ON ABSOLUTE VALUES OF $Q_K$ FUNCTIONS

GUANLONG BAO, ZENGJIAN LOU, RUISHEN QIAN, AND HASI WULAN

Abstract. In this paper, the effect of absolute values on the behavior of functions $f$ in the spaces $Q_K$ is investigated. It is clear that $g \in Q_K(\partial \mathbb{D}) \Rightarrow |g| \in Q_K(\partial \mathbb{D})$, but the converse is not always true. For $f$ in the Hardy space $H^2$, we give a condition involving the modulus of the function only, such that the condition together with $|f| \in Q_K(\partial \mathbb{D})$ is equivalent to $f \in Q_K$. As an application, a new criterion for inner-outer factorisation of $Q_K$ spaces is given. These results are also new for $Q_p$ spaces.

1. Introduction

Denote by $\partial \mathbb{D}$ the boundary of the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ be the space of functions analytic in $\mathbb{D}$. Throughout this paper, we assume that $K : [0, \infty) \to [0, \infty)$ is a right-continuous and increasing function. A function $f \in H(\mathbb{D})$ belongs to the space $Q_K$ if

$$\|f\|_{Q_K}^2 = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(g(a, z)) \, dA(z) < \infty,$$

where $dA$ is the area measure on $\mathbb{D}$ and $g(a, z)$ is the Green function in $\mathbb{D}$ with singularity at $a \in \mathbb{D}$. By [5, Theorem 2.1], we know that $\|f\|_{Q_K}^2$ is equivalent to

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K \left(1 - |\sigma_a(z)|^2\right) \, dA(z),$$

where $\sigma_a(z) = \frac{a - z}{1 - \bar{a} z}$ is a Möbius transformation of $\mathbb{D}$. If $K(t) = t^p$, $0 \leq p < \infty$, then the space $Q_K$ gives the space $Q_p$ (cf. [11, 13]). In particular, $Q_0$ is the Dirichlet space; $Q_1 = BMOA$, the space of functions with bounded mean oscillation on $\mathbb{D}$; $Q_p$ is the Bloch space for all $p > 1$. See [5] and [6] for more

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results on $Q_K$ spaces. Let $Q_K(\partial \mathbb{D})$ be the space of $f \in L^2(\partial \mathbb{D})$ with
\[
\|f\|^2_{Q_K(\partial \mathbb{D})} = \sup_{I \subset \partial \mathbb{D}} \int_I \int f(\zeta) - f(\eta)^2 K\left(|\zeta - \eta|\right) |d\zeta||d\eta| < \infty.
\]
Clearly, if $K(t) = t^2$, then $Q_K(\partial \mathbb{D})$ is equal to $BMO(\partial \mathbb{D})$, the space of functions having bounded mean oscillation on $\partial \mathbb{D}$ (see [7]).

To study $Q_K$ and $Q_K(\partial \mathbb{D})$, we usually need two constraints on $K$ as follows.

(1.1) \[ \int_0^1 \frac{\varphi_K(s)}{s} ds < \infty \]
and

(1.2) \[ \int_1^\infty \frac{\varphi_K(s)}{s^2} ds < \infty, \]
where
\[ \varphi_K(s) = \sup_{0 < t \leq 1} K(st)/K(t), \quad 0 < s < \infty. \]

If $K$ satisfies (1.2), then $Q_K \subseteq BMOA \subseteq H^2$, where $H^2$ denotes the Hardy space in $\mathbb{D}$ (see [3, 7]). Thus, if $K$ satisfies (1.2), then the function $f \in Q_K$ has its non-tangential limit $\tilde{f}$ almost everywhere on $\partial \mathbb{D}$. We also know that for $f \in H^2$ if $K$ satisfies (1.1) and (1.2), then $f \in Q_K$ if and only if $\tilde{f} \in Q_K(\partial \mathbb{D})$.

Using the triangle inequality, one gets that if $g \in Q_K(\partial \mathbb{D})$, then $|g|$ also belongs to $Q_K(\partial \mathbb{D})$. In general, the converse is not true. Consider
\[ g(e^{it}) = \begin{cases} \log t, & 0 < t < \pi, \\ -\log |t|, & -\pi < t < 0. \end{cases} \]
By [8, p. 66], $|g| \in BMO(\partial \mathbb{D})$, but $g \not\in BMO(\partial \mathbb{D})$. For $g \in H^2$, it is natural to seek a condition which together with $|\tilde{g}| \in Q_K(\partial \mathbb{D})$ is equivalent to $\tilde{g} \in Q_K(\partial \mathbb{D})$. Our main result, Theorem 1.1, is even new for $Q_K$ spaces.

**Theorem 1.1.** Suppose that $K$ satisfies (1.1) and (1.2). Let $f \in H^2$. Set
\[ d\mu_z(\zeta) = \frac{1 - |z|^2}{2\pi|\zeta - z|^2} |d\zeta|, \quad z \in \mathbb{D}, \quad \zeta \in \partial \mathbb{D}. \]

Then the following conditions are equivalent.

(i) $f \in Q_K$.

(ii) $\tilde{f} \in Q_K(\partial \mathbb{D})$.

(iii) $|f| \in Q_K(\partial \mathbb{D})$ and

(1.3) \[ \sup_{z \in \partial \mathbb{D}} \left( \int_{\partial \mathbb{D}} |\tilde{f}(\zeta)|d\mu_z(\zeta) - |f(z)| \right)^2 \frac{K(1 - |\sigma_{\alpha}(z)|^2)}{(1 - |z|^2)^2}dA(z) < \infty. \]

Applying Theorem 1.1, in Section 4, we will show a new criterion for inner-outer factorisation of $Q_K$ spaces.

In this article, the symbol $A \approx B$ means that $A \lesssim B \lesssim A$. We say that $A \lesssim B$ if there exists a constant $C$ such that $A \leq CB$. 

2. Preliminaries

Given \( f \in L^2(\partial \mathbb{D}) \), let \( \hat{f} \) be the Poisson extension of \( f \). Namely,
\[
\hat{f}(z) = \int_{\partial \mathbb{D}} f(\zeta) d\mu_z(\zeta), \quad z \in \mathbb{D}.
\]

We first give the following characterization of \( Q_K(\partial \mathbb{D}) \) spaces. In particular, if \( K(t) = t^p \), \( 0 < p < 1 \), the corresponding result was proved in [12].

**Theorem 2.1.** Suppose that \( K \) satisfies (1.1) and (1.2). Let \( f \in L^2(\partial \mathbb{D}) \). Then \( f \in Q_K(\partial \mathbb{D}) \) if and only if

\[
\sup_{a \in \mathbb{D}} \int_{\partial \mathbb{D}} \left( \int_{\partial \mathbb{D}} |f(\zeta)|^2 d\mu_z(\zeta) - |\hat{f}(z)|^2 \right) \frac{K(1 - |\sigma_z(z)|^2)}{(1 - |z|^2)^2} dA(z) < \infty.
\]

To prove Theorem 2.1, we need the following estimate.

**Lemma 2.2.** Let (1.1) and (1.2) hold for \( K \). If \( s < 1 + c \) and \( 2s + r - 4 \geq 0 \), then
\[
\int_{\mathbb{D}} \frac{K(1 - |\sigma_a(w)|^2)}{(1 - |w|^2)^s(1 - |\bar{w}z|^r)} dA(w) \approx \frac{K(1 - |\sigma_a(z)|^2)}{(1 - |z|^2)^{s+r-2}}
\]
for all \( a, z \in \mathbb{D} \). Here \( c \) is a small enough positive constant which depends only on (1.1) and (1.2).

**Proof.** We point out that
\[
\int_{\mathbb{D}} \frac{K(1 - |\sigma_a(w)|^2)}{(1 - |w|^2)^s(1 - |\bar{w}z|^r)} dA(w) \lesssim \frac{K(1 - |\sigma_a(z)|^2)}{(1 - |z|^2)^{s+r-2}}
\]
was proved in [1]. So we need only to prove the reverse. For any \( z \in \mathbb{D} \), let
\[
E(z, 1/2) = \{ w \in \mathbb{D} : |\sigma_z(w)| < 1/2 \}
\]
be the pseudo-hyperbolic disk. It is well known that
\[
1 - |z| \approx 1 - |w| \approx |1 - \bar{w}z|
\]
for all \( w \in E(z, 1/2) \). Furthermore, by [14, Lemma 4.30], we have that \( |1 - a\bar{w}| \approx |1 - a\bar{z}| \) for all \( a \in \mathbb{D} \) and \( w \in E(z, 1/2) \). Since \( K \) satisfies (1.2), \( K(2t) \approx K(t) \) for all \( t \in (0, 1) \). We obtain
\[
\int_{\mathbb{D}} \frac{K(1 - |\sigma_a(w)|^2)}{(1 - |w|^2)^s(1 - |\bar{w}z|^r)} dA(w) \gtrsim \int_{E(z, 1/2)} \frac{K(1 - |\sigma_a(w)|^2)}{(1 - |w|^2)^s(1 - |\bar{w}z|^r)} dA(w) \approx \frac{K(1 - |\sigma_a(z)|^2)}{(1 - |z|^2)^{s+r-2}},
\]
which gives the desired result. \( \Box \)

**Proof of Theorem 2.1.** For any \( f \in L^2(\partial \mathbb{D}) \), the Littlewood-Paley identity ([7, p. 228]) shows that
\[
(2.2) \quad \int_{\mathbb{D}} |
abla \hat{f}(w)|^2 \log \frac{1}{|w|} dA(w) = \frac{1}{2\pi} \int_{\partial \mathbb{D}} |f(\zeta) - \hat{f}(0)|^2 |d\zeta|.
\]
Replacing \( \hat{f} \) by \( \hat{f} \circ \sigma_z \) in (2.2) for \( z \in D \), one obtains
\[
\int_{\partial D} |f(\zeta)|^2 d\mu_z(\zeta) - |\hat{f}(z)|^2 \approx \int_D |\nabla \hat{f}(w)|^2 (1 - |\sigma_z(w)|^2) dA(w).
\]
Using Fubini’s theorem and Lemma 2.2, we obtain, for all \( a \in D \), that
\[
\int_B \left( \int_{\partial D} |f(\zeta)|^2 d\mu_z(\zeta) - |\hat{f}(z)|^2 \right) \frac{K(1 - |\sigma_a(z)|^2)}{(1 - |z|)^2} dA(z)
\approx \int_B \left( \int_B |\nabla \hat{f}(w)|^2 (1 - |\sigma_z(w)|^2) dA(w) \right) \frac{K(1 - |\sigma_a(z)|^2)}{(1 - |z|^2)^2} dA(z)
\approx \int_B |\nabla \hat{f}(w)|^2 K(1 - |\sigma_a(z)|^2) dA(w).
\]
By [9], we know that \( f \in Q_K(\partial D) \) if and only if
\[
\sup_{a \in D} \int_{\partial D} |\nabla \hat{f}(z)|^2 K(1 - |\sigma_a(z)|^2) dA(z) < \infty.
\]
Therefore, \( f \in Q_K(\partial D) \) if and only if
\[
\sup_{a \in D} \int_{\partial D} \left( \int_{\partial D} |f(\zeta)|^2 d\mu_z(\zeta) - |\hat{f}(z)|^2 \right) \frac{K(1 - |\sigma_a(z)|^2)}{(1 - |z|^2)^2} dA(z) < \infty. \quad \square
\]

By [6], for \( f \in H^2 \), if (1.1) and (1.2) hold for \( K \), then \( f \in Q_K \) if and only if \( \hat{f} \in Q_K(\partial D) \). This, together with Theorem 2.1, gives the following result immediately which was also obtained in [10] by a different method.

**Corollary 2.3.** Suppose that \( K \) satisfies (1.1) and (1.2). Let \( f \in H^2 \). Then \( f \in Q_K \) if and only if
\[
\sup_{a \in D} \int_{\partial D} \left( \int_{\partial D} |\hat{f}(\zeta)|^2 d\mu_z(\zeta) - |f(z)|^2 \right) \frac{K(1 - |\sigma_a(z)|^2)}{(1 - |z|^2)^2} dA(z) < \infty.
\]

### 3. Proof of Theorem 1.1

Recall that \( B \in H(D) \) is called an inner function if \( B \) is bounded in \( D \) and \( |B(\zeta)| = 1 \) for almost every \( \zeta \in \partial D \). An outer function for the Hardy space \( H^2 \) is the function of the form
\[
O(z) = \eta \exp \left( \int_{\partial D} \frac{\zeta + z}{\zeta - z} \log \psi(\zeta) \left| \frac{d\zeta}{2\pi} \right| \right), \quad \eta \in \partial D,
\]
where \( \psi > 0 \) a.e. on \( \partial D \), \( \log \psi \in L^1(\partial D) \) and \( \psi \in L^2(\partial D) \). See [3] for more results on inner and outer functions. Using a technique in [2], we give the proof of Theorem 1.1 as follows.
Proof of Theorem 1.1. Note that (i)\(\Leftrightarrow\) (ii) was proved in [6].

(i)\(\Rightarrow\) (iii). For \(f \in \mathcal{Q}_K\), we have that \(\tilde{f} \in \mathcal{Q}_K(\partial \mathbb{D})\). The triangle inequality gives that \(|\tilde{f}| \in \mathcal{Q}_K(\partial \mathbb{D})\). For any \(z \in \mathbb{D}\), it follows by Hölder’s inequality that

\[
\left( \int_{\partial \mathbb{D}} |\tilde{f}(\zeta)|d\mu_z(\zeta) - |f(z)| \right)^2 \leq \left( \int_{\partial \mathbb{D}} |\tilde{f}(\zeta) - f(z)|d\mu_z(\zeta) \right)^2 \\
\leq \int_{\partial \mathbb{D}} |\tilde{f}(\zeta) - f(z)|^2d\mu_z(\zeta) \\
= \int_{\partial \mathbb{D}} |\tilde{f}(\zeta)|^2d\mu_z(\zeta) - |f(z)|^2.
\]

Since \(f \in \mathcal{Q}_K\), the above estimate, together with Corollary 2.3, gives (1.3).

(iii)\(\Rightarrow\) (i). If \(f \equiv 0\), the result is true. Note that \(f \in H^2\). If \(f \not\equiv 0\), then \(f\) must be of the form \(BO\), where \(B\) is an inner function and \(O\) is an outer function of \(H^2\) (see [3]). By the estimates of \(B\) and \(O\) respectively, Böe [2, p. 237] gave that for any \(z \in \mathbb{D}\),

\[
|f'(z)| \leq \frac{4}{1 - |z|} \left( \int_{\partial \mathbb{D}} \left| \tilde{f}(\zeta) \right| - \left| \tilde{f}(z) \right| d\mu_z(\zeta) + \left| \tilde{f}(z) \right| - |f(z)| \right).
\]

Here we remind that

\[
\tilde{f}(z) = \int_{\partial \mathbb{D}} \tilde{f}(\zeta)d\mu_z(\zeta).
\]

Thus, for any \(a \in \mathbb{D}\), by Hölder’s inequality, we deduce that

\[
\int_{\mathbb{D}} |f'(z)|^2K(1 - |\sigma_a(z)|^2)dA(z) \\
\leq \int_{\mathbb{D}} \left( \int_{\partial \mathbb{D}} \left| \tilde{f}(\zeta) \right| - \left| \tilde{f}(z) \right| d\mu_z(\zeta) \right)^2 \frac{K(1 - |\sigma_a(z)|^2)}{(1 - |z|^2)^2}dA(z) \\
+ \int_{\mathbb{D}} \left( \left| \tilde{f}(z) \right| - |f(z)| \right)^2 \frac{K(1 - |\sigma_a(z)|^2)}{(1 - |z|^2)^2}dA(z) \\
\leq \int_{\mathbb{D}} \left( \int_{\partial \mathbb{D}} \left| \tilde{f}(\zeta) \right| - \left| \tilde{f}(z) \right| d\mu_z(\zeta) \right)^2 \frac{K(1 - |\sigma_a(z)|^2)}{(1 - |z|^2)^2}dA(z) \\
+ \int_{\mathbb{D}} \left( \left| \tilde{f}(z) \right| - |f(z)| \right)^2 \frac{K(1 - |\sigma_a(z)|^2)}{(1 - |z|^2)^2}dA(z) \\
\approx \int_{\mathbb{D}} \left( \int_{\partial \mathbb{D}} \left| \tilde{f}(\zeta) \right|^2d\mu_z(\zeta) - \left( \left| \tilde{f}(z) \right| \right)^2 \frac{K(1 - |\sigma_a(z)|^2)}{(1 - |z|^2)^2}dA(z) \\
+ \int_{\mathbb{D}} \left( \left| \tilde{f}(z) \right| - |f(z)| \right)^2 \frac{K(1 - |\sigma_a(z)|^2)}{(1 - |z|^2)^2}dA(z).
\]

By Theorem 2.1 and (1.3), we have that \(f \in \mathcal{Q}_K\). The proof is complete.
Remark. J. Xiao [12] gave an interesting characterization of $Q_p$ spaces in terms of functions with absolute values. Namely, for $f \in H^2$, if $0 < p < 1$, then $f \in Q_p$ if and only if $|\tilde{f}| \in Q_p(\partial \mathbb{D})$ and

$$\sup_{a \in \mathbb{D}} \int_D \left( \int_{\partial \mathbb{D}} |\tilde{f}(\zeta)| d\mu_z(\zeta) \right)^2 - |f(z)|^2 \left( \frac{(1 - |\sigma_a(z)|^2)^p}{(1 - |z|^2)^2} \right) dA(z) < \infty.$$  

We show that our Theorem 1.1 implies Xiao’s result above. In fact, set $K(t) = t^p$, $0 < p < 1$, in our Theorem 1.1 and Corollary 2.3. Note that

$$\left( \int_{\partial \mathbb{D}} |\tilde{f}(\zeta)| d\mu_z(\zeta) \right)^2 - |f(z)|^2 \geq \left( \int_{\partial \mathbb{D}} |\tilde{f}(\zeta)| d\mu_z(\zeta) - |f(z)| \right)^2$$

and

$$\left( \int_{\partial \mathbb{D}} |\tilde{f}(\zeta)| d\mu_z(\zeta) \right)^2 \leq \int_{\partial \mathbb{D}} |\tilde{f}(\zeta)|^2 d\mu_z(\zeta).$$

Thus, one can obtain Xiao’s result directly.

4. An application to inner-outer factorisation of $Q_K$ spaces

In this section, we will show a new criterion for inner-outer decomposition of $Q_K$ spaces. In fact, an inner-outer factorisation characterization of $Q_K$ spaces has been obtained in [6] as follows.

**Theorem A.** Let $K$ satisfy (1.1) and (1.2) with

$$\tilde{K}(|z|^2) = -\frac{\partial^2 K(1 - |z|^2)}{\partial z \partial \bar{z}}, \quad z \in \mathbb{D}.$$  

Let $f \in H^2$ with $f \not\equiv 0$. Then $f \in Q_K$ if and only if $f = BO$, where $B$ is an inner function and $O$ is an outer function in $Q_K$ for which

(4.1) $\sup_{a \in \mathbb{D}} \int_D |O(z)|^2 (1 - |B(z)|^2) \tilde{K} \left( |\sigma_a(z)|^2 \right) |\sigma'_a(z)|^2 dA(z) < \infty.$

As an application of Theorem 1.1, we obtain the following result.

**Theorem 4.1.** Let $K$ satisfy (1.1) and (1.2) with

$$\tilde{K}(|z|^2) = -\frac{\partial^2 K(1 - |z|^2)}{\partial z \partial \bar{z}}, \quad z \in \mathbb{D}.$$  

Let $f \in H^2$ with $f \not\equiv 0$. Then $f \in Q_K$ if and only if $f = BO$, where $B$ is an inner function and $O$ is an outer function in $Q_K$ for which

(4.2) $\sup_{a \in \mathbb{D}} \int_D |O(z)|^2 (1 - |B(z)|^2) \tilde{K} \left( |\sigma_a(z)|^2 \right) |\sigma'_a(z)|^2 dA(z) < \infty.$

Remark. Theorem 4.1 shows that formula (4.1) in Theorem A can be replaced by the weaker condition (4.2), and this result is also new for $Q_p$ spaces.
Proof. Necessity. This is a direct result from Theorem A.

Sufficiency. Let \( f = BO \) and \( O \in Q_K \). Note that \( O \in Q_K \) is equivalent to \( \tilde{O} \in Q_K(\partial \mathbb{D}) \). By the triangle inequality, one gets \( |\tilde{O}| \in Q_K(\partial \mathbb{D}) \). Hence \( |\tilde{f}| \in Q_K(\partial \mathbb{D}) \). Observe that

\[
\int_{\partial \mathbb{D}} |\tilde{f}(\zeta)|d\mu(z) - |f(z)| = \int_{\partial \mathbb{D}} |\tilde{O}(\zeta)|d\mu(z) - |O(z)| + |O(z)| - |B(z)O(z)|.
\]

(4.3)

Wulan and Ye [10] gave that if \( K \) satisfies (1.1) and (1.2), then for all \( z \in \mathbb{D} \)

\[
\tilde{K}(\|z\|^2) \approx \frac{K(1 - |z|^2)}{(1 - |z|^2)^2}.
\]

(4.4)

By Hölder’s inequality, \( \tilde{O} \in Q_K(\partial \mathbb{D}) \) and Corollary 2.3, we show that for any \( a \in \mathbb{D} \),

\[
\int_{\partial \mathbb{D}} \left( \int_{\partial \mathbb{D}} |\tilde{O}(\zeta)|d\mu(z) - |O(z)| \right)^2 K(1 - |\sigma_a(z)|^2) dA(z) \leq \int_{\partial \mathbb{D}} \left( \int_{\partial \mathbb{D}} |\tilde{O}(\zeta) - O(z)|^2 d\mu(z) \right)^2 K(1 - |\sigma_a(z)|^2) dA(z)
\]

\[
= \int_{\partial \mathbb{D}} \left( \int_{\partial \mathbb{D}} |\tilde{O}(\zeta)|^2 d\mu(z) - |O(z)|^2 \right) K(1 - |\sigma_a(z)|^2) dA(z) < \infty.
\]

Combining the above inequality, (4.2), (4.3) and (4.4), we get

\[
\sup_{a \in \partial \mathbb{D}} \int_{\partial \mathbb{D}} \left( \int_{\partial \mathbb{D}} |\tilde{f}(\zeta)|d\mu(z) - |f(z)| \right)^2 \frac{K(1 - |\sigma_a(z)|^2)}{(1 - |z|^2)^2} dA(z) < \infty.
\]

Applying Theorem 1.1, we get \( f \in Q_K \). The proof is complete. \( \square \)

For \( f \in Q_K \subseteq H^2 \), if we ignore the choice of a constant with modulus one, then \( f \) has a unique decomposition with the form \( f(z) = B(z)O(z) \), where \( B \) is an inner function and \( O \) is an outer function. Combining this with Theorem A and Theorem 4.1, we obtain an interesting result as follows.

**Corollary 4.2.** Suppose that \( K \) satisfies (1.1) and (1.2). Let \( B \) be an inner function and let \( O \) be an outer function in \( Q_K \). Then the following conditions are equivalent.

(i) For some \( p \in [1, 2] \),

\[
\sup_{a \in \partial \mathbb{D}} \int_{\partial \mathbb{D}} |O(z)|^2 (1 - |B(z)|^2)^p \tilde{K} (|\sigma_a(z)|^2) |\sigma'_a(z)|^2 dA(z) < \infty.
\]

(ii) For all \( p \in [1, 2] \),

\[
\sup_{a \in \partial \mathbb{D}} \int_{\partial \mathbb{D}} |O(z)|^2 (1 - |B(z)|^2)^p \tilde{K} (|\sigma_a(z)|^2) |\sigma'_a(z)|^2 dA(z) < \infty.
\]
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