SPLINE DIFFERENCE SCHEME FOR TWO-PARAMETER SINGULARLY PERTURBED PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we construct a numerical method to solve singularly perturbed one-dimensional parabolic convection–diffusion problems. We use Euler method with uniform step size for temporal discretization and exponential-spline scheme on spatial uniform mesh of Shishkin type for full discretization. We show that the resulting method is uniformly convergent with respect to diffusion parameter. An extensive amount of analysis has been carried out to prove the uniform convergence with respect to the singular perturbation parameter. The obtained numerical results show that the method is efficient, stable and reliable for solving convection–diffusion problem accurately even involving diffusion parameter.

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1. Introduction

In the present paper, we are concerned with the numerical approximation of the partial differential equation:

$$u_t - \varepsilon_d u_{xx} + \varepsilon_c p(x)u_x + q(x)u = f(x, t), \quad (x, t) \in Q_T. \quad (1)$$

Here $Q_T = \Omega \times I$, $\Omega \equiv (a, b)$, $\partial\Omega \equiv \{a, b\}$, $I \equiv (0, T)$, and a and b are real positive constants. We consider equation (1) associated with the Dirichlet boundary conditions

$$u = 0, (x, t) \in \partial\Omega \times I, \quad (2)$$

and the initial condition

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (3)$$

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where $u(x, t)$ is the dependent variable, x and t are the independent variables and f , p , q and u_0 are given functions of their arguments. It is usually assumed that $0 < \varepsilon_c \ll 1$ and $0 < \varepsilon_d \ll 1$.

This problem encloses both the reaction–diffusion problem when $\varepsilon_c = 0$ and the convection–diffusion problem when $\varepsilon_c = 1$. These problems occur naturally in various fields of science and engineering, such as, heat transfer with large Peclet numbers, combustion, nuclear engineering, control theory, elasticity, fluid mechanics, aerodynamics, quantum mechanics, optimal control, chemical-reactor theory, convection–diffusion process and geophysics; see [9,10,18].

There has been a lot of effort in developing numerical methods for the solution of singular perturbation problems that are uniformly convergent. Clavero et al. [1] and Kadalbajoo et al. [4] gives a uniform convergent numerical method with respect to the diffusion parameter to solve the one-dimensional time-dependent convection–diffusion problem. They used the implicit Euler method for the time discretization and the simple upwind finite difference scheme on a Shishkin mesh for the spatial discretization. Ramos [7] presented an exponentially fitted method for singularly perturbed, one-dimensional (convection–diffusion) parabolic problems, and showed its uniform convergence in the perturbation parameter. Surla and Jerkovic [13] considered a singularly perturbed boundary value problem using a spline collocation method. Sakai and Usmani [11,12] gave a new concept of B-spline in terms of hyperbolic and trigonometric splines which are different from the earlier ones. It is proved that the hyperbolic and trigonometric B-splines are characterized by a convolution of some special exponential functions and a characteristic function on the interval $[0, 1]$. Sharma and Kaushik [14] presented a numerical approximation of a singularly perturbed time delayed convection diffusion problem on a rectangular domain. Finally, Zahra and El Mhlawy [18] developed a numerical method to solve two-parameter singularly perturbed semi-Linear boundary value problems. This method is based on exponential spline with a piecewise uniform Shishkin mesh which is shown to be uniformly convergent independent of mesh parameters and perturbation parameter ε_c and ε_d .

Our contribution in this paper is to introduce a full discretization scheme based on Rothe's approximation in time discretization and on the exponential spline approximation in the spatial discretization for solving singularly perturbed, one-dimensional time-dependent convection–diffusion problem associated with homogeneous Dirichlet boundary conditions. It starts with the discretization in time by the 2-point Euler backward finite difference method in the time variable. After that we deal with the exponential spline method for the solution of the time discretized problem to compute the unknown function and the obtained system of algebraic equations is solved by iterative methods. Moreover, we obtain error estimates for the approximation process. The proposed technique is programmed using *Matlab 7*, and used in solving the proposed problem.

The rest of the paper is organized as follows: In section 2, we give some notations, assumptions and definitions, In §3, a time discretization scheme for the continuous problem is proposed in addition to the derivation of some a priori estimates. In §4, we prove the convergence results and error estimates of the semidiscretization scheme described in §3. We describe in §5 a full discretization scheme based on Rothe- exponential spline methods and the error estimates of the full discretized solution are derived. Finally, numerical results and discussions are presented and comparisons are made with other solutions in §6.

2. Notations, assumptions and definitions

In the sequel, we will denote by $(\cdot, \cdot)$ either the standard inner product in $L_2 = L_2(\Omega)$ or the pairing between $V \equiv H_0^1(\Omega)$ and $V^* \equiv H^{-1}$ (see e.g. [11]). we use the symbols $|\cdot|$, $\|\cdot\|$ and $\|\cdot\|_*$ as the norms in $L_2(\Omega)$, V , V^* , respectively. By $\rightarrow$, $\xrightarrow{w}$, we mean the strong and weak convergence. Also, we introduce some notations concerning the time discretization of our problem.

$$\delta z_i = \frac{z_i - z_{i-1}}{\tau}, \quad \bar{z}^i = \frac{1}{\tau} \int_{I_i} z(\cdot, t) dt, \quad 1 \leq i \leq n,$$

for any given family $\{z_i\}_{i=0}^n$. The following elementary relations will be used in the following analysis:

$$\sum_{i=1}^s (u_i - u_{i-1}, u_i) = |u_s|^2 - |u_0|^2 + \sum_{i=1}^s |u_i - u_{i-1}|^2, \quad (4)$$

and Young's inequality

$$ab \leq \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2 \quad \forall a, b \in R, \quad (5)$$

where ε is a small constant. We will assume, throughout this work, the following hypotheses on the given data.

(H1) The function $f : \Omega \times I \rightarrow R$ is Lipschitz continuous in the sense of

$$|f(x, t) - f(x, t')| \leq c |t - t'|, \quad \forall t, t' \in I, \quad (6)$$

with $f(0, 0) = 0$.

(H2) The functions $p(x)$ and $q(x)$ are smooth functions and satisfy

$$p_l \leq p(x) \leq p_u, q_l \leq q(x) \leq q_u \quad \forall x \in \Omega, \quad (7)$$

(H3) $u_0 \in H_0^1(\Omega)$

Under these assumptions, we can define the weak solution of problem (1)-(2).

Definition 2.1. The measurable function $u \in C(I; L_2(\Omega)) \cap L_2(I; V)$ with $\partial_t u \in L_2(I; V^*)$ and $u(x, 0) = u_0(x)$ is said to be a weak (variational) solution of (1)–(3) if and only if the integral identity

$$(u_t, \varphi) + \varepsilon_d(u_x, \varphi_x) + \varepsilon_c(p(x)u_x, \varphi) + (q(x)u, \varphi) = (f, \varphi), \quad (8)$$

holds for all $\varphi \in V$ and a.e. $t \in I$.

Remark. It is evident that for any functions $\alpha, \beta \in H^1(\Omega)$ and $\gamma \in L_2(\Omega)$

$$|(\alpha\beta_x, \gamma)| \leq \|\alpha\|^{1/2} |\alpha|^{1/2} |\beta_x| |\gamma| \leq \|\alpha\| \|\beta\| \|\gamma\|, \quad (9)$$

3. The temporal discretization scheme, A prior estimate

Our main goal is to approximate (1)–(3) from a numerical point of view and to prove its convergence. The scheme starts with the discretization in time by the 2-point Euler backward finite difference method in the time variable, so that the problem is converted into a linear system of differential equations that easily solved numerically at each subsequent time level. Let m be a positive integer and subdivide the time interval I by the points t_j , where $t_j = j\tau$, $\tau = T/m$, $j = 0, 1, \dots, m$. The suggested discretization scheme of problem (8) consists of the following problem (in the weak sense):

$$\text{Find } w_j \cong u(., t_j) \in V, \quad j = 1, \dots, m \text{ such that } w_0 = u_0 \text{ in } \Omega, \quad (10)$$

$$(\delta w_j, \varphi) + \varepsilon_d(\partial_x w_j, \partial_x \varphi) + \varepsilon_c(p(x)\partial_x w_j, \varphi) + (q(x)w_j, \varphi) = (f_j, \varphi) \forall \varphi \in V, \quad (11)$$

where ∂_x denotes to the derivative with respect to x and $f_j = f(x, t_j)$.

The existence of a weak solution $w_j \in V$ is proved in [2]. By means of w_j , ($j = 0, 1, \dots, m$) determined by the proposed scheme 10 -11 in each time step t_j , we introduce the following piecewise linear functions (Rothe functions)

$$w^m(0) = u_0, \quad w^m(t) = w_{j-1} + (t - t_{j-1})\delta w_j, \text{ for } t \in (t_{j-1}, t_j], \quad j = 0, 1, \dots, m \quad (12)$$

and the corresponding step function

$$\bar{w}^m(t) = \begin{cases} w_j & \text{for } t \in (t_{j-1}, t_j] \\ u_0 & \text{for } t = 0 \end{cases} \quad (13)$$

Using the notation of Rothe function and its corresponding step function, a piecewise constant interpolation of equation 11 over I for each $\varphi \in V$ yields

$$(\partial_t w^m, \varphi) + \varepsilon_d(\partial_x \bar{w}^m, \varphi) + \varepsilon_c(p\partial_x \bar{w}^m, \partial_x \varphi) + (q\bar{w}^m, \varphi) = (\bar{f}^m, \varphi), \quad (14)$$

where $t_{j-1} \leq t \leq t_j$ and $\bar{f}^m = f(., \bar{t}^m)$ with $\bar{t}^m = t_j$.

In order to show the stability of the discrete solution and prove the convergence results, we shall derive some a priori estimates.

Lemma 3.1. *Under the assumptions (H1)-(H3), there exists a positive constant C such that for any s ,*

$$\sum_{j=1}^s \tau |\delta w_j|^2 + \max_s \|w_s\|^2 + \max_s |w_s|^2 + \sum_{j=1}^s \|w_j - w_{j-1}\|^2 + \sum_{j=1}^s |w_j - w_{j-1}|^2 \leq C. \quad (15)$$

Proof. Let us choose $\varphi = \tau \delta w_j$ in 11 and summing over j from 1 to s , we obtain

$$\begin{aligned} \sum_{j=1}^s (\delta w_j, \tau \delta w_j) + \varepsilon_d \sum_{j=1}^s (\partial_x w_j, \partial_x (w_j - w_{j-1})) + \varepsilon_c \sum_{j=1}^s (p(x)\partial_x w_j, \tau \delta w_j) \\ + \sum_{j=1}^s (q(x)w_j, w_j - w_{j-1}) = \sum_{j=1}^s \tau (f_j, \delta w_j) \end{aligned} \quad (16)$$

Taking into consideration (4), the second and the fourth terms are estimated by

$$\begin{aligned} \varepsilon_d \sum_{j=1}^s (\partial_x w_j, \partial_x (w_j - w_{j-1})) &= \frac{\varepsilon_d}{2} \left(|\partial_x w_s|^2 - |\partial_x u_0|^2 + \sum_{j=1}^s |\partial_x (w_j - w_{j-1})|^2 \right) \\ &= \frac{\varepsilon_d}{2} \left(\|w_s\|^2 - \|u_0\|^2 + \sum_{j=1}^s \|w_j - w_{j-1}\|^2 \right), \end{aligned} \quad (17)$$

$$\left| \sum_{j=1}^s (q(x)w_j, w_j - w_{j-1}) \right| \geq \frac{q_l}{2} \left(|w_s|^2 - |u_0|^2 + \sum_{j=1}^s |w_j - w_{j-1}|^2 \right) \quad (18)$$

With the aid of 11, Young's inequality (5) and (9), we estimate the third term by

$$\left| \varepsilon_c \sum_{j=1}^s (p(x)\partial_x w_j, \tau \delta w_j) \right| \leq \frac{\varepsilon_c p_u}{2\varepsilon} \sum_{j=1}^s \tau \|w_j\|^2 + 2\varepsilon \varepsilon_c \sum_{j=1}^s \tau |\delta w_j|^2 \quad (19)$$

The last term is estimated by the use of (6) as follows

$$\left| \sum_{j=1}^s \tau (f_j, \delta w_j) \right| \leq \sum_{j=1}^s \tau |f_j| |\delta w_j| \leq \frac{1}{2\varepsilon} \sum_{j=1}^s \tau |f_j|^2 + 2\varepsilon \sum_{j=1}^s \tau |\delta w_j|^2 \quad (20)$$

Collecting (16)-(20), using **(H3)**, and choosing ε sufficiently small, we get

$$\begin{aligned} &\sum_{j=1}^s \tau |\delta w_j|^2 + \frac{\varepsilon_d}{2} \|w_s\|^2 + \frac{\varepsilon_d}{2} \sum_{j=1}^s \|w_j - w_{j-1}\|^2 \\ &\leq c_\varepsilon \sum_{j=1}^s \tau |f_j|^2 + \varepsilon \sum_{j=1}^s \tau |\delta w_j|^2 + \frac{\varepsilon_c p_u}{2\varepsilon} \sum_{j=1}^s \tau \|w_j\|^2 + 2\varepsilon \varepsilon_c \sum_{j=1}^s \tau |\delta w_j|^2 \end{aligned} \quad (21)$$

Applying Gronwall's inequality, we conclude the proof. $\square$

Lemma 3.2. *Uniformly with respect to n one has*

$$\|\partial_t w^m\|_{L_2(I; L_2(\Omega))}^2 \leq C \|\bar{w}^m\|_{C(I; V)}^2 \leq C, \|w^m\|_{C(I; V)}^2 \leq C \quad (22)$$

$$\|w^m - \bar{w}^m\|_{L_2(I; L_2(\Omega))}^2 \leq \frac{C}{m^2}, \|w^m - \bar{w}^m\|_{L_2(I; V)}^2 \leq \frac{C}{m} \quad (23)$$

Proof. The estimates (22) and (23) are a consequence of (15). By the use of the identity

$$|w^m - \bar{w}^m| \leq 2|w_j - w_{j-1}|, \quad (24)$$

the estimates (22)₃, and (23) are a consequence of (15)₂, (15)₃ and the definitions of w^m and $\bar{w}^m$ and thus the proof completes. $\square$

4. Convergence results

This section is devoted to proving the convergence of the proposed scheme and estimating its accuracy. Before we are able to prove convergence we need to prove the compactness of $\bar{w}^m$ in $L_2(I; L_2(\Omega))$ which is a consequence of the following assertion.

Lemma 4.1. *The estimate*

$$\int_0^{T-z} |\bar{w}^m(t+z) - \bar{w}^m(t)|^2 \leq C \left(z + \frac{1}{m} \right), \quad (25)$$

holds uniformly for $0 < z < z_0$ and n .

Proof. [3] We sum up (12) for $i = s+1, \dots, s+k$ considering $\varphi = (w_{s+k} - w_s)\tau$. Then we sum it up for $s = 1, \dots, m-k$ and obtain the estimate

$$\sum_{s=0}^{m-k} |u_{s+k} - u_s|^2 \tau \leq Ck\tau, \quad (26)$$

Hence for $k\tau \leq z \leq (k+1)\tau$ we conclude the desired estimate. $\square$

Theorem 4.1. *There exists $u \in L_2(I; L_2(\Omega)) \cap H^1(I; L_2(\Omega))$ such that*

$$\left. \begin{aligned} w^m &\rightharpoonup u && \text{in } L_2(I; L_2(\Omega)) \\ w^m &\xrightarrow{w} u, & \bar{w}^m &\xrightarrow{w} u && \text{in } L_2(I; V) \\ \partial_t w^m &\xrightarrow{w} \partial_t u && \text{in } L_2(I; L_2(\Omega)) \end{aligned} \right\} \quad (27)$$

(in the sense of subsequences). Moreover, we have

$$\|u - \bar{w}^m\|_{L_2(I; L_2(\Omega))}^2 + \|u - \bar{w}^m\|_{L_2(I; V)}^2 \leq \frac{C}{m^2}. \quad (28)$$

Proof. The estimate (22)₂ implies

$$\int_Q (\bar{w}^m(x+y, t) - \bar{w}^m(x, t))^2 \leq C|y|, \forall |y| \leq y_0$$

Hence, from lemma 4.1, $\{\bar{w}^m\}$ is compact in $L_2(I; L_2(\Omega))$ because of Kolmogorov's compactness argument. So we can conclude that $w^n \rightarrow u$ and $\bar{w}^m \rightarrow u$ in $L_2(I; L_2(\Omega))$ and also pointwise in Q . Also by the fact that $\bar{w}^m \in L_2(I; L_2(\Omega)) \cap C(I; V)$, we obtain that $\bar{w}^m$ is weakly convergent in $L_2(I; V)$; i.e.

$$\partial_x \bar{w}^m \xrightarrow{w} \partial_x u \text{ in } L_2(I; V), \quad (29)$$

and this implies that

$$(\partial_x \bar{w}_\tau^m, \varphi) \rightarrow (u_x, \varphi), \forall \varphi \in V. \quad (30)$$

It remains to prove that $\partial_t w^m = \partial_t u$. For each $t \in I$, by lemma 3.3, $\partial_t w^m$ is uniformly bounded in the reflexive Banach space $L_2(I; L_2(\Omega))$ and hence has a subsequence which converges weakly to an element $\xi \in V^*$ (Eberlin-Smulian theorem [6]). Thus

$$\partial_t w^m \xrightarrow{w} \xi \text{ in } V^* \forall t \in I. \quad (31)$$

Using Fubini theorem, we get

$$(w^m(t) - u_0, \chi) = \int_0^t (\partial_t w^m, \chi) dt. \quad (32)$$

Taking the limit as $n \rightarrow \infty$, we obtain

$$(u - u_0, \chi) = \int_0^t (\xi, \chi) dt, \quad (33)$$

which implies

$$\left(u - u_0 - \int_0^t \xi(t) dt, \chi \right) = 0. \quad (34)$$

Therefore, we have $\partial_t u = \xi$. Using the above discussions and the fact that and hence $\bar{f}^n \rightarrow f$ in $L_2(I; L_2(\Omega))$ and passing with $n \rightarrow \infty$ in (14) the proof is complete. $\square$

To obtain an error estimate, let us start by introducing the following additional notation $e_u = u(t) - \bar{w}^m$. We now take the difference between (8) and (14) we easily obtain the equality for every $\varphi \in V$

$$\begin{aligned} & (\partial_t(u - \bar{w}^m), \varphi) + \varepsilon_d(\nabla(u - \bar{w}^m), \nabla \varphi) + \varepsilon_c(p(\partial_x u - \bar{w}^m), \varphi) + (q(u - \bar{w}^m), \varphi) \\ & = ((f - \bar{f}^m), \varphi). \end{aligned} \quad (35)$$

Take $\varphi = e_u$ and write the new equation as $I + II + III + IV = V$. To begin with, we split the parabolic term into two parts $I = I_1 + I_2$, where

$$I_1 = \frac{1}{2} \frac{d}{dt} |e_u|^2, \quad (36)$$

$$|I_2| \leq |w^m - \bar{w}^m| |e_u| \leq 2\xi |w^m - \bar{w}^m|^2 + \frac{1}{2\xi} |e_u|^2 \quad (37)$$

The second term of equation (37) is bounded by

$$|II| \leq |\varepsilon_d(\nabla(u - \bar{w}^m), \nabla e_u)| \leq \|e_u\|^2 \quad (38)$$

The third and fourth terms of equation (37) are bounded by

$$|III| \leq |\varepsilon_c(p\partial_x(u - \bar{w}^m), e_u)| \leq 2\varepsilon_c \varepsilon \|u - \bar{w}^m\|^2 + \frac{\varepsilon_c}{2\varepsilon} |e_u|^2 \quad (39)$$

$$|IV| \leq q_0 |u - \bar{w}^m|^2 \leq q_0 |e_u|^2 \quad (40)$$

In view of **(H1)** and Young's inequality we can estimate the right hand by

$$|V| \leq \frac{\kappa}{2} |f - \bar{f}^m|^2 + \frac{1}{2\kappa} |e_u|^2 \leq \frac{\kappa}{2} \tau^2 + \frac{1}{2\kappa} |e_u|^2 \quad (41)$$

Collecting all the previous bounds, choosing ξ , ε , and κ sufficiently small and applying Gronwall lemma we obtain (28).

5. A Full discretization scheme

In this section, we use the exponential spline for the solution of discretization of problem (1-3) which is written in the form (denoting $y(x) \equiv w_j(x)$):

$$-\varepsilon_d y'' + \varepsilon_c p(x) y' + (q(x) + 1/\tau) y = f_i + w_{j-1}/\tau, \quad x \in \Omega, \quad (42)$$

with boundary conditions

$$y = 0, x \in \partial\Omega, \quad (43)$$

in the time step t_j , $j = 1, 2, \dots, n$, w_{j-1} is known and we have to solve the two parameter singularly perturbed boundary value problem

$$-\varepsilon_d y'' + \varepsilon_c p(x) y' + G(x) y = H(x), \quad x \in \Omega, \quad (44)$$

associated with the boundary condition (43). Here we use $G(x)$ and $H(x)$ to denote $(q(x) + 1/\tau)$ and $(f_j + w_{j-1}/\tau)$ respectively. To this end we consider a uniform mesh Δ with nodal point x_i on $[a, b]$ such that $\Delta : a = x_1 < x_2 < \dots < x_{m-1} < x_m = b$ where

$$x_i = a + i h, h = \frac{b-a}{m}, i = 0, 1, 2, \dots, m. \quad (45)$$

Let $y(x)$ be the exact solution of the problem (44)-(43) and S_i be an approximation to $y_i = y(x_i)$ obtained by the segment $Q_i(x)$ passing through the points (x_i, S_i) and (x_{i+1}, S_{i+1}) . Each mixed spline segment $Q_i(x)$ has the form, (for more details see [15-18]):

$$Q_i(x) = a_i e^{k(x-x_i)} + b_i e^{-k(x-x_i)} + c_i(x-x_i) + d_i, \quad i = 0, 1, 2, \dots, m, \quad (46)$$

where a_i, b_i, c_i and d_i are constants and k is a free parameter.

Following [18], we get the following relation for $i = 1, 2, \dots, m-1$,

$$(S_{i+1} - 2S_i + S_{i-1}) = h^2(\alpha M_{i+1} + \beta M_i + \alpha M_{i-1}), \quad (47)$$

where:

$\theta = kh$, $\alpha = (\sinh(\theta) - \theta)/\theta^2 \sinh(\theta)$ and $\beta = (2\theta \cosh(\theta) - 2\sinh(\theta))/\theta^2 \sinh(\theta)$, when $k \rightarrow 0$ that $\theta \rightarrow 0$ then $(\alpha, \beta) = \frac{1}{6}(1, 4)$ and the relation defined by (47) reduces to an ordinary cubic spline relation:

$$(S_{i+1} - 2S_i + S_{i-1}) = \frac{h^2}{6}(M_{i+1} + 4M_i + M_{i-1}), \quad (48)$$

at the point x_i the proposed singularly perturbed problem may discretized by:

$$M_i = \frac{1}{\varepsilon_d}(\varepsilon_c p_i S_i^{(1)} + G_i S_i + H_i), \quad (49)$$

where, $S_i^{(1)} = \frac{S_{i+1} - S_{i-1}}{2h}$, $S_{i+1}^{(1)} = \frac{3S_{i+1} - 4S_i + S_{i-1}}{2h}$, $S_{i-1}^{(1)} = \frac{-S_{i+1} + 4S_i - 3S_{i-1}}{2h}$, $G_i = G(x_i)$, $H_i = H(x_i)$ and $p_i = p(x_i)$.

Substituting (49) into (47), we get the following linear equation for every $i = 1, 2, \dots, m-1$

$$\begin{aligned} \varepsilon_c(S_{i-1} - 2S_i + S_{i+1}) + \frac{\varepsilon_d h}{2}(D_i S_{i-1} + E_i S_i + A_i S_{i+1}) \\ - h^2(\alpha G_{i-1} S_{i-1} + \beta G_i S_i + \alpha G_{i+1} S_{i+1}) = -h^2(\alpha H_{i-1} + \beta H_i + \alpha H_{i+1}). \end{aligned} \quad (50)$$

Equation (50) gives $n-1$ linear algebraic equations in $n-1$ unknowns S_i , where

$$A_i = \alpha p_{i-1} - \beta p_i - 3\alpha p_{i+1}, D_i = 3\alpha p_{i-1} + \beta p_i - \alpha p_{i+1}, E_i = 4\alpha(p_{i+1} - p_{i-1}).$$

Now we investigate the convergence analysis of the suggested algorithm. The exponential spline solution of (1)-(3) is based on the linear equation given by (50). Let $Y = (y_i)$, $S = (s_i)$, $C = (c_i)$, $T = (t_i)$ and $E = (e_i) = Y - S$ be $m-1$ dimensional column vectors. It is easily seen that the system of equation (50) gives $n-1$ linear algebraic equations in the $n-1$ unknowns S_i , $i = 1, 2, \dots, m-1$. Then we can rewrite equation (50) in the standard matrix equation as:

$$BY = C + T. \quad (51)$$

Now the matrix B may be written as

$$B = \varepsilon_d B_0 + h^2 B_1 F + \frac{\varepsilon_c h}{2} B_2, \text{ and } F = \text{diag } G_i \quad (52)$$

where

$$B_0 = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}, B_1 = \begin{pmatrix} \beta & \alpha & & & \\ \alpha & \beta & \alpha & & \\ & & \ddots & & \\ & & \alpha & \beta & \alpha \\ & & & \alpha & \beta \end{pmatrix},$$

$$B_2 = \begin{pmatrix} E_1 & A_1 & & & \\ D_2 & E_2 & A_2 & & \\ & & \ddots & & \\ & & D_{n-2} & E_{n-2} & A_{n-2} \\ & & & D_{n-1} & E_{n-1} \end{pmatrix}$$

$$\text{and } C_i = \begin{cases} -h^2(\alpha H_0 + \beta H_1 + \alpha H_2), & i = 1 \\ -h^2(\alpha H_{i-1} + \beta H_i + \alpha H_{i+1}), & i = 2, 3, \dots, n-2 \\ -h^2(\alpha H_{n-2} + \beta H_{n-1} + \alpha H_n), & i = n-1 \end{cases}$$

We have the local truncation error are

$$\|T\| = \begin{cases} \frac{h^6}{240} L_i + O(h)^7, i = 6 & (\alpha, \beta) = \frac{1}{12}(1, 10) \\ \frac{h^4}{24} L_i + O(h)^5, i = 4 & (\alpha, \beta) = \frac{1}{24}(1, 22) \\ \frac{h^4}{12} L_i + O(h)^5, i = 4 & (\alpha, \beta) = \frac{1}{6}(1, 4), \end{cases} \text{ where } L_i = \max_{a \leq x \leq b} |y^i(x)|.$$

From Eq. (52), then the row sum of the matrix B satisfies: for $|p_i| \leq P \geq 0$,

$$\begin{aligned} B &= \varepsilon_d B_0 + \frac{\varepsilon_c h}{2} B_1 + h^2 B_2 F, \\ B_1 &= \varepsilon_d + \frac{\varepsilon_c h}{2} (2\alpha + \beta) P + h^2 (2\alpha + \beta) F \quad i = 1, n-1, \\ B_i &= h^2 (2\alpha + \beta) F, \quad i = 2, 3, \dots, n-2, \end{aligned} \quad (53)$$

for small values of ε_d and ε_c then the matrix B is irreducible and monotone and it follows that B^{-1} exist thus the problem (51) has a unique solution.

Lemma 5.1 ([5]). *For any $0 < d < 1$, we have up to a certain order q that it depends on the smoothness of the data*

$$|y^k(x)| \leq C \left\{ 1 + \psi_1^k e^{-d\psi_1 x} + \psi_2^k e^{-d\psi_2(1-x)} \right\}, \text{ for } 0 \leq k \leq q \quad (54)$$

Shishkin Mesh Strategy

We form the piecewise-uniform grid in such a way that more points are generated in the boundary layer regions than outside of it. We divide the interval $\bar{\Omega} = [0, 1]$ into three subintervals:

$$\Lambda_0 = [0, \gamma_1], \Lambda_c = [\gamma_1, 1 - \gamma_2] \text{ and } \Lambda_1 = [1 - \gamma_2, 1],$$

where transition parameters are given by

$$\gamma_1 = \min\left(\frac{1}{4}, \frac{2}{\psi_1} \ln n\right), \quad \gamma_2 = \min\left(\frac{1}{4}, \frac{2}{\psi_2} \ln n\right),$$

where

$$\psi_1 = -\max_{x \in [0,1]} \eta_1(x) \text{ and } \psi_2 = \min_{x \in [0,1]} \eta_2(x), \quad \eta_1(x) < 0 \text{ and } \eta_2(x) > 0.$$

the real solutions of characteristic equation

$$-\varepsilon_d \eta(x)^2 + \varepsilon_c p(x) \eta(x) + G(x) = 0.$$

with n to be the number of subdivision points of the interval $[0,1]$ and we place $n/4$, $n/2$ and $n/4$ mesh points, respectively, in $[0, \gamma_1]$, $[\gamma_1, 1 - \gamma_2]$ and $[1 - \gamma_2, 1]$. Denote the step sizes in each subinterval by $h_1 = \frac{4\gamma_1}{n}$, $h_2 = \frac{2(1-\gamma_1-\gamma_2)}{n}$ and $h_3 = \frac{4\gamma_2}{n}$, respectively.

Where the resulting piecewise-uniform Shishkin mesh may be represented by:

$$\tilde{h} = \begin{cases} h_1 = h_i = \frac{4\gamma_1}{n} & \text{if } i = 1, 2, \dots, n/4, \\ h_2 = h_i = \frac{2(1-\gamma_1-\gamma_2)}{n} & \text{if } i = n/4 + 1, \dots, 3n/4, \\ h_3 = h_i = \frac{4\gamma_2}{n} & \text{if } i = 3n/4 + 1, \dots, n, \end{cases}$$

where n is the number of discretization points and set of mesh points $\bar{\Omega}_n = \{x_i\}_{i=0}^n$ with:

$$x_i = \begin{cases} x_{i-1} + h_1 & \text{if } i = 1, 2, \dots, n/4, \\ x_{i-1} + h_2 & \text{if } i = n/4 + 1, \dots, 3n/4, \\ x_{i-1} + h_3 & \text{if } i = 3n/4 + 1, \dots, n, \end{cases}$$

Thus, a uniform mesh is placed on each of these subintervals.

Lemma 5.2 ([5]). *The solution $y(x)$ of (42)-(43) has the representation*

$$y(x) = U(x) + E_0(x) + E_1(x).$$

Theorem 5.1. *Let $S(x)$ be the approximate solution of the solution $y(x)$ of (42), then*

$$\|S(x) - y(x)\|_\infty \leq Cn^{-2}(\ln n)^2,$$

where C is a constant independent of ε_d and ε_c .

Proof. The estimate is obtained on each subinterval $\Omega_i = [0, 1]$ separately. Let any function z on Ω_i , see [5]

$$\bar{z}(x) = z_{i-1}(x)\phi_{i-1}(x) + z_i(x)\phi_i(x), \quad (55)$$

and so it is obvious that, on Ω_i

$$\bar{z}(x) \leq \max_{\Omega_i} z(x)[\phi_{i-1}(x) + \phi_i(x)] \quad (56)$$

Taking maximum norm on both sides, we get

$$|\bar{z}(x)| \leq \max_{\Omega_i} |z(x)|, \quad (57)$$

and by appropriate Taylor expansions it is easy to see that

$$|\bar{z}(x) - z(x)| \leq Ch_i^2 \max_{\Omega_i} |z''(x)|. \quad (58)$$

From (58) and Lemma 5.1, on Ω_i ,

$$\begin{aligned} |S(x) - y(x)| &\leq Ch_i^2 \max_{\Omega_i} |y''(x)| \leq Ch_i^2 \{1 + \psi_1^2 e^{-j\psi_1 x} + \psi_2^2 e^{-j\psi_2(1-x)}\} \\ &\leq Ch_i^2 \{1 + \psi_1^2 + \psi_2^2\} \end{aligned}$$

Also, using Lemma 5.3 and equation (58) on Ω_i ,

$$\begin{aligned} |S(x) - y(x)| &\leq |\bar{U} + \bar{E}_0 + \bar{E}_1 - U - E_0 - E_1| \\ &\leq |\bar{U} - U| + |\bar{E}_0 - E_0| + |\bar{E}_1 - E_1| \\ &\leq Ch_i^2 \max_{\Omega_i} |U''(x)| + 2 \max_{\Omega_i} |E_0(x)| + 2 \max_{\Omega_i} |E_1(x)| \\ &\leq C(h_i^2 + e^{-j\psi_1 x_i} + e^{-j\psi_2(1-x_i)}) \end{aligned} \quad (59)$$

Case 1. The argument now depends on whether $\frac{2\ln n}{\psi_1} \leq \frac{1}{4}$ and $\frac{2\ln n}{\psi_2} \leq \frac{1}{4}$. In this case $\psi_1 \leq C \ln n$ and $\psi_2 \leq C \ln n$. Then the result follows at once from (58)

$$|S(x) - y(x)| \leq Cn^{-2}(\ln n)^2$$

Case 2. When $\gamma_1 = \frac{2}{\psi_1} \ln n$ and $\gamma_2 = \frac{2}{\psi_2} \ln n$. Suppose that i satisfies $1 \leq i \leq \frac{n}{4}$ and $\frac{3n}{4} \leq i \leq n$. Then $h_i = \frac{4\gamma_1}{n} = \frac{Cn^{-1} \ln n}{\psi_1}$ and $h_i = \frac{4\gamma_2}{n} = \frac{Cn^{-1} \ln n}{\psi_2}$ respectively. Now from (58)

$$|S(x) - y(x)| \leq Cn^{-2}(\ln n)^2$$

If i satisfied $\frac{n}{4} \leq i \leq \frac{3n}{4}$. Then $\gamma_1 \leq x_i$ and $x_i \leq 1 - \gamma_2$ or $\gamma_2 \leq 1 - x_i$ and so

$$e^{-\psi_1 x_i} \leq e^{-\psi_1 \gamma_1} = e^{-2 \ln n} = n^{-2}, \text{ since } \gamma_1 = \frac{2 \ln n}{\psi_1}$$

$$e^{-\psi_2 (1-x_i)} \leq e^{-\psi_2 \gamma_2} = e^{-2 \ln n} = n^{-2}, \text{ since } \gamma_2 = \frac{2 \ln n}{\psi_2}$$

Using the above values in (59), we get the required result. $\square$

We are now in a position to formulate the main contribution of this work by combining (28) and (59) in the following theorem.

Theorem 5.2. *Let the assumptions (H1)-(H2) be satisfied. Then $u^\alpha \rightarrow u$ in $u \in H^1(I; L_2(\Omega)) \cap L_2(I; V)$ for $\alpha \rightarrow 0$ where u is the variational solution of problem (1)-(3) and u^α is constructed from the solution of problem (42)-(43). Moreover, we have the error estimates*

$$\|u - u^\alpha\|_{L_2(I; L_2(\Omega))} \leq C(h^2 + \tau).$$

6. Numerical Results

We present a case study to illustrate the performance of the error estimation procedures of section 5. For purposes of comparison, contrast and performance, two cases with known solutions were chosen.

Case study 1: Consider the following singularly perturbed parabolic partial differential equation, see [1].

$u_t - \varepsilon_d u_{xx} + u_x = f(x, t)$, $T = 1$, in the interval $[0, 1]$.
associated with the Dirichlet boundary conditions $u(0, t) = 0$, $u(1, t) = 0$. and the initial condition $u_0(x) = e^{-1/\varepsilon} + (1 - e^{-1/\varepsilon})x - e^{-(1-x)/\varepsilon}$, where $f(x, t) = e^{-t}(-c_1 + c_2(1 - x) + e^{-(1-x)/\varepsilon})$. The exact solution is given by $u(x, t) = e^{-t}(c_1 + c_2x - e^{-(1-x)/\varepsilon})$, where $c_1 = e^{-1/\varepsilon}$, $c_2 = 1 - e^{-1/\varepsilon}$.

Case study 2: Consider the following singularly perturbed parabolic partial differential equation.

$u_t - \varepsilon_d u_{xx} + \varepsilon_c u_x = f(x, t)$, $T = 1$, in the interval $[0, 1]$.
associated with the Dirichlet boundary conditions $u(0, t) = 0$, $u(1, t) = 0$. and the initial condition

$$u_0(x) = [\rho_1 \cos(\pi x) + \rho_2 \sin(\pi x) + \psi_1 \exp(\lambda_1 x) + \psi_2 \exp(-\lambda_2(1 - x))],$$

where

$$f(x, t) = e^{-t} \left\{ -\rho_1 \cos(\pi x) - \rho_2 \sin(\pi x) - \psi_1 \exp(\lambda_1 x) - \psi_2 \exp(-\lambda_2(1 - x)) \right\}$$

$$+ \varepsilon_d \left\{ \rho_1 \pi^2 \cos(\pi x) + \rho_2 \pi^2 \sin(\pi x) - \frac{\psi_1}{\lambda_1^2} \exp(\lambda_1 x) - \frac{\psi_2}{\lambda_2^2} \exp(-\lambda_2(1 - x)) \right\}$$

$$+ \varepsilon_c \left\{ -\rho_1 \pi \sin(\pi x) + \rho_2 \pi \cos(\pi x) + \frac{\psi_1}{\lambda_1} \exp(\lambda_1 x) + \frac{\psi_2}{\lambda_2} \exp(-\lambda_2(1 - x)) \right\}.$$

TABLE 1. Maximum pointwise errors for Case study1

n ε_d	16	32	64	128	256	512
2^{-6}	2.9748-2	1.4647-2	8.4430-3	9.1708-3	9.3305-3	9.1833-3
2^{-8}	3.5638-2	1.3000-2	9.3378-3	8.4218-3	8.3579-3	8.7259-3
2^{-10}	5.1972-2	1.4234-2	8.3305-3	7.9579-3	8.1124-3	8.0812-3
2^{-12}	6.7088-2	2.3185-2	8.2129-3	7.6243-3	7.9618-3	8.0662-3
2^{-24}	7.3818-2	3.4126-2	1.5756-2	9.1052-3	8.4470-3	8.3314-3
2^{-26}	7.3819-2	3.4128-2	1.5761-2	9.1078-3	8.4491-3	8.3337-3
2^{-28}	7.3819-2	3.4129-2	1.5763-2	9.1084-3	8.4496-3	8.3343-3
2^{-30}	7.3819-2	3.4129-2	1.5763-2	9.1086-3	8.4497-3	8.3344-3
2^{-32}	7.3819-2	3.4129-2	1.5763-2	9.1086-3	8.4498-3	8.3345-3

TABLE 2. Maximum pointwise errors and order of convergence for Case study1

n ε_d	16	32	64	128
2^{-10}	5.1972-2	1.4234-2	8.3305-3	7.9579-3
	1.8684	0.7729	0.0660	
2^{-12}	6.7088-2	2.3185-2	8.2129-3	7.6243-3
	1.5328	1.4972	0.1073	
2^{-24}	7.3818-2	3.4126-2	1.5756-2	9.1052-3
	1.1131	1.1150	0.7911	
2^{-26}	7.3819-2	3.4128-2	1.5761-2	9.1078-3
	1.1130	1.1146	0.7912	
2^{-28}	7.3819-2	3.4129-2	1.5763-2	9.1084-3
	1.1130	1.1144	0.7913	
2^{-30}	7.3819-2	3.4129-2	1.5763-2	9.1086-3
	1.1130	1.1144	0.7912	
2^{-32}	7.3819-2	3.4129-2	1.5763-2	9.1086-3
	1.1130	1.1144	0.7912	

The exact solution is given by

$$u(x, t) = e^{-t}[\rho_1 \cos(\pi x) + \rho_2 \sin(\pi x) + \psi_1 \exp(\lambda_1 x) + \psi_2 \exp(-\lambda_2(1 - x))],$$

$$\text{where } \rho_1 = \frac{\varepsilon_d \pi^2 + 1}{\varepsilon_c^2 \pi^2 + (\varepsilon_d \pi^2 + 1)^2}, \rho_2 = \frac{\varepsilon_c \pi}{\varepsilon_c^2 \pi^2 + (\varepsilon_d \pi^2 + 1)^2}, \psi_1 = -\rho_1 \frac{1 + \exp(-\lambda_2)}{1 - \exp(\lambda_1 - \lambda_2)}, \psi_2 = \rho_1 \frac{1 + \exp(\lambda_1)}{1 - \exp(\lambda_1 - \lambda_2)}, \lambda_1 = \frac{\varepsilon_c - \sqrt{\varepsilon_c^2 + 4\varepsilon_d}}{2\varepsilon_d}, \lambda_2 = \frac{\varepsilon_c + \sqrt{\varepsilon_c^2 + 4\varepsilon_d}}{2\varepsilon_d}.$$

All the computations were performed by using MATLAB 7.

TABLE 3. Comparison of maximum pointwise errors for Case study1

n	32		64		128		256	
ε_d	Proposed scheme	Scheme in [1]	Proposed scheme	Scheme in [1]	Proposed scheme	Scheme in [1]	Proposed scheme	Scheme in [1]
2^{-8}	1.4647-2	4.5409-2	8.4430-3	2.7112-2	9.1708-3	1.5275-2	9.3305-3	8.3396-3
2^{-10}	1.3000-2	4.8331-2	9.3378-3	2.9637-2	8.4218-3	1.7085-2	8.3579-3	9.4669-3
2^{-12}	1.4234-2	4.9125-2	8.3305-3	3.0386-2	7.9579-3	1.7720-2	8.1124-3	9.4669-3
2^{-24}	2.3185-2	4.9395-2	8.2129-3	3.0647-2	7.6243-3	1.7952-2	7.9618-3	1.0184-2
2^{-26}	3.4126-2	4.9395-2	1.5756-2	3.0647-2	9.1052-3	1.7952-2	8.4470-3	1.0184-2

TABLE 4. Maximum errors for Case study2 $n = 128$ and $\tau = 1.6/(n+1)^2$

ε_c	$\varepsilon_d = 10^{-2}$	$\varepsilon_d = 10^{-4}$	$\varepsilon_d = 10^{-6}$
10^{-3}	3.2784-2	2.9841-2	2.8807-2
10^{-4}	4.5702-2	4.7121-2	4.6675-2
10^{-5}	5.3734-2	6.6223-2	6.8517-2

To compute the experimental rates of convergence Ord^n for every fixed ε , we use the rate of convergence from:

$$Ord^n = \frac{\ln E^n - \ln E^{2n}}{\ln 2}$$

The numerical results displayed in Tables 1-3 clearly indicate that the proposed method based on an exponential-spline function with Shishkin mesh is ε -uniformly convergent. Numerical solution profiles for case study are given in figure 1, respectively, for $n = 128$, $\tau = 0.02$ and different values of $\varepsilon_d = 1$, 2^{-6} and 2^{-12} . For comparison, our results are better than the numerical solution presented in Clavero et al. [1]. Numerical solution profiles for case study2 are given in table 4 and also figure 2 and 3 for different values of ε_d and ε_c . It is clearly that all solutions are in good harmony with each approximate and exact solution and the convergence obtained is parameter uniform and satisfies the theoretical predictions.

7. Conclusion

In this paper, we have investigated the application of Rothe's approximation in time discretization and the exponential spline approximation in the spatial discretization to solve the convection-diffusion problems. The method is shown to be uniformly convergent i.e., independent of mesh parameters and perturbation parameters ε_d and ε_c . It has also been seen that the accuracy in the numerical results for the proposed scheme is comparable to that obtained by the uniformly convergent method with a uniform mesh [1]. It has been found that the proposed algorithm gives highly accurate numerical results and higher order of convergence than an upwind finite difference scheme. The present method

provides a very simple and accurate method for solving convection–diffusion problems.

FIGURE 1. Exact and approximate solutions profiles of the case study1 for $n = 128$, $\tau = 0.02$ and $\varepsilon_d = 2^{-6}$ and $\varepsilon_d = 2^{-12}$ respectively

FIGURE 2. Exact and approximate solutions profiles of the case study2 for $n = 128, 256$, $\tau = 1.6/(n + 1)^2$, $\varepsilon_c = 2^{-6}$, and $\varepsilon_d = 2^{-6}$.

FIGURE 3. Exact and approximate solutions profiles of the case study2 for $n = 64, 128$, $\tau = 0.002$, $\varepsilon_c = 2^{-6}$, and $\varepsilon_d = 2^{-6}$.

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