A REMARK OF ODD DIVISOR FUNCTIONS AND WEIERSTRASS $\wp$ -FUNCTIONS

KWANGCHUL LEE, DAEYEOL KIM, AND GYEONG-SIG SEO*

Abstract. In this paper, we study the convolution sums involving odd divisor functions, and their relations to Weierstrass $\wp$ -functions.

1. Introduction

For $N, m, d \in \mathbb{N}$ with $r, s \in \mathbb{N} \cup \{0\}$, we define some necessary divisor functions and infinite sums for later use, which also appear in many areas of number theory:

$$\sigma_s(N) = \sum_{d|N} d^s, \quad \sigma_{s,r}(N; m) = \sum_{\substack{d|N \\ d \equiv r \pmod{m}}} d^s,$$

$$\sigma_{s,oo}(n) = \sum_{\substack{d|n \\ d, \frac{n}{d} \text{ odd}}} d^s,$$

$$S_1 = \sum_{N \text{ odd}} \sigma_{1,1}(N; 2)q^N, \quad S_2 = \sum_{N \geq 2 \text{ even}} \sigma_{1,1}(N; 2)q^N.$$

We also make use of the following convention:

$$\sigma_s(N) = 0 \quad \text{if } N \notin \mathbb{Z} \text{ or } N < 0.$$

The history of the convolution sums involving the divisor functions $\sigma_s(N)$ goes back to Glaisher [6, 7, 8]. Many recent works on convolution formulas for divisor functions can be found in B. C. Berndt [3]; H. Hahn [9]; J. G. Huard, Z. M. Ou, B. K. Spearman, and K. S. Williams [10]; G. Melfi [16]; B. Cho, D. Kim, and J.-K. Koo [4, 5]; and A.

Received November 27, 2013. Accepted December 19, 2013.

2010 Mathematics Subject Classification. 11A67.

Key words and phrases. Weierstrass $\wp(x)$ functions, divisor functions, convolution sums.

*Corresponding author

Alaca, S. Alaca, and K. S. Williams [1, 2].

The paper is organized as follows. In section 2, we derive the convolution sums of the odd divisor function $\sigma_{1,1}(n; 2)$. In section 3, we obtain the algebraic relations of infinite products formular for $\sigma_5(n)$.

2. Preliminaries

Proposition 2.1. ([15, p.4]) *Let n be an odd positive integer. Let $f : \mathbb{Z} \rightarrow \mathbb{C}$ be an odd function. Then,*

$$\begin{aligned} & \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c)) \\ &= \frac{1}{8} \sum_{d|n} (d^2 - 1)F(d) - 3 \sum_{\substack{ax < n \\ a, x \text{ odd}}} \sigma_1(o(n-ax))F(a), \end{aligned}$$

where $o(m)$ is an odd part of $n \in \mathbb{N}$, that is, if $m = 2^\alpha n$ with $(n, 2) = 1$. And then, $o(m) = n$.

Proposition 2.2. ([13, Theorem 3.7]) *For $N \geq 3$,*

$$\begin{aligned} & \sum_{\substack{k+l+h=N \\ k,l,h>0}} \sigma_{1,1}(k; 2)\sigma_{1,1}(l; 2)\sigma_{1,1}(h; 2) \\ &= -\frac{29}{768}\sigma_5(N) + \frac{1}{768}\sigma_5(2N) - \frac{11}{192}\sigma_3(N) + \frac{1}{192}\sigma_3(2N) + \frac{1}{192}\sigma_{1,1}(N; 2). \end{aligned}$$

Lemma 2.3. *Let $N \geq 3$. Then,*

$$\begin{aligned} & \sum_{m_1+m_2+m_3=N} \sigma_{1,oo}(m_1; 2)\sigma_{1,oo}(m_2; 2)\sigma_{1,oo}(m_3; 2) \\ &= \frac{1}{192}(\sigma_5(N) - \sigma_3(N)) - \frac{1}{8} \sum_{\substack{ax < N \\ a, x \text{ odd}}} a^3 \sigma_{1,1}(N-ax; 2). \end{aligned}$$

Proof. Put $F(x) = x^3$ in Proposition 2.1. Let $m_1 = ax$, $m_2 = by$ and $m_3 = cz$. Then, we obtain Lemma 2.3. $\square$

Lemma 2.4. *Let $N \geq 5$ be an odd positive integer. Then,*

$$\begin{aligned}
& \sum_{\substack{m_1+m_2+m_3=N \\ m_1 \text{ odd} \\ m_2, m_3 \text{ even}}} \sigma_{1,1}(m_1; 2) \sigma_{1,1}(m_2; 2) \sigma_{1,1}(m_3; 2) \\
&= \sum_{\substack{m'_1+m'_2+m'_3=N \\ m'_2 \text{ odd} \\ m'_1, m'_3 \text{ even}}} \sigma_{1,1}(m'_1; 2) \sigma_{1,1}(m'_2; 2) \sigma_{1,1}(m'_3; 2) \\
&= \sum_{\substack{m''_1+m''_2+m''_3=N \\ m''_3 \text{ odd} \\ m''_1, m''_2 \text{ even}}} \sigma_{1,1}(m''_1; 2) \sigma_{1,1}(m''_2; 2) \sigma_{1,1}(m''_3; 2).
\end{aligned}$$

Proof. Consider,

$$S_1 = \{(m_1, m_2, m_3) \in \mathbb{N}^3 \mid m_1 : \text{odd}, m_2, m_3 : \text{even}, m_1+m_2+m_3 = N\}$$

and

$$S_2 = \{(m'_1, m'_2, m'_3) \in \mathbb{N}^3 \mid m'_2 : \text{odd}, m'_1, m'_3 : \text{even}, m'_1+m'_2+m'_3 = N\}.$$

We easily check that

$$f : S_1 \rightarrow S_2, \text{ bijective, via } (m_1, m_2, m_3) \mapsto (m'_2, m'_1, m'_3).$$

And thus,

$$\begin{aligned}
& \sum_{\substack{m_1+m_2+m_3=N \\ m_1 \text{ odd} \\ m_2, m_3 \text{ even}}} \sigma_{1,1}(m_1; 2) \sigma_{1,1}(m_2; 2) \sigma_{1,1}(m_3; 2) \\
&= \sum_{\substack{m'_1+m'_2+m'_3=N \\ m'_2 \text{ odd} \\ m'_1, m'_3 \text{ even}}} \sigma_{1,1}(m'_1; 2) \sigma_{1,1}(m'_2; 2) \sigma_{1,1}(m'_3; 2).
\end{aligned}$$

And, let,

$$S_3 = \{(m''_1, m''_2, m''_3) \in \mathbb{N}^3 \mid m''_3 : \text{odd}, m''_1, m''_2 : \text{even}, m''_1 + m''_2 + m''_3 = n\}.$$

Then, by the similar method, we know that

$$\begin{aligned}
& \sum_{\substack{m_1+m_2+m_3=N \\ m_1 \text{ odd} \\ m_2, m_3 \text{ even}}} \sigma_{1,1}(m_1; 2) \sigma_{1,1}(m_2; 2) \sigma_{1,1}(m_3; 2) \\
&= \sum_{\substack{m_1''+m_2''+m_3''=N \\ m_3'' \text{ odd} \\ m_1'', m_2'' \text{ even}}} \sigma_{1,1}(m_1''; 2) \sigma_{1,1}(m_2''; 2) \sigma_{1,1}(m_3''; 2).
\end{aligned}$$

So, we can get the Lemma 2.4. $\square$

Lemma 2.5. *Let $N \geq 3$ be odd. Then,*

$$\sum_{\substack{ax < N \\ a, x \text{ odd}}} a^3 \sigma_{1,1}(N - ax; 2) = \frac{1}{96} (\sigma_5(N) - 4\sigma_3(N) + 3c(N)),$$

where

$$\sum_{N=1}^{\infty} c(N) q^N = q \prod_{N=1}^{\infty} (1 - q^{2N})^{12}.$$

Proof. By [12, Theorem 1.3] and Lemma 2.4,

$$\begin{aligned}
(1) \quad & \sum_{m_1+m_2+m_3=N} \sigma_{1,oo}(m_1; 2) \sigma_{1,oo}(m_2; 2) \sigma_{1,oo}(m_3; 2) \\
&= \sum_{\substack{n+l+m-1=N \\ n, l, m \geq 1}} \sigma_{1,1}(2n-1; 2) \sigma_{1,1}(2l-1; 2) \sigma_{1,1}(2m-1; 2).
\end{aligned}$$

The relation (1) yields the following arithmetic result;

$$\frac{1}{192} (\sigma_5(N) - \sigma_3(N)) - \frac{1}{8} \sum_{\substack{ax < N \\ a, x \text{ odd}}} a^3 \sigma_{1,1}(N - ax; 2) = \frac{1}{256} (\sigma_5(N) - c(N)).$$

Thus, we are keeping in simple, and then we get the Lemma 2.5. $\square$

Theorem 2.6. *If $N \geq 5$ be odd, then*

$$(2) \quad \sum_{\substack{m_1+m_2+m_3=N \\ m_1 \text{ odd} \\ m_2, m_3 \text{ even}}} \sigma_{1,1}(m_1; 2) \sigma_{1,1}(m_2; 2) \sigma_{1,1}(m_3; 2) \\ = \frac{1}{2304} (\sigma_5(N) - 8\sigma_3(N) + 4\sigma_1(N) + 3c(N)),$$

where

$$\sum_{N=1}^{\infty} c(N) q^N = q \prod_{N=1}^{\infty} (1 - q^{2N})^{12}.$$

Proof. By Proposition 2.2, we get

$$(3) \quad \sum_{\substack{m_1+m_2+m_3=N \\ N \text{ odd}}} \sigma_{1,1}(m_1; 2) \sigma_{1,1}(m_2; 2) \sigma_{1,1}(m_3; 2) \\ = \sum_{\substack{m_1+m_2+m_3=N \\ m_1, m_2, m_3 \text{ odd}}} \sigma_{1,1}(m_1; 2) \sigma_{1,1}(m_2; 2) \sigma_{1,1}(m_3; 2) \\ + \sum_{\substack{m_1+m_2+m_3=N \\ m_1 \text{ odd} \\ m_2, m_3 \text{ even}}} \sigma_{1,1}(m_1; 2) \sigma_{1,1}(m_2; 2) \sigma_{1,1}(m_3; 2) \\ + \sum_{\substack{m_1+m_2+m_3=N \\ m_2 \text{ odd} \\ m_1, m_3 \text{ even}}} \sigma_{1,1}(m_1; 2) \sigma_{1,1}(m_2; 2) \sigma_{1,1}(m_3; 2) \\ + \sum_{\substack{m_1+m_2+m_3=N \\ m_3 \text{ odd} \\ m_1, m_2 \text{ even}}} \sigma_{1,1}(m_1; 2) \sigma_{1,1}(m_2; 2) \sigma_{1,1}(m_3; 2).$$

It is well known that

$$(4) \quad \sigma_s(mn) = \sigma_s(m) \sigma_s(n) \text{ with } (m, n) = 1.$$

First, if N is odd, then $(N, 2) = 1$ and we obtain the following by (4).

$$(5) \quad \begin{aligned} \sigma_5(2N) &= \sigma_5(2) \sigma_5(N) = (1 + 5^5) \sigma_5(N) = 33\sigma_5(N), \\ \sigma_3(2N) &= \sigma_3(2) \sigma_3(N) = (1 + 2^3) \sigma_3(N) = 9\sigma_3(N), \\ \sigma_{1,1}(N; 2) &= \sigma_1(N) \end{aligned}$$

and by (5), we derive that

$$\begin{aligned}
(6) \quad & \sum_{\substack{k+l+h=N \\ k,l,h>0 \\ N \text{ odd}}} \sigma_{1,1}(k;2)\sigma_{1,1}(l;2)\sigma_{1,1}(h;2) \\
&= -\frac{29}{768}\sigma_5(N) + \frac{1}{768}\sigma_5(2N) - \frac{11}{192}\sigma_3(N) + \frac{1}{192}\sigma_3(2N) + \frac{1}{192}\sigma_{1,1}(N;2) \\
&= -\frac{29}{768}\sigma_5(N) + \frac{33}{768}\sigma_5(N) - \frac{11}{192}\sigma_3(N) + \frac{9}{192}\sigma_3(N) + \frac{1}{192}\sigma_1(N) \\
&= \frac{1}{192}\sigma_5(N) - \frac{1}{96}\sigma_3(N) + \frac{1}{192}\sigma_1(N)
\end{aligned}$$

using Proposition 2.2. So, by Lemma 2.4, we get

$$\begin{aligned}
(7) \quad & \sum_{\substack{m_1+m_2+m_3=N \\ m_1 \text{ odd} \\ m_2, m_3 \text{ even}}} \sigma_{1,1}(m_1;2)\sigma_{1,1}(m_2;2)\sigma_{1,1}(m_3;2) \\
&= \sum_{\substack{m'_1+m'_2+m'_3=N \\ m'_2 \text{ odd} \\ m'_1, m'_3 \text{ even}}} \sigma_{1,1}(m'_1;2)\sigma_{1,1}(m'_2;2)\sigma_{1,1}(m'_3;2) \\
&= \sum_{\substack{m''_1+m''_2+m''_3=N \\ m''_3 \text{ odd} \\ m''_1, m''_2 \text{ even}}} \sigma_{1,1}(m''_1;2)\sigma_{1,1}(m''_2;2)\sigma_{1,1}(m''_3;2).
\end{aligned}$$

Comparing (3) with (7), we have

$$\begin{aligned}
(8) \quad & \sum_{\substack{m_1+m_2+m_3=N \\ N \text{ odd}}} \sigma_{1,1}(m_1;2)\sigma_{1,1}(m_2;2)\sigma_{1,1}(m_3;2) \\
&= \sum_{\substack{m_1+m_2+m_3=N \\ m_1, m_2, m_3 \text{ odd}}} \sigma_{1,1}(m_1;2)\sigma_{1,1}(m_2;2)\sigma_{1,1}(m_3;2) \\
&\quad + 3 \sum_{\substack{m_1+m_2+m_3=N \\ m_1 \text{ odd} \\ m_2, m_3 \text{ even}}} \sigma_{1,1}(m_1;2)\sigma_{1,1}(m_2;2)\sigma_{1,1}(m_3;2).
\end{aligned}$$

From (6) and (8), we obtain

$$\begin{aligned} & \frac{1}{192}\sigma_5(N) - \frac{1}{96}\sigma_3(N) + \frac{1}{192}\sigma_1(N) \\ &= \frac{1}{192}\sigma_5(N) - \frac{1}{192}\sigma_3(N) - \frac{1}{8} \sum_{\substack{ax < N \\ a, x \text{ odd}}} a^3 \sigma_{1,1}(N - ax; 2) \\ & \quad + 3 \sum_{\substack{m_1+m_2+m_3=N \\ m_1 \text{ odd} \\ m_2, m_3 \text{ even}}} \sigma_{1,1}(m_1; 2) \sigma_{1,1}(m_2; 2) \sigma_{1,1}(m_3; 2), \end{aligned}$$

and

$$\begin{aligned} & \sum_{\substack{m_1+m_2+m_3=N \\ m_1 \text{ odd} \\ m_2, m_3 \text{ even}}} \sigma_{1,1}(m_1; 2) \sigma_{1,1}(m_2; 2) \sigma_{1,1}(m_3; 2) \\ &= -\frac{1}{576}\sigma_3(N) + \frac{1}{576}\sigma_1(N) + \frac{1}{24} \sum_{\substack{ax < N \\ a, x \text{ odd}}} a^3 \sigma_{1,1}(N - ax; 2). \end{aligned}$$

And by Lemma 2.5, we get

$$\begin{aligned} & \sum_{\substack{m_1+m_2+m_3=N \\ m_1 \text{ odd} \\ m_2, m_3 \text{ even}}} \sigma_{1,1}(m_1; 2) \sigma_{1,1}(m_2; 2) \sigma_{1,1}(m_3; 2) \\ &= \frac{1}{2304}(\sigma_5(N) - 8\sigma_3(N) + 4\sigma_1(N) + 3c(N)), \end{aligned}$$

where

$$\sum_{N=1}^{\infty} c(N)q^N = q \prod_{N=1}^{\infty} (1 - q^{2N})^{12}.$$

□

3. Approach to infinite product identities

In this section we shall study the relations of infinite product formula derived from Weierstrass $\wp$ -functions. Let $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ ($\tau \in \mathfrak{h}$ the complex upper half plane) be a lattice and $z \in \mathbb{C}$. The Weierstrass $\wp$ function relative to Λ_τ is defined by the series

$$\wp(z; \Lambda_\tau) = \frac{1}{z^2} + \sum_{\substack{\omega \in \Lambda_\tau \\ \omega \neq 0}} \left\{ \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right\},$$

and the Eisenstein series of weight $2k$ for Λ_τ with $k > 1$ is the series

$$G_{2k}(\Lambda_\tau) = \sum_{\substack{\omega \in \Lambda_\tau \\ \omega \neq 0}} \omega^{-2k}.$$

We shall use the notation $\wp(z)$ and G_{2k} instead of $\wp(z; \Lambda_\tau)$ and $G_{2k}(\Lambda_\tau)$, respectively, when the lattice Λ_τ has been fixed. Then the Laurent series for $\wp(z)$ about $z = 0$ is given by

$$\wp(z) = z^{-2} + \sum_{k=1}^{\infty} (2k+1)G_{2k+2}z^{2k}.$$

To prove our theorem, we need the following propositions.

Proposition 3.1. ([14, p.251]) *Let $e_1 = \wp(\frac{\tau}{2})$, $e_2 = \wp(\frac{1}{2})$, and $e_3 = \wp(\frac{\tau+1}{2})$, where $P_0 = \prod_{n=1}^{\infty} (1 - q^{2n})$, $P_1 = \prod_{n=1}^{\infty} (1 - q^{2n-1})$, $P_2 = \prod_{n=1}^{\infty} (1 + q^{2n})$, and $P_3 = \prod_{n=1}^{\infty} (1 + q^{2n-1})$. Then,*

$$\begin{aligned} (a) \quad e_2 - e_1 &= \pi^2 P_0^4 P_3^8. \\ (b) \quad e_2 - e_3 &= \pi^2 P_0^4 P_1^8. \\ (c) \quad e_3 - e_1 &= 2^4 \pi^2 q P_0^4 P_2^8. \end{aligned}$$

Proposition 3.2. ([11, p.252]) *Let $(\wp(\frac{1}{2}) - \wp(\frac{\tau}{2}))^2 = \pi^4 \sum_{n=0}^{\infty} \alpha_n q^n$, $(\wp(\frac{1}{2}) - \wp(\frac{\tau+1}{2}))^2 = \pi^4 \sum_{n=0}^{\infty} \beta_n q^n$ and $(\wp(\frac{\tau+1}{2}) - \wp(\frac{\tau}{2}))^2 = \pi^4 \sum_{n=0}^{\infty} \gamma_n q^n$. Then, we get the following.*

(a)

$$\alpha_n = \begin{cases} 1 & n = 0 \\ 16\sigma_3(n) & n \text{ odd} \\ 256\sigma_3(\frac{n}{2}) - 16\sigma_3(n) & n \text{ even.} \end{cases}$$

(b)

$$\beta_n = \begin{cases} 1 & n = 0 \\ -16\sigma_3(n) & n \text{ odd} \\ 256\sigma_3(\frac{n}{2}) - 16\sigma_3(n) & n \text{ even.} \end{cases}$$

(c)

$$\gamma_n = \begin{cases} 32\sigma_3(n) - 32\sigma_3(\frac{n}{2}) & n \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 3.3. *Let $G(q) = \sum_{n: \text{odd}} \sigma_5(n)q^n$. Then,*

$$\begin{aligned} G(q) = & 4q \left\{ \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{36}}{(1-q^n)^{16}(1-q^{4n})^8} + q^2 \prod_{n=1}^{\infty} \frac{(1-q^{4n})^{32}}{(1-q^{2n})^{12}} \right. \\ & - 16q^2 \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{12}(1-q^{4n})^8}{(1-q^n)^8} - 2 \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{16}}{(1-q^n)^8} \\ & \left. + 16q \prod_{n=1}^{\infty} \frac{(1-q^{4n})^{16}}{(1-q^{2n})^8} \right\} + \frac{1}{4} \left\{ \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{32}}{(1-q^n)^{16}(1-q^{4n})^{16}} \right. \\ & \left. - \prod_{n=1}^{\infty} \frac{(1-q^n)^{16}}{(1-q^{2n})^8} \right\} - 3q \prod_{n=1}^{\infty} (1-q^{2n})^{12}. \end{aligned}$$

Proof. For $N \in \mathbb{N}$, let $S_1 = \sum_{n: \text{odd}} \sigma_{1,1}(n; 2)q^n$, $S_2 = \sum_{n: \text{even}} \sigma_{1,1}(n; 2)q^n$, $(a; q)_{\infty} = (a)_{\infty} = \prod_{n \geq 0} (1 - aq^n)$. For $|q| < 1$, we have

$$\begin{aligned} (9) \quad S_1 \cdot S_2^2 = & \frac{1}{576} q \left\{ \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{36}}{(1-q^n)^{16}(1-q^{4n})^8} + q^2 \prod_{n=1}^{\infty} \frac{(1-q^{4n})^{32}}{(1-q^{2n})^{12}} \right. \\ & - 16q^2 \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{12}(1-q^{4n})^8}{(1-q^n)^8} - 2 \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{16}}{(1-q^n)^8} \\ & \left. + 16q \prod_{n=1}^{\infty} \frac{(1-q^{4n})^{16}}{(1-q^{2n})^8} + \prod_{n=1}^{\infty} \frac{(1-q^{4n})^8}{(1-q^{2n})^4} \right\} \end{aligned}$$

and by Proposition 3.1 and 3.2,

$$(10) \quad \sum_{n: \text{odd}} \sigma_3(n)q^n = \frac{1}{32} \left\{ \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{32}}{(1-q^n)^{16}(1-q^{4n})^{16}} - \prod_{n=1}^{\infty} \frac{(1-q^n)^{16}}{(1-q^{2n})^8} \right\}.$$

By (2), we get

$$\begin{aligned} (11) \quad & \sum_{N=1}^{\infty} \sigma_5(N)q^N \\ & = 2304 \sum_{N=1}^{\infty} \sum_{\substack{m_1+m_2+m_3=N \\ m_1: \text{odd} \\ m_2, m_3: \text{even}}} \sigma_{1,1}(m_1; 2)\sigma_{1,1}(m_2; 2)\sigma_{1,1}(m_3; 2)q^N \\ & \quad + 8 \sum_{N=1}^{\infty} \sigma_3(N)q^N - 4 \sum_{N=1}^{\infty} \sigma_1(N)q^N - 3 \sum_{N=1}^{\infty} c(N)q^N. \end{aligned}$$

By the definition of S_1 and S_2 , we derive that

$$(12) \quad \sum_{N=1}^{\infty} \sum_{\substack{m_1+m_2+m_3=N \\ m_1 \text{ odd} \\ m_2, m_3 \text{ even}}} \sigma_{1,1}(m_1; 2) \sigma_{1,1}(m_2; 2) \sigma_{1,1}(m_3; 2) q^N = S_1 \cdot S_2^2.$$

And if n is odd, then $\sigma_{1,1}(n; 2) = \sigma_1(n)$. So, $S_1 = \sum_{N=1}^{\infty} \sigma_1(N) q^N$. Therefore by (9), (10), (11) and (12), we get

$$\begin{aligned} G(q) = \sum_{n \text{ odd}} \sigma_5(n) q^n &= 4q \left\{ \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{36}}{(1-q^n)^{16}(1-q^{4n})^8} + q^2 \prod_{n=1}^{\infty} \frac{(1-q^{4n})^{32}}{(1-q^{2n})^{12}} \right. \\ &\quad - 16q^2 \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{12}(1-q^{4n})^8}{(1-q^n)^8} - 2 \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{16}}{(1-q^n)^8} \\ &\quad \left. + 16q \prod_{n=1}^{\infty} \frac{(1-q^{4n})^{16}}{(1-q^{2n})^8} \right\} + \frac{1}{4} \left\{ \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{32}}{(1-q^n)^{16}(1-q^{4n})^{16}} \right. \\ &\quad \left. - \prod_{n=1}^{\infty} \frac{(1-q^n)^{16}}{(1-q^{2n})^8} \right\} - 3q \prod_{n=1}^{\infty} (1-q^{2n})^{12}. \quad \square \end{aligned}$$

4. Appendix

Let

$$h(n) = \sum_{\substack{ax < n \\ a, x \text{ odd}}} a^3 \sigma_{1,1}(n - ax; 2).$$

The value of $h(n)$ (for $5 \leq n \leq 51$) are given in Table 1, if n is an odd positive integer.

n	$h(n)$	n	$h(n)$	n	$h(n)$	n	$h(n)$	n	$h(n)$	n	$h(n)$
5	29	13	3763	21	48946	29	76694	37	801994	45	2086568
7	158	15	8810	23	76694	31	115837	39	1030674	47	2569278
9	583	17	16875	25	115837	33	454837	41	1316390	49	3144192
11	1639	19	29529	27	169942	35	654689	43	1657679	51	3841031

Table 1 Values of $h(n)$

References

- [1] A. Alaca, S. Alaca, and K. S. Williams, *The convolution sum* $\sum_{m < n/16} \sigma(m) \sigma(n - 16m)$, *Canad. Math. Bull.* **51**(1) (2008), 3-14.

- [2] A. Alaca, S. Alaca, and K. S. Williams, *The convolution sum $\sum_{l+24m=n} \sigma(l)\sigma(m)$ and $\sum_{3l+8m=n} \sigma(l)\sigma(m)$* , Math. J. Okayama Univ. **49** (2007), 93-111.
- [3] B. C. Berndt, *Ramanujan's Notebooks. Part II*, Springer-Verlag, New York, 1989.
- [4] B. Cho, D. Kim, and J.-K. Koo, *Divisor functions arising from q -series*, Publ. Math. Debrecen **76(3-4)** (2010), 495-508.
- [5] B. Cho, D. Kim, and J.-K. Koo, *Modula forms arising from divisor functions*, J. Math. Anal. Appl. **356(2)** (2009), 537-547.
- [6] J. W. L. Glaisher, *On the square of the series in which the coefficients are the sums of the divisors of the exponents*, Mess. Math. **14** (1884), 156-163.
- [7] J. W. L. Glaisher, *On certain sums of products of quantities depending upon the divisors of a number*, Mess. Math. **15** (1885), 1-20.
- [8] J. W. L. Glaisher, *Expressions for the five powers of the series in which the coefficients are the sums of the divisors of the exponents*, Mess. Math. **15** (1885), 33-36.
- [9] H. Hahn, *Convolution sums of some functions on divisors*, Rocky Mountain J. Math. **37** (2007), 1593-1622.
- [10] J. G. Huard, Z. M. Ou, B. K. Spearman, and K. S. Williams, *Elementary Evaluation of Certain Convolution Sums Involving Divisor Functions*, Number theory for the millennium, II, (Urbana, IL, 2000), 229-274, A K Peters, Natick, MA, 2002.
- [11] D. Kim, A. Kim, and H. Park, *Congruences of the Weierstrass $\wp(x)$ and $\wp''(x)(x = \frac{1}{2}, \frac{\tau}{2}, \frac{\tau+1}{2})$ -Functions on divisors*, Bull. Korean Math. Soc. **50** (2013), 241-261.
- [12] D. Kim, A. Kim, and A. Sankaranarayanan, *Eisenstein seires and their applications to some arithmetic identities and congruences*, Advances in Difference Equations 2013, **2013**:84.
- [13] A. Kim, D. Kim and L. Yan, *Convolution sums arising from divisor functions*, J. Korean Math. Soc. **50** (2013), 331-360.
- [14] S. Lang, *elliptic Functions*, Addison-Wesly, 1973.
- [15] Erin McAfee, *A three term arithmetic formula of liouville type with application to sums of six squares*, B. Math.(Honors), Carleton University, 2004.
- [16] G. Melfi, *On some modular identities*, Number theory (Eger, 1996), 371-382, de Gruyter, Berlin, 1998, 371-382.

Kwangchul Lee

Department of Mathematics, Chonbuk National University,
567 Baekje-daero, Deokjin-gu, Jeonju-si,
Jeollabuk-do 561-756, Republic of Korea.
E-mail : bee-777@hanmail.net

Daeyeoul Kim

National Institute for Mathematical Sciences,
Yuseong-daero 1689-gil, Yuseong-gu, Daejeon 305-811, South Korea.

E-mail : daeyeoul@nims.re.kr

Gyeong-Sig Seo

Department of Mathematics and Institute of Pure and Applied Mathematics, Chonbuk National University,

567 Baekje-daero, Deokjin-gu, Jeonju-si,

Jeollabuk-do 561-756, Republic of Korea.

E-mail : sgs0303@jbnu.ac.kr