THE GREATEST EXPANDED NUMBER EXPANDED
BY SUMMING OF POWERS OF ITS DIGITS

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Abstract. In this paper, we proved some properties of the greatest expanded numbers, and give the method to determine the greatest expanded numbers and find the integer $x$ for which $S_{q,p}(x) - x$ is the largest. Additionally, we provide an algorithm to find the greatest expanded number.

1. Introduction

For two positive integers $q$, $p (\geq 2)$, let $S_{q,p} : Z^+ \to Z^+$ be the map defined by, to each positive integer $x = \sum_{i=0}^{n} a_ip^i$, assigning the sum of the $q$th powers of its $p$-base digits. In other words, for $x = \sum_{i=0}^{n} a_ip^i, 0 \leq a_i \leq p - 1$,

$$S_{q,p}(x) = S_{q,p}\left(\sum_{i=0}^{n} a_ip^i\right) = \sum_{i=0}^{n} a_i^q.$$

On the sum of powers of digits of an integer, Singh identified fixed points and periodic orbits in the dynamical system defined by summing the $r$th powers of the digits of a positive integer repeatedly [2].

Grundman and Teeple presented a method for determining the fixed points and cycles for $S_{q,p}(x)$ and apply it to $S_{5,p}(x)$ with $2 \leq p \leq 10$ [1]. Grundman and Teeple also proved in [1] that

$$\text{if } x \geq p^{q+1}, \text{ then } S_{q,p}(x) < x. \quad (1.1)$$

Thus there can exist the integers with $x \leq S_{q,p}(x)$ for $x < p^{q+1}$. In fact, such integers exist and are much smaller than $p^{q+1}$.

In this paper, we proved some properties of such numbers and give the method to determine the greatest such numbers and to find the
integer $x$ for which $S_{q,p}(x) - x$ is the largest. Additionally, we provide an algorithm to find the greatest expanded number.

2. Analysis of the difference $S_{q,p}(x) - x$

**Definition 2.1.** 1. A positive integer $x = \sum_{i=0}^{n} a_{i}p^{i}, \text{ } 0 \leq a_{i} \leq p - 1,$ is called an expanded number in $q$th powers in $p$-base if $x \leq S_{q,p}(x)$, where $S_{q,p}(x)$ is the sum of the $q$th powers of its $p$-base digits.
2. The greatest expanded number in $q$th powers in $p$-base is the largest number among all of expanded numbers in $q$th powers in $p$-base, and denoted by $M(S_{q,p})$.
3. A positive integer $x = \sum_{i=0}^{n} a_{i}p^{i}, \text{ } 0 \leq a_{i} \leq p - 1,$ is called a circulated number in $q$th powers in $p$-base if $x = S_{q,p}(x)$.

**Example 2.1.** If $p = 10, q = 2$, the expanded numbers in 2th powers (squares) in 10-base are 1, 2, 3, ..., 99 of all 51 integers, and 1 is the only circulated number in 2th powers (squares) in 10-base.

**Example 2.2.** For all natural numbers $p(\geq 2), q$, 1 is a circulated number in $q$th powers in $p$-base.

From (1.1), we know that if $x \leq S_{q,p}(x)$ then $x < p^{q+1}[1]$. But, if $x = p^{q+1} - 1$, then

$$x = ((p - 1) + 1)^{q+1} - 1$$
$$> (p - 1)^{q+1} + (q + 1)(p + 1)^{q} - 1$$
$$\geq (q + 1)(p + 1)^{q} = S_{q,p}(x).$$

Thus, in any case, $p^{q+1} - 1$ can't be the expanded number. Therefore, we need to analyze the characteristics of the expanded number.

To do this we observe about $S_{q,p}(x) - x$. Since

$$S_{q,p}(x) - x = \sum_{i=0}^{q} a_{i}^{q} - \sum_{i=0}^{q} a_{i}p^{i} = \sum_{i=0}^{q} (a_{i}^{q} - a_{i}p^{i}),$$

for nonnegative integer $i$, let $h_{i}(t) = t^{q} - pt^{i}(0 \leq t \leq p - 1)$. This function $h_{i}$ gives an important clue to find the greatest expanded number.

In the next lemma we can obtain the required tool to analyze $S_{q,p}(x) - x$.

**Lemma 2.1.** When $p, q$ are the integers with $p \geq 3, q \geq 2, p \geq q$, about the integer $i(\leq q)$, let $h_{i}(t) = t^{q} - pt^{i}(0 \leq t \leq p - 1)$. Then

1. if $\kappa = [(q - 1)\log_{p}(p - 1)] + 1$, the maximum of $h_{i}(t)$ is
(a) when \( i < \kappa \), \( h_i(p - 1) \).
(b) when \( i \geq \kappa \), 0.

2. \( i > j \Rightarrow h_i(p - 1) < h_j(p - 1) \).

3. if \( \lambda = \log_p \frac{(p - 1)^q - 1}{p - 2} \),
   (a) when \( i < \lambda \), \( h_i(p - 1) > h_i(1) \).
   (b) when \( i \geq \lambda \), \( h_i(p - 1) \leq h_i(1) \).

4. for \( i, \kappa \leq i < n \), if the real root less than \( p - 1 \) of \( h_i(t) = (p - 1)^q - (p - 1)p^i \) is \( \alpha_i \),
   (a) if \( m_0 = \sqrt[q-1]{\frac{p^i}{q}} \), \( m_{n+1} = \frac{h_i(p - 1) \cdot m_n}{h_i(m_n)} = \frac{(p - 1)^q - (p - 1)p^i}{p^i - m_{n-1}^q} \),
   for \( n \),
   \[ \alpha_i < m_{n+1} < m_n \text{ and } \lim_{n \to \infty} m_n = \alpha_i. \]
   (b) if \( f(t) = h_i(t) - h_i(p - 1), r_0 = 0, r - n + 1 = r_n - \frac{f(r_n)}{f'(r_n)}, \)
   \[ [r_n] = [m_n], \text{ then } [\alpha_i] = [m_n]. \]

Proof. 1. \( h_i'(t) = 0 \Rightarrow t = \sqrt[q-1]{\frac{p^i}{q}}. \)
   When \( i \leq q - 1 \),
   \[ \sqrt[q-1]{\frac{p^i}{q}} \leq \sqrt[q-1]{\frac{p^{q-1}}{q}} = p \sqrt[q-1]{\frac{1}{q}} < p, \]
   so \( h_i(t) \) has a minimum less than one in case of \( 0 \leq t < p \), and doesn’t have a maximum. Therefore the global maximum of \( h_i(t) \)
   is \( \max\{h_i(0) = 0, h_i(p - 1)\} \). (In case of \( t > 0 \), \( h_i(t) \) has one extremum which is a minimum.)
   On the other hand,
   \( h_i(p - 1) < h_i(0) \Leftrightarrow (p - 1)^q - p^i(p - 1) < 0 \)
   \[ \Leftrightarrow p^i > (p - 1)^q - 1 \]
   \[ \Leftrightarrow i > (q - 1) \log_p(p - 1). \]
   Therefore, if \( \kappa = \lfloor (q - 1) \log_p(p - 1) \rfloor + 1 \), the maximum of \( h_i(t) \) is
   when \( i < \kappa \), \( h_i(p - 1) \), when \( i \geq \kappa \), 0.
i) $i < \kappa$, $h_i(p - 1) \geq 0 = h_i(0)$  

ii) $i \geq \kappa$, $h_i(p - 1) < 0 = h_i(0)$

Case 1: $i < \kappa$, $h_i(p - 1)$

Case 2: $i \geq \kappa$, $h_i(p - 1)$

(The graph of $y = h_i(t)$ by the size of $i$)

2. If $i = j + a (a$ is a natural number),

$$h_i(p - 1) = (p - 1)^q - p^j a^a(p - 1)$$

$$= (p - 1)^q - p^j(1 - 1) - (p^j + a - p^j)(p - 1)$$

$$= h_j(p - 1) - (p^j + a - p^j)(p - 1)$$

$$< h_j(p - 1).$$

3.

$$h_i(p - 1) \leq h_i(1) \iff (p - 1)^q - p^j(p - 1) \leq 1 - p^j$$

$$\iff p^j(p - 2) \geq (p - 1)^q - 1$$

$$\iff i \geq \log_p \left( \frac{(p - 1)^q - 1}{p - 2} \right) = \lambda.$$  

4. If $m = \sqrt[q]{ \frac{p^j}{q}}$, $h_i(t) = t^q - p^j t (0 \leq t \leq p - 1)$ is a minimum at $t = m$, curved down. And if $i \geq \kappa$, $h_i(p - 1) < 0$.

In this case, the line connecting the origin and the point $(m, h_i(m))$ meet $y = h_i(p - 1)$ and $x$ coordinate of intersection is $m_1$,

$$m_1 : m = (-h_i(p - 1)) : (-h_i(m)).$$
Greatest expanded number expanded by summing of powers of its digits

Therefore,
\[ m_1 = \frac{-h_i(p-1) \cdot m}{-h_i(m)} = \frac{(p-1)^i - (p-1)^q}{p^i - m^{q-1}} = \frac{q(p-1)}{(q-1)} \left( 1 - \frac{(p-1)^{q-1}}{p^i} \right). \]

In the same way, for a natural number \( n \), if
\[ m_n+1 = \frac{(p-1)^i - (p-1)^q}{m_n^{q-1} - p^i}, \]
\( \alpha_i < m_{n+1} < m_n \) and
\[ \lim_{n \to \infty} m_n = \alpha_i. \]

On the other hand, if
\[ f(t) = h_i(t) - h_i(p-1) = t^q - p^i t + (p-1)p^i - (p-1)^q, \]
\[ f'(t) = qt^{q-1} - p^i. \]

Let \( r_0 = 0 \), by applying the method of Newton,
\[ r_1 = 0 - \frac{-((p-1)^q + (p-1)p^i)}{-p^i} = \frac{-(p-1)^q + (p-1)p^i}{p^i} = p - 1 - \frac{(p-1)^q}{p^i}. \]
\[ r_{n+1} = r_n - \frac{f(r_n)}{f'(r_n)} = r_n - \frac{r_n^q - r_n p^i + (p-1)^i - (p-1)^q}{q r_n^{q-1} - p^i}. \]

At this moment, \( \alpha_i > r_{n+1} > r_n \), and \( \lim_{n \to \infty} r_n = \alpha_i \). Therefore, by repeating the above two processes, if \([r_n] = [m_n], [\alpha_i] = [m_n] \).

3. Properties of the greatest expanded number

Now using Lemma 2.1, let us find the useful way to get the greatest expanded number. The next theorem is useful in case of \( p \geq q \).
Lemma 3.1. If \( x = \sum_{i=0}^{n} a_i p^i (a \neq 0, p > 2, q \geq 2 \) are natural numbers, \( 0 \leq a_i \leq p - 1 \), then

\[
x \leq S_{q,p}(x) \Rightarrow x < (q-1)p^q.
\]

Proof. 1. In case of \( q > p \), since \((q-1)p^q \geq p \cdot p^q = p^{q+1}\), by (1.1), \( x < p^{q+1} \leq (q-1)p^q \).

2. Now let us look into the case of \( q \leq p \). By (1.1), it is enough to consider \( n \leq q \), let \( x = a_q p^q + a_{q-1} p^{q-1} + \cdots + a_1 p + a_0 = \sum_{i=1}^{q} a_i p^i \).

Then

\[
S_{q,p}(x) - x = \sum_{i=0}^{q} a_i^q - \sum_{i=0}^{q} a_i p^i = \sum_{i=0}^{q} (a_i^q - a_i p^i),
\]

\[
S_{q,p}(x) - x - (a_q^q - a_q p^q) = \sum_{i=0}^{q-1} (a_i^q - a_i p^i).
\]

If \( h_i(t) = t^q - p^q \) \((0 \leq t \leq p - 1)\), \( \kappa \), in Lemma 2.1, is

\[
\kappa = [(q-1) \log_q (p-1)] + q < (q-1) + 1 = q.
\]

Thus, by Lemma 2.1, the maximum of \( S_{q,p}(x) - x - (a_q^q - a_q p^q) = \sum_{i=0}^{q-1} (a_i^q - a_i p^i) \) is

\[
\sum_{i=0}^{\kappa-1} h_i(p-1) + \sum_{i=0}^{q-1} h_i(0) = \sum_{i=0}^{\kappa-1} ((p-1)^q - (p-1)p^i) = \kappa (p-1)^q - p^{\kappa} + 1.
\]

Therefore, when \( a_q = q - 1 \), the maximum of \( S_{q,p}(x) - x \) is

\[
(q-1)^q - (q-1)p^q + \kappa (p-1)^q - p^{\kappa} + 1 < (q-1)^q - (q-1)p^q + (q-1)(p-1)^q - p^{\kappa} + 1
\]

\[
< (q-1)^q - (q-1) \{ (p-1)^q + q(p-1)^{q-1} \} + (q-1)(p-1)^q - p^{\kappa} + 1 = (q-1)^q - q(q-1)(p-1)^{q-1} - p^{\kappa} + 1 < 0
\]

\( \therefore 2 \leq q \leq p \), so \((q-1)^{q-1} < q(p-1)^{q-1}\).
Also, if \( h_q(t) = t^q - p^q t \) has a minimum at \( t = m \),

\[
m = \sqrt[q-1]{\frac{p^q}{q}} \geq \sqrt[q-1]{\frac{p^q}{p}} = p,
\]

so \( h_q(t) = t^q - p^q t \) is decreased at \( 0 \leq t \leq p - 1 \). Thus, when \( a_q \geq q - 1 \), \( S_{q,p}(x) - x < 0 \).

Therefore, \( S_{q,p}(x) \geq x \Rightarrow x < (q - 1)p^q \).

\[\begin{align*}
S_{q,p}(x) - x &= (q - 2)^q - (q - 2)p^q + \sum_{i=0}^{q-1} (p - 1)p^i \\
&= (q - 2)^q - (q - 2)p^q + q(p - 1)^q - (p^q - 1) \\
&= (q - 2)^q - (q - 1)p^q + q(p - 1)^q + 1 \\
&= h(p, q).
\end{align*}\]

Thus, a necessary and sufficient condition for \( x = (q - 1)p^q - 1 \) to be the greatest expanded number is \( h(p, q) \geq 0 \).  

\[\Box\]

**Example 3.2.** Since

\[
h(5, 2) = (q - 2)^q - (q - 1)p^q + q(p - 1)^q + 1 \\
= (2 - 2)^2 - (2 - 1) \cdot 5^2 + 2 \cdot (5 - 1)^2 + 1 \\
= 8 > 0,
\]
the sum of the squares of digits in 5 base is
\[(q - 1)p^q - 1 = (2 - 1) \cdot 5^2 - 1 = 24 = 44_{(5)}.
\]

When \( q \) is a constant, \( h(p,q) \) is a monic polynomial of \( p \). For given \( q \), if \( p \) is big enough, \((q - 1)p^q - 1 \) is the greatest expanded number. Here, let us investigate, in case of \( q \) is small, according as how big \( p \) is, \((q - 1)p^q - 1 \) comes to be the greatest expanded number.

Corollary 3.3. 1. If \( p \geq 3 \), \( M(S_{2,p}) \) is \( p^2 - 1 \).
2. If \( p \geq 8 \), \( M(S_{3,p}) \) is \( 2p^3 - 1 \).
3. If \( p \geq 15 \), \( M(S_{4,p}) \) is \( 3p^4 - 1 \).

Proof. 1. \( h(p,2) = -p^2 + 2(p - 1)^2 + 1 = p^2 - 4p + 3 = (p - 2)^2 - 1 \).

If \( p \geq 3 \), \( h(p,2) \geq 0 \). By Corollary 3.2, \( p^2 - 1 \) is \( M(S_{2,p}) \).

2. \( h(p,3) = (q - 2)^q - (q - 1)p^q + q(p - 1)^q + 1 \)
   \[= 1 - 2p^3 + 3(p - 1)^3 + 1 \]
   \[= p^3 - 9p^2 + 9p - 1 \]
   \[= (p - 1)((p - 4)^2 - 15). \]

Thus, if \( p \geq 8 \), then \( h(p,3) > 0 \).

3. \( h(p,4) = 2^2 - 3p^4 + 4(p - 1)^4 + 1 = p^4 - 16p^3 + 24p^2 - 16p + 9, \)
\[
\frac{d}{dp} h(p,4) = 4p^3 - 16 \cdot 3p^2 + 24 \cdot 2p - 16 = s(p^3 - 12p^2 + 12p - 4).
\]

If \( p \geq 15 \), \( \frac{d}{dp} h(p,4) > 0 \), and \( h(15,4) = 1794 > 0 \).

Therefore, if \( p \geq 15 \), \( h(p,4) > 0 \).

When the greatest expanded number is expressed in their base, at the end part of digits of the greatest expanded number, the same digits, \( p - 1 \), are repeated. For example, \( M(S_{2,10}) \) is 99, so 2 times 9, \( M(S_{3,10}) \) is 1999, so 3 times 9, \( M(S_{4,5}) \) is \( 10444_{(5)} \), so 3 times 4. This means the greatest expanded number can be denoted like \( a \cdot p^b - 1(a, b \) are natural numbers.).

Therefore how many times \((p - 1)\) are repeated there like this? In other words, when ‘the greatest expanded number\( M(S_{q,p}) \)’ is denoted like \( a \cdot p^b - 1(a, b \) are natural numbers.), let us think about the maximum of \( b \) (in other words, when \( a \) is not multiple of \( p \), the value of \( b \)).
Theorem 3.4. When the greatest expanded number \((M(S_{q,p}))\) is denoted in base \(p\), the number of \((p - 1)\) at the end part satisfies the next inequality.

\[
\kappa \leq \text{the number of } p - 1 \leq q,
\]

where \(\kappa = \lfloor (q - 1) \log_p (p - 1) \rfloor + 1\).

Proof. In the proof of Lemma 2.1, if \(i < \kappa\), for two natural numbers
\(x = x_1 p^{i+1} + x_0,\)
\(y = x_1 p^{i+1} + p^{i+1} - 1,\)
where \(x_0 = a_i p^i + a_{i-1} p^{i-1} + \cdots + a_1 p + a_0 < p^{i+1} - 1, 0 \leq a_m < p, m = 0, 1, \ldots, i - 1,\) we have

\[
S_{q,p}(x) - x \geq 0 \Rightarrow S_{q,p}(y) - y \geq 0.
\]

\[
\because 0 \leq S_{q,p}(x) - x = S_{q,p}(x_1) - x_1 p^{i+1} + \sum_{n=0}^{i} (a_n^q - a_n p^n)
\]

\[
\leq S_{q,p}(x_1) - x_1 p^{i+1} + \sum_{n=0}^{i} ((p - 1)^q - (p - 1) p^n)
\]

\[
= S_{q,p}(y) - y.
\]

Here, if \(x < y, x\) can’t be \(M(S_{p,q}).\) In other words, \(M(S_{p,q})\) should be the form like \(y.\) At this time, the maximum of \(i\) is \(\kappa - 1.\) Therefore, when \(M(S_{p,q})\) is denoted in \(p\) base, the digits from the position of \(1(=p^0)\) to the position of \(p^{\kappa - 1}\) are all \(p - 1.\)

Therefore, if \(\kappa = \lfloor (q - 1) \log_p (p - 1) \rfloor + 1,\)

\[
\kappa \leq \text{the number of } p - 1 \leq q.
\]

\[\square\]

In case \(p\) is not too much less compared \(q, \kappa = q - 1.\) (Table 1.)

Now, let us look into about the digits whose position is more than \(p\) of the greatest expanded number.

Theorem 3.5. For the integers \(p \geq 3, q \geq 2,\) when
\(\kappa = \lfloor (q - 1) \log_p (p - 1) \rfloor + 1, x = \sum_{i=0}^{n} a_i p^i (a_n \neq 0, 0 \leq a_i \leq p - 1)\) and an integer \(j, \kappa \leq j \leq n,\)

1. if \(x\) is the expanded number, \(y = \sum_{i=j}^{n} a_i p^i + p^\kappa - 1\) is the expanded number.
2. if \(x\) is \(M(S_{q,p}), a_j\) is the maximum of \(b\) to \(\sum_{i=j+1}^{n} a_i p^i + b p^j + p^\kappa - 1\) be the expanded number \((0 \leq b \leq p - 1,\) if \(j = n, \sum_{i=j+1}^{n} a_i p^i = \sum_{i=n+1}^{n} a_i p^i = 0).\)
Proof. 1. By Lemma 2.1, if \( h_i(t) = t^q - p^i t \), \( 0 \leq t \leq p - 1 \). When \( i \geq \kappa \), the maximum of \( h_i(t) \) is 0. Therefore, if \( i \geq \kappa \),
\[
h_i(a_i) = a_i^q - a_ip^i \leq 0.
\]
Thus, \( S_{q,p}(y) - y \geq S_{q,p}(x) - x > 0 \).

2. If the maximum of \( b \) to make \( \sum_{i=j+1}^{n} a_ip^i + bp^j + p^\kappa - 1 \) the expanded number is \( m \), when \( z = \sum_{i=j+1}^{n} a_ip^i + mp^j + p^\kappa - 1 \), \( x \) is the greatest expanded number, so \( x \geq z \). Therefore \( a_j \geq m \).

By (1), \( \sum_{i=j}^{n} a_ip^i + p^\kappa - 1 = \sum_{i=j+1}^{n} a_ip^i + a_jp^i + p^\kappa - 1 \) is the expanded number, \( a_j \leq m \), therefore, \( a_j = m \).

\( \Box \)

Theorem 3.5 tells that each digits of the greatest expanded number could be determined one by one.

Now, for the given numbers \( p, q \), and the sum of \( q \)-powered in \( p \)-base, let us find what number becomes the biggest. In other words, let us find what \( x \) is the biggest in \( S_{q,p}(x) - x \). This also gives us the information concerning the greatest expanded number. We can find the answer about this in the next theorem.

**Theorem 3.6.** When \( \kappa = [(q-1) \log_p(p-1)] + 1 \), \( \lambda = \log_p \left(\frac{(p-1)^{q} - 1}{p-2}\right) \), \( p \geq 3 \),

1. A natural number that \( S_{q,p}(x) - x \) is the maximum is \( p^\kappa - 1 \).
2. When it is denoted in base \( p \), among the natural number \( x \) of which \( n+1 \) is the position number, a natural number that is \( S_{q,p}(x) - x \) is the maximum
   (a) if \( n < \kappa \), \( p^n - 1 \),
   (b) if \( \kappa \leq n < \lambda \), \( (p-1)p^n + p^\kappa - 1 \),
   (c) if \( n = \lambda \) (in case of \( \lambda \) is an integer), \( (p-1)p^n + p^\kappa - 1 \) and \( p^n + p^\kappa - 1 \),
   (d) if \( n > \lambda \), \( p^n + p^\kappa - 1 \).

**Proof.** 1. By Lemma 2.1, when \( h_i(t) = t^q - p^i t \) \( 0 \leq t \leq p - 1 \), if \( i \geq \kappa \), about \( t \) as \( 1 \leq t \leq q - 1 \), \( h_i(t) < h_i(0) = 0 \), if \( i < \kappa \), the maximum of \( h_i(t) \) is \( h_i(p-1) \geq 0 \). Therefore, when \( x = \sum_{i=0}^{\kappa} a_ip^i(a_i \text{ is nonnegative integer that is less than } p) \), the cases \( S_{q,p}(x) - x = \sum_{i=0}^{\kappa} (a_i^q - a_ip^i) \) becomes the maximum are ‘for \( i \) as \( i \geq \kappa \), \( a_i = 0 \), and for \( i \) as \( i < \kappa \), \( a_i = p - 1 \).’ Therefore, the natural number that \( S_{q,p}(x) - x \) is the maximum is \( p^\kappa - 1 \).

2. In the proof of Lemma 2.1, when \( h_i(t) = t^q - p^i t \) \( 0 \leq t \leq p - 1 \), if \( n < \kappa \), \( h_i(p-1) > h_i(0) > h_i(1) \),
if \( \kappa \leq n \leq \lambda \), \( h_i(0) > h_i(p - 1) > h_i(1) \) (but, in case of \( n = \lambda \), \( h_i(0) > h_i(p - 1) = h_i(1) \)),

if \( n \geq \lambda \), \( h_i(0) > h_i(1) \geq h_i(p - 1) \).

From these we get the result. When \( x = \sum_{i=0}^{n} a_ip^i \) (\( a_i \) is nonnegative integer that is less than \( p \)), if \( x \) is the natural number of \((n + 1)\) position, it should be \( a_n \geq 1 \). Therefore, the natural number that \( S_{q,p}(x) - x \) is the maximum is like the result above.

Lemma 3.1 is very meaningful in case that \( p \) is much bigger than given \( q \). Therefore, we need to study the meaningful content to apply this case. About this, let us look into the next theorem.

**Lemma 3.7.** About the integers \( p, q \) as \( p \geq 3, q \geq 2 \), when \( \kappa = [(q - 1) \log_p(p - 1)] + 1, \alpha = \log_p(\kappa(p - 1)^q - p^\kappa + 2), \eta = [\alpha] \), if \( x = \sum_{i=0}^{n} a_ip^i \) \((a_n \neq 0, 0 \leq a_i \leq p - 1)\) is \( M(S_{q,p}) \), \( n = \eta \).

**Proof.** (i) \( \alpha \geq \lambda \),

If \( p^\kappa \leq p(p - 1)^{q-1} \) in \( \kappa = [(q - 1) \log_p(p - 1)] + 1 \),

\[
\lambda = \log_p \left( \frac{(p - 1)^q - 1}{p - 2} \right), \quad \alpha = \log_p(\kappa(p - 10^\eta - p^\kappa + 2),
\]

\[
\frac{p^\kappa - p^\lambda}{p - 2} = (p - 2)(\kappa(p - 1)^q - p^\kappa + 2) - ((p - 1)^q - 1)
\]

\[
\geq (p - 2)(\kappa(p - 1)^q - p(p - 1)^{q-1} + 2) - ((p - 1)^q - 1)
\]

\[
= ((p - 2)((\kappa - 1)(p - 1) - (p - 1))(p - 1)^{q-1} + 2p - 3)
\]

\[
= ((p - 2)(p - 1)(\kappa - 1) - 1)(p - 1)^{q-1} + 2p - 3,
\]

if \( q \geq 3 \), in \( \kappa = [2 \log_p(p - 1)] + 1 \) when \( p \geq 3, \kappa \geq 2 \).

Therefore, when \( p \geq 3, q \geq 3 \), \( \frac{p^\kappa - p^\lambda}{p - 2} \geq 0 \). Also, if \( q = 2 \), since \( \kappa = 1 \),

\[
\frac{p^\kappa - p^\lambda}{p - 2} \geq -(p - 1) + 2(p - 2) + 1 = p - 2 \geq 0 \).

Here, when \( p \geq 3, q \geq 2 \),

\[
\frac{p^\kappa - p^\lambda}{p - 2} \geq 0 \). In other words, \( \alpha \geq \lambda \). .........................(1)

(ii) If there is an integer \( j \) as \( \lambda \leq j \leq, n = [\alpha] = \eta \).

By Theorem 3.5, if \( n \geq \lambda, p^\kappa + p^\kappa - 1 \) becomes the \((n+1)\) position natural number that \( S_{q,p}(y) - y \) is the maximum. In other words,

\[
0 \leq S_{q,p}(x) - x \leq S_{q,p}(y) - y.
\]
In

\[ y = p^n + (p-1)p^{\kappa-1} + \cdots + (p-1)p + (p-1) \]

\[ = p^n + \sum_{i=0}^{\kappa-1} (p-1)p^i, \]

if \( S_{q,p}(y) - y = 1 - p^n + \kappa(p-1)^q - p^\kappa + 1, \)

\( S_{q,p}(y) - y \geq 0, \) \( p^n \leq \kappa(p-1)^q - p^\kappa + 2. \)

In other words, \( n \leq \log_{p,q}(\kappa(p-1)^q - p^\kappa + 2) = \alpha. \) Therefore, if there is an integer \( j \) as \( \lambda \leq j \leq \alpha, \) about such \( j, \) when

\[ y = p^j + (p-1)p^{\kappa-1} + \cdots + (p-1)p + (p-1) \]

\[ = p^j + \sum_{i=0}^{\kappa-1} (p-1)p^i, \]

\( S_{q,p}(y) - y = \kappa(p-1)^q - p^\kappa + 2 - p^j \geq 0. \)

At this time, if \( x = \sum_{i=0}^{n} a_ip^i \) (\( a_n \neq 0, 0 \leq a_i \leq p-1 \)) is the greatest expanded number \((M(S_{q,p})), \) \( n \) is the maximum of such \( j. \) Therefore, if there is an integer \( j \) as \( \lambda \leq j \leq \alpha, \) \( n = [\alpha] = \eta. \) …….. (2)

(iii) And, if there isn’t an integer \( j \) as \( \lambda \leq j \leq \alpha, \) when \( n > \lambda, \) about \( x = \sum_{i=0}^{n} a_ip^i \) (\( a_n \neq 0, 0 \leq a_i \leq p-1 \)), \( S_{q,p}(x) - x < 0. \)

Therefore, if there is not an integer \( j \) as \( \lambda \leq j \leq \alpha, \) to become

\( S_{q,p}(x) - x \geq 0 \) it should be \( n \leq \lambda \leq \alpha. \) …………………………… (3)

By (1), (2), (3), \( n \leq \eta = [\alpha]. \) ……………………………… (4)

(iv) \( \alpha \geq \kappa. \)

\[ p^\kappa - p^\kappa = (\kappa(p-1)^q - p^\kappa + 2) - p^\kappa = \kappa(p-1)^q - 2p^\kappa + 2 \]

\[ \geq \kappa(p-1)^q - 2p(p-1)^{q-1} + 2 \]

\[ = ((\kappa - 2)(p-1) - 2)(p-1)^{q-1} + 2. \]

Since \( p \geq 3, \) if \( \kappa \geq 3, \) \( p^\alpha - p^\kappa > 0, \)

if \( \alpha \geq 4 \) (except \( p = 3, q = 4 \)), since \( \kappa \geq 3, \) \( p^\alpha - p^\kappa > 0. \)

if \( p = 3, q = 4, \) since \( \kappa = 2, \) \( p^\alpha - p^\kappa = 2 \cdot (3-1)^4 - 3 \cdot 2^2 + 2 = 32 - 18 + 2 > 0. \)

if \( q = 2, \) \( \kappa = 2, \)

\[ p^\alpha - p^\kappa = 2(p-1)^3 - 2p^2 + 2 = 2(p-1)((p-1)^2 - (p+1)) \]

\[ = 2(p-1)(p^2 - 3p) \geq 0. \]

if \( q = 2, \) since \( \kappa = 1, \)

\[ p^\alpha - p^\kappa = (p-1)^2 - 2p + 2 = p^2 - 4p + 3 = (p-1)(p-3) \geq 0. \]

From the above, since \( p^\alpha - p^\kappa \geq 0, \) \( \alpha \geq \kappa. \) …………………………. (5)
(v) By (5), since \( \eta \geq \kappa \), let us look into

\[ x = p^n + p^\kappa - 1 = p^n + \sum_{i=0}^{\kappa-1} (p - 1)p^i. \]

Since \( p^n = p^{[\alpha]} \leq p^\kappa = \kappa(p - 1)^q - p^\kappa + 2 \),

Here, \( n \geq \eta \). \( \square \)

By (3), \( n = \eta \).

The above lemma decides which digit is the maximum of \( x \leq S_{q,p}(x) \). Compared with the given \( q \), when \( p \) is big enough, \( q = \eta \), in that case it is more meaningful to be explained by Lemma 3.1. If not, this theorem is more meaningful.

And, in the proof of Lemma 3.7, in case \( \alpha = \log_p(\kappa(p - 1)^q - p^\kappa + 2) \) becomes \( n = \alpha \), in (ii) since \( S_{q,p}(y) - y = 0 \), \( (n + 1)\text{digit} \) which is not \( x \) is the circulated number and the greatest expanded number. When \( (q, p) \) is \((3, 2), (3, 3), (6, 3)\), it comes to be.

In the above proof of Lemma 3.7, instead of (iv), it can be explained more simply 'though there is the case of \( \eta = \kappa - 1 \), when \( x = p - 1 \) through \( S_{q,p}(x) - x > 0 \), it should be \( n \geq \eta' \). However, \( \alpha \geq \kappa \) is also valuable to know, so it is proved like the above.

Here let us denote the range of \( \eta \) with \( p, \kappa \).

Since \( \kappa^{\alpha-1} \leq (p - 1)q^{-1} < p^\kappa \),

\( \kappa(p - 1)p^\kappa - 2 \leq \kappa(p - 1)^q - p^\kappa + 2 < \kappa(p - 1)p^\kappa - p^\kappa + 2, \)

\( \kappa p^\kappa - \kappa p^{\kappa-1} - 2 \leq \kappa(p - 1)^q - p^\kappa + 2 < \kappa p^{\kappa+1}(\kappa + 1)p^\kappa + 2, \)

\( p^{\kappa-1}(-\kappa - p) < \kappa(p - 1)^q - p\kappa + 2 < \kappa p^{\kappa+1}, \)

\( \log_p(\kappa - p) + \kappa - 1 < \log_p \kappa + \kappa + 1, \)

\( [\log_p(\kappa - p)] + \kappa - 1 \leq \eta \leq [\log_p \kappa] + \kappa + 1. \)

In other words, the digit of the greatest expanded number is less than or equal to \( [\log_p \kappa] + \kappa + 2, \), \( \eta - \kappa \) is less than or equal to \( [\log_p \kappa] + 1 \).

**Example 3.3.** In case of \( p = 3, q = 10 \),

\( \kappa = [(q - 1) \log_p(p - 1)] + 1 = [9 \log_3 2] + 1 = [5.678] + 1 = 6, \)

\( \eta = [\log_p(\kappa(p - 1)^q - p^\kappa + 2)] = [7.83] = 7. \)

Thus, \( M(S_{10,3}) \) is \( \eta + 1 = 8 \text{ digit number} \) (Actually, \( M(S_{10,3}) \) is \( 3^8 - 1 = 2222222223 \)).

Similarly to Corollary 3.2, let us find in what case \( p^{n+1} - 1 \) becomes the greatest expanded number.
Since \( S_{q,p}(p^{q+1} - 1) = (\eta + 1)(p - 1)^q \),

\[
S_{q,p}(p^{q+1} - 1) - (p^{q+1} - 1) \geq 0 \iff (\eta + 1)(p - 1)^q - (p^{q+1} - 1) \geq 0
\]
\[\iff p^{q+1} \geq (\eta + 1)(p - 1)^q + 1.\]

However, the case that the above inequality is satisfied is not common, so it is useful only in some cases. And, when in the above the value of \( \kappa \) and the greatest expanded number are denoted in base \( p \), the number of \( (p - 1) \) and the \((\eta + 1)\) digit number of the greatest expanded number at the end are shown like the next.

Table 1. The value of \( \kappa \) according to \( p \), \( q \) and the amount and digit of \((p - 1)\) at the end of the greatest equally expanded number.

<table>
<thead>
<tr>
<th>( p )</th>
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<th>4</th>
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<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
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<td>( \text{am}_{\text{dig}} )</td>
<td>( \text{am}_{\text{int}} )</td>
<td>( \text{am}_{\text{dig}} )</td>
<td>( \text{am}_{\text{int}} )</td>
<td>( \text{am}_{\text{dig}} )</td>
<td>( \text{am}_{\text{int}} )</td>
<td>( \text{am}_{\text{dig}} )</td>
<td>( \text{am}_{\text{int}} )</td>
<td>( \text{am}_{\text{dig}} )</td>
</tr>
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<td>2</td>
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<td>2</td>
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<td>2</td>
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<td>3</td>
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<td>3</td>
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</tr>
</tbody>
</table>

Table 1. The value of \( \kappa \) according to \( p, q \) and the amount and digit of \((p - 1)\) at the end of the greatest equally expanded number.

And, when we clarify the above Lemma 2.1, Lemma 3.1, Lemma 3.7 the next theorem is completed.

**Theorem 3.8.** About the integer \( p, q \) as \( p \geq 3, q \geq 2 \), when \( \kappa = [(q - 1)\log_p(p - 1)] + 1, \alpha = \log_p([p - 1]^q - p^\kappa + 2], \eta = [\alpha], \)

if \( x = \sum_{i=0}^n a_i p^i (a_n \neq 0, 0 \leq a_i \leq p - 1) \) is the greatest expanded number, \( \kappa \leq n = \eta \leq q \), and the next makes sense.

1. About the integer \( i \) as \( 0 \leq i < \kappa, a_i = p - 1 \).
2. About the integer \( i \) as \( \kappa \leq i \leq n \), if the real root that is less than \( p - 1 \) of \( t^q - p^i t = (p - 1)^q - (p - 1)p^i \) is \( \alpha_i \), \( a_i = p - 1 \) or \( a_i \leq \alpha_i \).
3. In case of \( n = q, a_n < q - 1 \).

**Proof.** 1. By Lemma 3.1, Lemma 3.7, \( \kappa \leq n = \eta \leq q \).
2. In case of \( i \geq \kappa, h_i(t) = t^q - p^i t(0 \leq t \leq p - 1) \) is the minimum

at \( t = [q - 1] \frac{p^i}{q} \). Thus, if the real root that is less than \( p - 1 \) of \( t^q - p^i t = (p - 1)^q - (p - 1)p^i \) is \( \alpha_i \), when \( a_i < t < p - 1 \), since
Greatest expanded number expanded by summing of powers of its digits

\[ h_i(t) = t^q - p^i t < h_i(p - 1), \]

\[ x = A \cdot p^{i+1} + a_i p^i + B, \quad y = A \cdot p^{i+1} + (p - 1)p^i + B \] (A is the natural number, B is the nonnegative integer less than \( p^i \), when \( a_i \) is the integer as \( \alpha_i < a_i < p - 1 \), if \( y \) is not the expanded number,

\[ S_{q,p}(x) - x = S_{q,p}(y) - y + (a^q_i - p^i a_i) - ((p - 1)^q - p^i(p - 1)) < 0. \]

In other words, if \( y \) is not the expanded number, \( x \) is also not the expanded number.

3. By Lemma 3.1, \( a_n < q - 1 \).

The above Theorem 3.8 explains if \( x \) is the greatest expanded number,

\[ p^\eta + \sum_{i=0}^{\kappa-1} (p - 1)p^i \leq x < \min\{(q - 1)p^\eta, p^{\eta+1}\}. \]

4. Finding algorithm of the greatest expanded number

To find \( M(S_{q,p}) \), we can check the natural number less than \( x = (q - 1)p^\eta - 1 \), or \( x = p^{\eta+1} - 1 \), but in case either \( p \) or \( q \) is big, actually it is not easy to calculate them, therefore, we need to find the way to find the greatest expanded number using the above characteristics.

First, by Corollary 3.2, in case of \( p \geq q \) and \( h(p, q) > 0 \), \( (q - 1)p^\eta - 1 \) becomes just the greatest expanded number. And, by Lemma 3.7 the digit of the greatest equally expanded number can be calculated. Of course, when \( \eta = q \) and \( p > q \) the highest digit value is less than \( (q - 1) \).

Also, by Theorem 3.4, the \( \kappa \) digit at the end can be decided by \( (p - 1) \). And, by using Theorem 3.5, from the value of \( p^\kappa \) digit to \( p^\eta \) digit, the value of \( \eta - \kappa + q \) digit only have to be decided in turn.
Finding algorithm of the greatest expanded number

1. Calculate $\kappa = [(q - 1) \log_p(p - 1)] + 1$, $\alpha = \log_p(\kappa(p - 1)^q - p^\kappa + 2)$, $\eta = \lceil \alpha \rceil$.

2. By Theorem 3.8 check the natural number $x$ as $x \leq p^{\eta+1} - 1$.
   In case of $\eta = q$ and $p \geq q$, check the natural number $x$ as $x \leq (q - 1)p^q - 1$.

3. If $x = \sum_{i=\kappa}^{\eta} a_i p^i + b p^\kappa - 1$, and it is decided by the order of $a_\eta, a_{\eta-1}, a_{\eta-2}, \ldots, a_\kappa$.
   (a) in case it is decided from $a_\eta$ to $a_{j+1}$, the maximum of $b$ to make $\sum_{i=j+1}^{\eta} a_i p^i + b p^\kappa - 1$ become the expanded number is decided by $a_j$.
   (b) We can substitute one by one one by one and check from $p - 1$ of $b$, but in case of $(p - 1)$ if it does not exist, calculate $\alpha_i$ in Lemma 2.1, and check from $\lceil \alpha_i \rceil$.

The above way is good to apply by using computer, and when $p \geq q$, in case of calculating it directly, like Example 3.2, it is useful to check $h(p,q)$ first.

In case that $\alpha_i$ is not used in the above way, since the value of $p^\eta$ digit is more than 1, the case that the checking time is the maximum is that the greatest expanded number is $p^\eta + p^\kappa - 1$, and the number of times is $\eta = q$ and when $p \geq q$, the number of times is $(q - 2) + p(q - \kappa)$, in the other case the number of times is $(p - 1)(\eta - \kappa + 1)$. Actually we can find the greatest expanded number by much less checking.

Example 4.1. Find $M(S_{10,20})$.
Here, we may not apply $a_\eta$.

First, we can calculate,
$\kappa = [(q - 1) \log_p(p - 1)] + 1 = [9 \cdot \log_{20} 19] + 1 = 9$.
$\eta = [10.56] = 10$. Therefore, $\eta = q$.
Thus, the greatest expanded number to solve is $x = a_{10} \cdot 20^{10} + a_9 \cdot 20^9 + 20^9 - 1$. Also, $a_{10} \leq q - 2 = 8$. Now, when $s_1 = b_1^{10} - 20^{10}b_1 + 9 \times 19^{10} - (20^9 - 1)$ is denoted, substitute 8, 7, \ldots to $b_1$ in turns and check whether $’s_1 > 0’$ exists or not. Then, when $b_1 = 5$, it exists first. Therefore, $a_{10} = 5$.

When $s_2 = b_2^9 - 20^9 b_2 + 5^{10} - 5 \times 20^{10} + 9 \times 19^{10} - (20^9 - 1)$ is denoted, substitute 19, 18, \ldots to $b_2$ in turns and check whether $’s_2 > 0’$ exists or not. Then, when $b_2 = 6$, it exists first. Therefore, $a_9 = 6$. Therefore $M(S_{10,20}) = 5 \cdot 20^{10} + 6 \ldots 20^{9} + 20^9 - 1 = 5 \cdot 20^{10} + 7 \cdot 20^9 - 1 = 5.7 \times 20^{10} - 1_{(20)}$. (We can find $(8-5+1)+(19-6+1) = 18$ the number of times by checking.)
When we apply $\alpha_i$ above, when $b_2 = 19$, ‘$s_2 > 0$’ doesn’t exist.

If $h_9(t) = t^{10} - 20^9 t$, $h_9(t)$ is the minimum at $m = \sqrt[9]{\frac{20^9}{10}}$. $m_1 = \frac{h_i(p - 1) \cdot m}{h_i(m)} = \frac{q(p - 1)}{q - 1} \left(1 - \frac{(p - 1)^{q-1}}{p^q}\right) \approx 7.806$

$r_1 = p - 1 - \frac{(p - 1)^q}{p^q} \approx 7.025$.

Since $[r_1] = [m_1] = 7$, $[\alpha_9 = 7]$.

Instead of finding $r_1$, by checking $t = [m_1] = 7$, you can find $[\alpha_9]$. Therefore you can start checking from the case of $b_2 = 7$. Including the calculation of $m_1$, $r_1$, you can find the greatest expanded number by checking the number of times $(8 - 5 + 1) + 2 + 2 = 9$.

The table shows the greatest expanded numbers obtained from the algorithm. (Each number is denoted in its base.)
Table 2. The greatest expanded number according to $p$, $q$.

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<th>$n$</th>
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</table>

Note: The values are expanded numbers according to the given $p$ and $q$ values.
Greatest expanded number expanded by summing of powers of its digits

References


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