of the closed-loop system (6) is deteriorated.

5. Conclusion

In this paper, the problems of stability analysis, stabilization, and design the delay-dependent robust H_∞ controller for time-delayed systems with parameter uncertainties and stochastic disturbances were investigated. It was assumed that the parameter uncertainties were norm bounded. In order to use stochastic characteristic of the disturbances, the mean and variance for disturbances were utilized. Main results were three separated subsections

which were the stability analysis criterion, controller design criterion and robust H_∞ controller design criterion. Firstly, in Theorem 1, the stability criterion for the nominal systems with time-varying delays was derived by utilizing the newly augmented Lyapunov-Krasovskii functional. Secondly, based on the results of Theorem 1, the new stabilization criterion for the nominal form of the systems was established in Theorem 2. Finally, Theorem 2 was extended to the problem of robust H_∞ controller design for the time-delayed systems with parameter uncertainties and stochastic disturbances in Theorem 3. Moreover, three examples were given to illustrate the effectiveness of the presented criterion.

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