MINIMAL BASICALLY DISCONNECTED COVERS OF $P'$-SPACES

CHANG IL KIM* AND KAP HUN JUNG**

Abstract. Observing that for any $P'$-space $X$, $\Lambda vX$ is a $P'$-space if $vX$ is a weakly Lindelöf space, $(\Lambda vX \times \Lambda Y, \Lambda X \times \Lambda Y)$ is the minimal basically disconnected cover of $X \times Y$ for a countably locally weakly Lindelöf space $Y$.

1. Introduction

All spaces in this paper are assumed to be Tychonoff and \((\beta X, \beta X)\) (resp. \((vX, vX)\), resp.) denotes the Stone-Čech compactification (Hewitt realcompactification, resp.) of $X$.

Iliadis constructed the absolute of a Hausdorff space $X$, which is the minimal extremally disconnected cover \((EX, \pi_X)\) of $X$ and they turn out to be the perfect onto projective covers ([5]). To generalize extremally disconnected spaces, basically disconnected spaces, quasi-$F$ spaces and cloz-spaces have been introduced and their minimal covers have been studied by various authors. In these ramifications, minimal covers of compact spaces can be nicely characterized.

In particular, Vermeer ([7]) showed that every Tychonoff space $X$ has the minimal basically disconnected cover \((AX, \Lambda X)\) and that for any compact space $X$, $AX$, is given by the Stone space $S(\sigma Z(X)^\#)$ of a $\sigma$-complete Boolean subalgebra $\sigma Z(X)^\#$ of $R(X)$.

In [1], Comfort, Hindman, and Negrepontis showed that $X$ is a $P'$-space and $Y$ is a countably locally weakly Lindelöf space, then $X \times Y$ is a basically disconnected space.

The purpose of this paper is to construct the minimal basically disconnected covers of $P'$-spaces.

In [4], it showed that if $X$ is a weakly $P$-space and $Y$ is a countably locally weakly Lindelöf space, then $(\Lambda X \times \Lambda Y, \Lambda X \times \Lambda Y)$ is the minimal basically disconnected cover of $X \times Y$.

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* First author.
** Corresponding author.
In this paper, we will show that for any $P'$-space ([6]) $X$ such that $vX$ is a weakly Lindelöf space, $\Lambda_X$ is a $P'$-space and that for any countably locally weakly Lindelöf space $Y$, $(\Lambda vX \times \Lambda Y, \Lambda_X \times \Lambda_Y)$ is the minimal basically disconnected cover of $X \times Y$.

For the terminology, we refer to [2] and [5].

2. Basically disconnected covers of $P'$-spaces

Let $X$ be a space. The collection $R(X)$ of all regular closed sets in $X$, when partially ordered by inclusion, becomes a complete Boolean algebra, in which the join, meet, and complementation operations are defined as follows:

For any $A \in R(X)$ and any $F \subseteq R(X)$,
\[
\bigvee \mathcal{F} = cl_X\left(\bigcup \{F \mid F \in \mathcal{F}\}\right),
\]
\[
\bigwedge \mathcal{F} = cl_X\left(int_X\left(\bigcap \{F \mid F \in \mathcal{F}\}\right)\right),
\]
and $A' = cl_X(X - A)$.

A sublattice of $R(X)$ is a subset of $R(X)$ that contains $\emptyset$, $X$ and is closed under finite joins and finite meets ([5]).

Let $Z(X)$ be the set of all zero-sets in $X$ and $Z(X)^\# = \{cl_X\left(int_X(A)\right) \mid A \in Z(X)\}$. Then $Z(X)^\#$ is a sublattice of $R(X)$.

Recall that a map $f : Y \to X$ is called a covering map if it is a continuous, onto, perfect, and irreducible map.

**Lemma 2.1.** ([3], [5])

(1) Let $f : Y \to X$ be a covering map. Then the map $\psi : R(Y) \to R(X)$, defined by $\psi(A) = f(A)$, is a Boolean isomorphism and the inverse map $\psi^{-1}$ of $\psi$ is given by $\psi^{-1}(B) = cl_Y\left(f^{-1}\left(int_X(B)\right)\right) = cl_Y\left(int_X(f^{-1}(B))\right)$.

(2) Let $X$ be a dense subspace of a space $K$. Then the map $\phi : R(K) \to R(X)$, defined by $\phi(A) = A \cap X$, is a Boolean isomorphism and the inverse map $\phi^{-1}$ of $\phi$ is given by $\phi^{-1}(B) = cl_K(B)$.

A lattice $L$ is called $\sigma$-complete if every countable subset of $L$ has join and meet. For any subset $M$ of a Boolean algebra $L$, there is the smallest $\sigma$-complete Boolean algebra $\sigma M$ of $L$ containing $M$.

Let $X$ be a space. Then $Z(X)^\#$ is a sublattice of $R(X)$. Note that for any zero-set $A$ in $X$, there is a zero-set $B$ in $\beta X$ such that $A = B \cap X$. Hence, by Lemma 2.1, $Z(X)^\#$, $Z(vX)^\#$ and $Z(\beta X)^\#$ are Boolean isomorphic. Moreover $\sigma Z(X)^\#$, $\sigma Z(vX)^\#$ and $\sigma Z(\beta X)^\#$ are Boolean isomorphic.

**Definition 1.** A space $X$ is called basically disconnected if for any zero-set $Z$ in $X$, $int_X(Z)$ is closed in $X$, equivalently, $Z(X)^\# = B(X)$, where $B(X)$ is the set of all clopen sets in $X$.

A space $X$ is a basically disconnected space if and only if $\beta X$ is a basically disconnected space.

Suppose that $X$ is a basically disconnected space. Then for any sequence $(B_n)_{n \in N}$ in $B(X)$,
\[
\bigwedge \{B_n \mid n \in N\} = cl_X\left(int_X\left(\bigcap \{B_n \mid n \in N\}\right)\right) \in Z(X)^\#
\]
and \( \bigvee \{B_n \mid n \in N \} = \text{cl}_X \left( \text{int}_X \left( \bigcup \{B_n \mid n \in N \} \right) \right) \in Z(X)^\# \). Hence \( X \) is a basically disconnected space if and only if \( Z(X)^\# \) is a \( \sigma \)-complete Boolean algebra.

**Lemma 2.2.** Let \( f : X \to Y \) be a covering map and \((A_n)\) a decreasing sequence of closed sets in \( X \). Then \( f(\bigcap \{A_n \mid n \in N\}) = \bigcap \{f(A_n) \mid n \in N\} \)

**Proof.** Clearly, we have \( f(\bigcap \{A_n \mid n \in N\}) \subseteq \bigcap \{f(A_n) \mid n \in N\} \). Let \( x \in \bigcap \{f(A_n) \mid n \in N\} \). Since \((A_n)\) is a decreasing sequence of closed sets in \( X \), \( \{A_n \cap f^{-1}(x) \mid n \in N\} \) has a family of closed sets in \( f^{-1}(x) \) with the finite intersection property. Since \( f^{-1}(x) \) is compact, \( \bigcap \{A_n \cap f^{-1}(x) \mid n \in N\} \neq \emptyset \) and so \( \bigcap \{A_n \mid n \in N\} \cap f^{-1}(x) \neq \emptyset \). Note that

\[
\emptyset \neq f(\bigcap \{A_n \mid n \in N\}) \cap x = f(\bigcap \{A_n \mid n \in N\} \cap f^{-1}(x)).
\]

Hence \( x \in f(\bigcap \{A_n \mid n \in N\}) \) and so \( f(\bigcap \{A_n \mid n \in N\}) \supseteq \bigcap \{f(A_n) \mid n \in N\} \). Thus we have the result.

\( \square \)

A space \( X \) is called a \( P \)-space if every zero-set in \( X \) is open in \( X \). The concept of \( P' \)-spaces is a generalization of the concept of \( P \)-spaces ([6]).

**Definition 2.** A space \( X \) is called a \( P' \)-space if every zero-set in \( X \) is a regular closed sets in \( X \), equivalently, for any non-empty zero set \( Z \) in \( X \), \( \text{int}_X(Z) = \emptyset \).

A space \( X \) is called a weakly Lindelöf space if every open cover \( U \) of \( X \) has a countable subset \( V \) of \( U \) such that \( \cup \{V \mid V \in V\} \) is dense in \( X \).

We recall that a covering map \( f : X \to Y \) is called \( z^\# \)-irreducible if \( f(Z(X)^\#) = Z(Y)^\# \) and that if \( Y \) is a weakly Lindelöf space, then \( f : X \to Y \) is a \( z^\# \)-irreducible map.

**Definition 3.** Let \( X \) be a space. Then a pair \((Y, f)\) is called

1. a cover of \( X \) if \( f : X \to Y \) is a covering map,
2. a basically disconnected cover of \( X \) if \((Y, f)\) is a cover of \( X \) and \( Y \) is a basically disconnected space, and
3. a minimal basically disconnected cover of \( X \) if \((Y, f)\) is a basically disconnected cover of \( X \) and for any basically disconnected cover \((Z, g)\) of \( X \), there is a covering map \( h : Z \to Y \) such that \( f \circ h = g \).

Vermeer([7]) showed that every space \( X \) has a minimal basically disconnected cover \((AX, \Lambda_X)\) and that if \( X \) is a compact space, then \( \Lambda_X \) is the Stone-space \( S(\sigma Z(X)^\#) \) of \( \sigma Z(X)^\# \) and \( \Lambda_X(\alpha) = \cap \{A \mid A \in \alpha\} \) (\( \alpha \in \Lambda_X \)).

Let \( X \) be a space. Since \( \sigma Z(X)^\# \) and \( \sigma Z(\beta X)^\# \) are Boolean isomorphic, \( S(\sigma Z(X)^\#) \) and \( S(\sigma Z(\beta X)^\#) \) are homeomorphic.

Let \( X, Y \) be spaces and \( f : Y \to X \) a map. For any \( U \subseteq X \), let \( f_U : f^{-1}(U) \to U \) denote the restriction and co-restriction of \( f \) with respect to \( f^{-1}(U) \) and \( U \), respectively.
For any space \(X\), let \((\Lambda_\beta X, \Lambda_\beta)\) denote the minimal basically disconnected cover of \(\beta X\).

**Lemma 2.3.** ([3], [5]) Let \(X\) be a space. Then we have the following:

1. if \(\Lambda_\beta^{-1}(X)\) is an basically disconnected space, then \((\Lambda_\beta^{-1}(X), \Lambda_\beta X)\) is the minimal basically disconnected cover of \(X\), and
2. if \(\Lambda_X : \Lambda X \to X\) is \(z^\#\)-irreducible, then \(\Lambda_\beta^{-1}(X) = \Lambda X\), \(\Lambda X = \Lambda_\beta X\), and \(\beta \Lambda X = \Lambda_\beta X\).

**Theorem 2.4.** Let \(X\) be a \(P^t\)-space such that \(vX\) is a weakly Lindelöf space. Then \(\Lambda vX\) is a \(P^t\)-space.

**Proof.** Take any zero-set \(Z\) in \(\Lambda vX\) such that \(\emptyset \neq Z\) and \(\text{int}_{\Lambda vX}(Z) = \emptyset\). Then there is a continuous function \(f : \Lambda vX \to R\) such that \(Z = f^{-1}(0)\). For any \(n \in N\), let \(Z_n = \text{cl}_{\Lambda vX}(\text{int}_{\Lambda vX}(f^{-1}([0, \frac{1}{n}))))\). Then for any \(n \in N\), \(Z_{n+1} \subseteq \text{int}_{\Lambda vX}(Z_n)\) and \((Z_n)\) is a decreasing sequence in \(Z(\Lambda vX)^\#\) such that \(Z = \cap\{Z_n \mid n \in N\}\). Since \(\Lambda vX\) is a covering map, by Lemma 2.2, \(\Lambda vX(Z) = \cap\{\Lambda vX(Z_n) \mid n \in N\}\). Since \(vX\) is a weakly Lindelöf space, \(\Lambda vX : \Lambda vX \to vX\) is \(z^\#\)-irreducible and so for any \(n \in N\), \(\Lambda vX(Z_n) \in Z(vX)^\#\).

Let \(n \in N\). Then there exists a zero-set \(A_n\) in \(Z(vX)^\#\) such that \(\Lambda vX(Z_n) = \text{cl}_{\Lambda vX}(\text{int}_{\Lambda vX}(A_n))\). Since \(vX\) is a \(P^t\)-space, \(\text{cl}_{\Lambda vX}(\text{int}_{\Lambda vX}(A_n)) = A_n\) and so \(\Lambda vX(Z_n) \in Z(vX)\). Hence \(\Lambda vX(Z) = \cap\{\Lambda vX(Z_n) \mid n \in N\} \in Z(vX)\). Since \(\Lambda vX : \Lambda vX \to vX\) is \(z^\#\)-irreducible, by Lemma 2.3, \(\Lambda vX^{-1}(vX) = \Lambda vX\) and \(\Lambda vX = \Lambda_\beta vX\). Note that

\[
\Lambda vX(Z \cap \Lambda vX) = \Lambda vX(\Lambda vX^{-1}vX)
= \Lambda_\beta(\Lambda vX^{-1}vX)
= \Lambda_\beta(Z) \cap vX.
\]

Since \(\text{int}_{\Lambda vX}(Z) = \emptyset\), \(\emptyset = \Lambda vX(Z \cap vX) = \Lambda vX(Z) \cap X = \emptyset\). Since \(vX\) is a \(P^t\)-space and \(\Lambda vX(Z) \in Z(vX)\), \(\Lambda vX(Z) = \emptyset\) and hence \(Z = \emptyset\). This is a contradiction and so \(\text{int}_{\Lambda vX}(Z) \neq \emptyset\). Therefore \(vX\) is a \(P^t\)-space.

\(\square\)

It is well-known that a basically disconnected \(P^t\)-space is a \(P\)-space. Using this, we have the following.

**Corollary 2.5.** Let \(X\) be a \(P^t\)-space such that \(X\) or \(vX\) has a dense weakly Lindelöf subspace. Then \(\Lambda vX\) is a \(P^t\)-space.

**Proof.** Suppose that \(D\) is a dense weakly Lindelöf subspace of \(vX\). Let \(U\) be an open cover of \(vX\). Then \(U_D = \{U \cap D \mid U \in U\}\) is an open cover of \(D\). Since \(D\) is a weakly Lindelöf space, there is a countable subset \(V\) of \(U\) such that \(\{V \cap D \mid V \in V\}\) is dense in \(D\). Since \(D\) is dense in \(vX\) and \(\cup\{V \mid V \in V\}\) is open in \(vX\), \(\cup\{V \mid V \in V\}\) is dense in \(vX\). Hence \(vX\) is a weakly Lindelöf space. By Theorem 2.4, \(\Lambda vX\) is a \(P^t\)-space.

\(\square\)
Suppose that $D$ is a dense weakly Lindelöf space of $X$. Then similarly we can show that $X$ is a dense weakly Lindelöf subspace of $vX$.

**Theorem 2.6.** ([4]) Let $X, Y$ be spaces such that $\Lambda_\beta^{-1}(X) = \Lambda X$ and $\Lambda_\beta^{-1}(Y) = \Lambda Y$. If $\Lambda X \times \Lambda Y$ is a basically disconnected space, then $(\Lambda X \times \Lambda Y, \Lambda_X \times \Lambda_Y)$ is the minimal basically disconnected cover of $X \times Y$, where $(\Lambda_X \times \Lambda_Y)(x, y) = (\Lambda_X(x) \times \Lambda_Y(y))$.

A space $X$ is called a countably locally weakly Lindelöf space if for any countable collection $\{U_n \mid n \in \mathbb{N}\}$ of open covers of $X$ and for any $x \in X$, there is a neighborhood $G$ of $x$ in $X$ and for any $n \in \mathbb{N}$, there is a subfamily $V_n$ of $U_n$ such that $G \subseteq \text{cl}_X(\bigcup V_n)$.

In [1], it was shown that if $X$ is a $P$-space and $Y$ is a countably locally weakly Lindelöf space, then $X \times Y$ is a basically disconnected space. By Corollary 2.5 and Theorem 2.6, we have the following corollary:

**Corollary 2.7.** Let $X$ be a $P'$-space such that $X$ or $vX$ has a weakly Lindelöf dense subspace and $Y$ a countably locally weakly Lindelöf space. Then $(\Lambda vX \times \Lambda Y, \Lambda_X \times \Lambda_Y)$ is the minimal basically disconnected cover of $X \times Y$.

**References**


Chang Il Kim
Department of Mathematics Education, Dankook University, 152, Jukjeon, Suji, Yongin, Gyeonggi, 448-701, Korea
E-mail address: kci206@hanmail.net

Kap Hun Jung
School of Liberal Arts, Seoul National University of Science and Technology, Seoul 139-743, Korea
E-mail address: jkh58@hanmail.net