SOME PROPERTIES OF GENERALIZED HYPERGEOMETRIC FUNCTION

SNEHAL B. RAO, AMIT D. PATEL, JYOTINDRA C. PRAJAPATI,
AND AJAY K. SHUKLA

ABSTRACT. In present paper, we obtain functions $R_t(c, \nu, a, b)$ and $R_t(c, -\mu, a, b)$ by using generalized hypergeometric function. A recurrence relation, integral representation of the generalized hypergeometric function ${}_2R_1(a, b; c; \tau; z)$ and some special cases have also been discussed.

1. Introduction and preliminaries

The special functions play very important role, particularly the hypergeometric function in solving numerous problems of mathematical physics, engineering and applied mathematics, is well-known ([4], [6], [9], [15]). This fact has inspired many mathematicians for investigations of several generalizations of hypergeometric function ([5], [11], [16], [17], [18], [19], [20]).

The Gauss hypergeometric function is defined [12] as,

$$(1) \quad {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k \quad (|z| < 1, c \neq 0, -1, -2, \dots)$$

and the generalized hypergeometric function, in a classical sense, has been defined [3] by

$$(2) \quad {}_pF_q \left[\begin{matrix} a_1, \dots, a_p; z \\ b_1, \dots, b_q \end{matrix} \right] = {}_pF_q [a_1, \dots, a_p; b_1, \dots, b_q; z] \\ = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!} \quad (p = q + 1, |z| < 1)$$

and no denominator parameter equals zero or negative integer.

Received May 9, 2012; Revised December 26, 2012.

2010 *Mathematics Subject Classification.* Primary 33C20, 33E20, 26A33.

Key words and phrases. generalized hypergeometric function, recurrence relation, integral representation, fractional integral and differential operators.

E. Wright [21] has further extended the generalization of the hypergeometric series in the following form:

$$(3) \quad {}_p\Psi_q(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + \beta_1 n) \dots \Gamma(\alpha_p + \beta_p n)}{\Gamma(\rho_1 + \mu_1 n) \dots \Gamma(\rho_q + \mu_q n)} \frac{z^n}{n!},$$

where β_r and μ_t are real positive numbers such that $1 + \sum_{t=1}^q \mu_t - \sum_{r=1}^p \beta_r > 0$.

When β_r and μ_t are equal to 1, Equation (3) differs from the generalized hypergeometric function ${}_pF_q$ by a constant multiplier only. This generalized form of the hypergeometric function has been investigated by M. Dotsenko [2], V. Malovichko [8] and others. One of the interesting special case considered in [2] has the following form:

$$(4) \quad {}_2R_1^{\omega, \mu}(z) = {}_2R_1(a, b; c; \omega, \mu; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma\left(b + \frac{\omega}{\mu}n\right)}{\Gamma\left(c + \frac{\omega}{\mu}n\right)} \frac{z^n}{n!};$$

$$(\Re(a) > 0, \Re(b) > 0, \Re(c) > 0).$$

Here ω, μ both either positive or negative simultaneously, $|z| < 1$.

The function ${}_2R_1^{\omega, \mu}(z)$ is not symmetric with respect to the parameters a and b . By letting $\frac{\omega}{\mu} = \tau > 0$ in Equation (4), Virchenko et al. [18] defined the generalized hypergeometric function in a different sense as:

$$(5) \quad {}_2R_1^{\tau}(z) = {}_2R_1(a, b; c; \tau; z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma(b + \tau k)}{\Gamma(c + \tau k) k!} z^k; \tau > 0, |z| < 1.$$

If $\tau = 1$, then (5) reduces to a Gauss's hypergeometric function ${}_2F_1(a, b; c; z)$.

For $\tau > 0$, on $|z| = 1$, the function ${}_2R_1(a, b; c; \tau; z)$ is defined provided $\Re(c - a - b) > 0$, as discussed in the proposition of Appendix A.

Rao et al. [13], [14] studied various properties of generalized hypergeometric function in the light of fractional calculus.

The Riemann-Liouville fractional integral of order ν is defined as [10]:

For $\Re(\nu) > 0$,

$$(6) \quad I^{\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t - \xi)^{\nu-1} f(\xi) d\xi$$

and the fractional differential operator of order μ defined as [10]:

$$(7) \quad D^{\mu} f(t) = D^n \{I^{n-\mu} f(t)\},$$

where $\Re(\mu) > 0$, and n is the smallest integer with the property that $n > \Re \mu$.

The Laplace transform of the function $f(z)$ is defined as [1]:

$$(8) \quad L\{f(z)\} = \int_0^{\infty} e^{-sz} f(z) dz.$$

2. Fractional operators and the generalized hypergeometric function ${}_2R_1(a, b; c; \tau; z)$

Consider the function $f(t) = \frac{1}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma(b+k) (ct)^k}{(k!)^2} = {}_2F_1(a, b; 1; ct)$, where $a, b \in \mathbb{C} (\Re(a) > 0, \Re(b) > 0)$ and c is arbitrary constant such that $|ct| < 1$.

On applying the fractional integral operator (6) of order ν on $f(t)$, we give

$$\begin{aligned} I^\nu f(t) &= \frac{1}{\Gamma(\nu)} \int_0^t \left[(t-\xi)^{\nu-1} f(\xi) \right] d\xi \\ &= \frac{1}{\Gamma(\nu)} \int_0^t \left[(t-\xi)^{\nu-1} \frac{1}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma(b+k) (c\xi)^k}{(k!)^2} \right] d\xi \\ &= \frac{t^\nu}{\Gamma(\nu+1)} \left\{ \frac{\Gamma(\nu+1)}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma(b+k) (ct)^k}{\Gamma(\nu+1+k) k!} \right\}, \end{aligned}$$

one can easily write this in following form:

$$(9) \quad \frac{t^\nu}{\Gamma(\nu+1)} {}_2R_1(a, b; \nu+1; 1; ct) = \frac{t^\nu}{\Gamma(\nu+1)} {}_2F_1(a, b; \nu+1; ct).$$

Here we denote (9) as $R_t(c, \nu, a, b)$, i.e.,

$$\begin{aligned} (10) \quad R_t(c, \nu, a, b) &= \frac{t^\nu}{\Gamma(\nu+1)} {}_2R_1(a, b; \nu+1; 1; ct) \\ &= \frac{t^\nu}{\Gamma(\nu+1)} {}_2F_1(a, b; \nu+1; ct). \end{aligned}$$

Now, applying the fractional differential operator (7) of order μ on $f(t)$, we get

$$\begin{aligned} D^\mu f(t) &= \left(\frac{d}{dt} \right)^n \left[I^{n-\mu} \frac{1}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma(b+k) (ct)^k}{(k!)^2} \right] \\ &= D^n \left[\frac{t^{n-\mu}}{\Gamma(b)} \left\{ \sum_{k=0}^{\infty} \frac{(a)_k \Gamma(b+k) (ct)^k}{\Gamma(1+n-\mu+k) k!} \right\} \right] \end{aligned}$$

which yields

$$\begin{aligned} (11) \quad D^\mu f(t) &= \frac{t^{-\mu}}{\Gamma(1-\mu)} {}_2R_1(a, b; 1-\mu; 1; ct) \\ &= \frac{t^{-\mu}}{\Gamma(1-\mu)} {}_2F_1(a, b; 1-\mu; ct). \end{aligned}$$

On denoting (11) as

$$(12) \quad \begin{aligned} R_t(c, -\mu, a, b) &= \frac{t^{-\mu}}{\Gamma(1-\mu)} {}_2R_1(a, b; 1-\mu; 1; ct) \\ &= \frac{t^{-\mu}}{\Gamma(1-\mu)} {}_2F_1(a, b; 1-\mu; ct). \end{aligned}$$

3. Properties of the functions $R_t(c, \nu, a, b)$ and $R_t(c, -\mu, a, b)$

Theorem 3.1. *If $a, b \in \mathbb{C}$ ($\Re(a) > 0, \Re(b) > 0$) and c is arbitrary constant such that $|ct| < 1$, then*

$$(13) \quad I^\lambda R_t(c, \nu, a, b) = R_t(c, \lambda + \nu, a, b),$$

$$(14) \quad D^\lambda R_t(c, \nu, a, b) = R_t(c, \nu - \lambda, a, b).$$

For $n \in \mathbb{N}$ and α as any constant;

The Laplace transform of $R_t(c, \nu, -n, \alpha + n - 1)$ is given as

$$(15) \quad L\{R_t(c, \nu, -n, \alpha + n - 1)\} = \frac{1}{s^{\nu+1}} \cdot y_n(c; \alpha, -s),$$

where $y_n(c; \alpha, -s)$ is the generalized Bessel polynomial [7].

Proof. From (6) and the left-hand side of (13), we get

$$\begin{aligned} I^\lambda R_t(c, \nu, a, b) &= \frac{1}{\Gamma(\lambda)} \int_0^t (t-\xi)^{\lambda-1} R_\xi(c, \nu, a, b) d\xi \\ &= \frac{1}{\Gamma(\lambda)} \int_0^t (t-\xi)^{\lambda-1} \left(\frac{\xi^\nu}{\Gamma(\nu+1)} {}_2R_1(a, b; \nu+1; 1; c\xi) \right) d\xi \\ &= \frac{1}{\Gamma(\lambda)} \int_0^t (t-\xi)^{\lambda-1} \left(\frac{\xi^\nu}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma(b+k) (c\xi)^k}{\Gamma(\nu+1+k) k!} \right) d\xi \end{aligned}$$

which gives

$$\begin{aligned} &I^\lambda R_t(c, \nu, a, b) \\ &= \frac{1}{\Gamma(\lambda)} \int_0^1 (1-x)^{\lambda-1} t^{\lambda-1} \left(\frac{(xt)^\nu}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma(b+k) (cxt)^k}{\Gamma(\nu+1+k) k!} \right) t dx \\ &= \frac{t^{\lambda+\nu}}{\Gamma(\lambda+\nu+1)} {}_2R_1(a, b; \lambda+\nu+1; 1; ct) = R_t(c, \lambda+\nu, a, b). \end{aligned}$$

This is the proof of (13).

From (7) and left hand side of (14);

$$\begin{aligned}
 D^\lambda R_t(c, \nu, a, b) &= D^n [I^{n-\lambda} R_t(c, \nu, a, b)] \\
 &= D^n [R_t(c, n - \lambda + \nu, a, b)] \\
 &= D^n \left[\frac{t^{n-\lambda+\nu}}{\Gamma(n - \lambda + \nu + 1)} {}_2R_1(a, b; n - \lambda + \nu + 1; 1; ct) \right] \\
 &= \frac{t^{\nu-\lambda}}{\Gamma(\nu - \lambda + 1)} {}_2R_1(a, b; \nu - \lambda + 1; 1; ct) \\
 &= R_t(c, \nu - \lambda, a, b)
 \end{aligned}$$

which is (14).

On setting $n \in \mathbb{N}$, replacing a by $-n$ and b by $\alpha + n - 1$ in $R_t(c, \nu, a, b)$, where α as any constant and then taking Laplace transform of (10), it yields

$$\begin{aligned}
 &L \{R_t(c, \nu, -n, \alpha + n - 1)\} \\
 &= L \left\{ \frac{t^\nu}{\Gamma(\nu + 1)} {}_2R_1(-n, \alpha + n - 1; \nu + 1; 1; ct) \right\} \\
 &= \int_0^\infty e^{-st} \left\{ \frac{t^\nu}{\Gamma(\nu + 1)} {}_2R_1(-n, \alpha + n - 1; \nu + 1; 1; ct) \right\} dt \\
 &= \frac{1}{s^{\nu+1}} \sum_{k=0}^n \frac{(-n)_k (\alpha + n - 1)_k}{k!} \left(\frac{c}{s}\right)^k \\
 &= \frac{1}{s^{\nu+1}} {}_2F_0 \left(-n, \alpha + n - 1; -; -\frac{c}{-s} \right) \\
 &= \frac{1}{s^{\nu+1}} \cdot y_n(c; \alpha, -s).
 \end{aligned}$$

Thus, the Laplace transform of $R_t(c, \nu, -n, \alpha + n - 1)$ is

$$L \{R_t(c, \nu, -n, \alpha + n - 1)\} = \frac{1}{s^{\nu+1}} \cdot y_n(c; \alpha, s),$$

where $y_n(c; \alpha, -s)$ is the generalized Bessel polynomial [7].

This proves (15). □

Theorem 3.2. Let $a, b \in \mathbb{C}$ ($\Re(a) > 0, \Re(b) > 0$) and c is arbitrary constant such that $|ct| < 1, \Re(\mu) < 1$. Then

$$(16) \quad I^\lambda R_t(c, -\mu, a, b) = R_t(c, \lambda - \mu, a, b),$$

$$(17) \quad D^\lambda R_t(c, -\mu, a, b) = R_t(c, -\lambda - \mu, a, b).$$

For $n \in \mathbb{N}$ and α as any constant;

The Laplace transform of $R_t(c, -\mu, -n, \alpha + n - 1)$ is given as

$$(18) \quad L \{R_t(c, -\mu, -n, \alpha + n - 1)\} = \frac{1}{s^{1-\mu}} \cdot y_n(c; \alpha, -s),$$

where $y_n(c; \alpha, -s)$ is the generalized Bessel polynomial [7].

Proof. From (6) and left hand side of (16), we get

$$\begin{aligned} I^\lambda R_t(c, -\mu, a, b) &= \frac{1}{\Gamma(\lambda)} \int_0^t (t-\xi)^{\lambda-1} R_\xi(c, -\mu, a, b) d\xi \\ &= \frac{1}{\Gamma(\lambda)} \int_0^t (t-\xi)^{\lambda-1} \left(\frac{\xi^{-\mu}}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma(b+k) (c\xi)^k}{\Gamma(1-\mu+k) k!} \right) d\xi \end{aligned}$$

which, upon substituting $\xi = xt$, yields

$$\begin{aligned} &I^\lambda R_t(c, -\mu, a, b) \\ &= \frac{1}{\Gamma(\lambda)} \int_0^1 (1-x)^{\lambda-1} t^{\lambda-1} \left(\frac{(xt)^{-\mu}}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma(b+k) (cxt)^k}{\Gamma(1-\mu+k) k!} \right) t dx \\ &= \frac{t^{\lambda-\mu}}{\Gamma(\lambda-\mu+1)} {}_2R_1(a, b; \lambda-\mu+1; 1; ct) = R_t(c, \lambda-\mu, a, b). \end{aligned}$$

This is the proof of (16).

From (7) and left hand side of (17), we get

$$\begin{aligned} &D^\lambda R_t(c, -\mu, a, b) \\ &= D^n [I^{n-\lambda} R_t(c, -\mu, a, b)] \\ &= D^n [R_t(c, n-\lambda-\mu, a, b)] \\ &= D^n \left[\frac{t^{n-\lambda-\mu}}{\Gamma(n-\lambda-\mu+1)} {}_2R_1(a, b; n-\lambda-\mu+1; 1; ct) \right] \\ &= \frac{t^{-\mu-\lambda}}{\Gamma(-\mu-\lambda+1)} \left[\frac{\Gamma(-\mu-\lambda+1)}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma(b+k) (ct)^k}{\Gamma(-\lambda-\mu+1+k) k!} \right] \\ &= \frac{t^{-\mu-\lambda}}{\Gamma(-\mu-\lambda+1)} {}_2R_1(a, b; -\mu-\lambda+1; 1; ct), \end{aligned}$$

this can also be written as $R_t(c, -\mu-\lambda, a, b)$. This leads to (17).

On setting $n \in \mathbb{N}$, replacing a by $-n$ and b by $\alpha+n-1$ in $R_t(c, -\mu, a, b)$, where α as any constant and then taking Laplace transform of (12), it yields

$$\begin{aligned} &L\{R_t(c, -\mu, -n, \alpha+n-1)\} \\ &= L\left\{ \frac{t^\nu}{\Gamma(1-\mu)} {}_2R_1(-n, \alpha+n-1; 1-\mu; 1; ct) \right\} \\ &= \int_0^\infty e^{-st} \left\{ \frac{t^\nu}{\Gamma(1-\mu)} {}_2R_1(-n, \alpha+n-1; 1-\mu; 1; ct) \right\} dt \\ &= \frac{1}{s^{-\mu+1}} \sum_{k=0}^n \frac{(-n)_k (\alpha+n-1)_k}{k!} \left(\frac{c}{s}\right)^k \\ &= \frac{1}{s^{1-\mu}} {}_2F_0\left(-n, \alpha+n-1; -; -\frac{c}{-s}\right) = \frac{1}{s^{1-\mu}} \cdot y_n(c; \alpha, -s). \end{aligned}$$

Thus, the Laplace transform of $R_t(c, -\mu, -n, \alpha + n - 1)$ is given as

$$L\{R_t(c, -\mu, -n, \alpha + n - 1)\} = \frac{1}{s^{1-\mu}} \cdot y_n(c; \alpha, -s),$$

where $y_n(c; \alpha, -s)$ is the generalized Bessel polynomial [7].

This is the proof of (18). $\square$

4. Recurrence relation for the generalized hypergeometric function ${}_2R_1(a, b; c; \tau; z)$

Theorem 4.1. If $\Re(k) > 0, \Re(a) > 0, \Re(b) > 0, \Re(m) > 0, |z| < 1$, then

$$(19) \quad (m+1) {}_2R_1(a, b; c+s+1; k; z) - {}_2R_1(a, b; m+2; k; z) \\ = \left\{ \frac{(k)^2}{(m+2)} \right\} z^2 {}_2\ddot{R}_1(a, b; m+3; k; z) + \frac{(k)}{(m+2)} \{(k) + 2(m+1)\} z \\ \times {}_2\dot{R}_1(a, b; m+3; k; z) + (m) {}_2R_1(a, b; m+3; k; z),$$

where ${}_2\dot{R}_1(a, b; c; \tau; z) = \frac{d}{dz} {}_2R_1(a, b; c; \tau; z)$ and ${}_2\ddot{R}_1(a, b; c; \tau; z) = \frac{d^2}{dz^2} {}_2R_1(a, b; c; \tau; z)$.

Proof. On applying the fundamental relation of the Gamma function $\Gamma(z+1) = z\Gamma(z)$ to (5), we can write

$$(20) \quad {}_2R_1(a, b; m+1; k; z) = \frac{\Gamma(m+1)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(b+kn)}{(m+kn) \Gamma(m+kn)} \frac{z^n}{n!}$$

and

$$(21) \quad {}_2R_1(a, b; m+2; k; z) = \frac{\Gamma(m+2)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(b+kn)}{(m+1+kn)(m+kn) \Gamma(m+kn)} \frac{z^n}{n!}.$$

On writing equation (21) as:

$$(22) \quad {}_2R_1(a, b; m+2; k; z) \\ = \frac{\Gamma(m+2)}{\Gamma(b)} \sum_{n=0}^{\infty} \left\{ \frac{1}{(m+kn)} - \frac{1}{(m+1+kn)} \right\} \frac{(a)_n \Gamma(b+kn)}{\Gamma(m+kn)} \frac{z^n}{n!} \\ = (m+1) {}_2R_1(a, b; m+1; k; z) - \frac{\Gamma(m+2)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(b+kn)}{(m+1+kn) \Gamma(m+kn)} \frac{z^n}{n!}.$$

For our convenience, we, denote the last summation in (22) by S as:

$$(23) \quad S = \frac{\Gamma(m+2)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(b+kn)}{(m+1+kn) \Gamma(m+kn)} \frac{z^n}{n!} \\ = (m+1) {}_2R_1(a, b; m+1; k; z) - {}_2R_1(a, b; m+2; k; z).$$

On applying a simple identity $\frac{1}{u} = \frac{1}{u(u+1)} + \frac{1}{u+1}$, ($u = kn + m + 1$) to (23), we obtain

$$S = \frac{\Gamma(m+2)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(m+kn)(a)_n \Gamma(b+kn)}{\Gamma(m+3+kn)} \frac{z^n}{n!} \\ + \frac{\Gamma(m+2)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(c+s+kn)(m+1+kn)(a)_n \Gamma(b+kn)}{\Gamma(m+3+kn)} \frac{z^n}{n!}$$

and

$$(24) \quad S = \frac{k}{(m+2)} \left\{ \frac{\Gamma(m+3)}{\Gamma(b)} \sum_{n=1}^{\infty} \frac{(a)_n \Gamma(b+kn)}{\Gamma(m+3+kn)} \frac{z^n}{(n-1)!} \right\} \\ + \frac{m}{(m+2)} \left\{ \frac{\Gamma(m+3)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(b+kn)}{\Gamma(m+3+kn)} \frac{z^n}{n!} \right\} \\ + \frac{k^2}{(m+2)} \left\{ \frac{\Gamma(m+3)}{\Gamma(b)} \sum_{n=1}^{\infty} \frac{n(a)_n \Gamma(b+kn)}{\Gamma(m+3+kn)} \frac{z^n}{(n-1)!} \right\} \\ + \frac{\alpha}{(m+2)} \left\{ \frac{\Gamma(m+3)}{\Gamma(b)} \sum_{n=1}^{\infty} \frac{(a)_n \Gamma(b+kn)}{\Gamma(m+3+kn)} \frac{z^n}{(n-1)!} \right\} \\ + \frac{\beta}{(m+2)} \left\{ \frac{\Gamma(m+3)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(b+kn)}{\Gamma(m+3+kn)} \frac{z^n}{n!} \right\},$$

where $\alpha = k(2m+1)$ and $\beta = m(m+1)$.

We now express each summation in the right hand side of (24) as follows:

$$(25) \quad \frac{d^2}{dz^2} (z^2 {}_2R_1(a, b; m+3; k; z)) \\ = {}_2R_1(a, b; m+3; k; z) + 4z {}_2\dot{R}_1(a, b; m+3; k; z) \\ + z^2 {}_2\ddot{R}_1(a, b; m+3; k; z)$$

and

$$(26) \quad \frac{d^2}{dz^2} (z^2 {}_2R_1(a, b; m+3; k; z)) \\ = \frac{\Gamma(m+3)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(n+2)(n+1)(a)_n \Gamma(b+kn)}{\Gamma(m+3+kn)} \frac{z^n}{n!} \\ = \frac{\Gamma(m+3)}{\Gamma(b)} \sum_{n=1}^{\infty} \frac{n(a)_n \Gamma(b+kn)}{\Gamma(m+3+kn)} \frac{z^n}{(n-1)!} \\ + 3 \frac{\Gamma(m+3)}{\Gamma(b)} \sum_{n=1}^{\infty} \frac{(a)_n \Gamma(b+kn)}{\Gamma(m+3+kn)} \frac{z^n}{(n-1)!} + {}_2R_1(a, b; m+3; k; z).$$

(25) and (26) imply

$$\begin{aligned}
 (27) \quad & \frac{\Gamma(m+3)}{\Gamma(b)} \sum_{n=1}^{\infty} \frac{n(a)_n \Gamma(b+kn)}{\Gamma(m+3+kn)} \frac{z^n}{(n-1)!} \\
 &= z^2 {}_2\ddot{R}_1(a, b; m+3; k; z) + 4z {}_2\dot{R}_1(a, b; m+3; k; z) \\
 &\quad - 3 \frac{\Gamma(m+3)}{\Gamma(b)} \sum_{n=1}^{\infty} \frac{(a)_n \Gamma(b+kn)}{\Gamma(m+3+kn)} \frac{z^n}{(n-1)!}.
 \end{aligned}$$

Let

$$\begin{aligned}
 (28) \quad & \frac{d}{dz} (z {}_2R_1(a, b; m+3; k; z)) \\
 &= {}_2R_1(a, b; m+3; k; z) + z {}_2\dot{R}_1(a, b; m+3; k; z)
 \end{aligned}$$

and

$$\begin{aligned}
 (29) \quad & \frac{d}{dz} (z {}_2R_1(a, b; m+3; k; z)) \\
 &= \frac{\Gamma(m+3)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(n+1)(a)_n \Gamma(b+kn)}{\Gamma(m+3+kn)} \frac{z^n}{n!} \\
 &= \frac{\Gamma(m+3)}{\Gamma(b)} \sum_{n=1}^{\infty} \frac{(a)_n \Gamma(b+kn)}{\Gamma(m+3+kn)} \frac{z^n}{(n-1)!} + {}_2R_1(a, b; m+3; k; z).
 \end{aligned}$$

From (28) and (29), we get

$$(30) \quad \frac{\Gamma(m+3)}{\Gamma(b)} \sum_{n=1}^{\infty} \frac{(a)_n \Gamma(b+kn)}{\Gamma(m+3+kn)} \frac{z^n}{(n-1)!} = z {}_2\dot{R}_1(a, b; m+3; k; z).$$

Combining (27) and (30), it yields

$$\begin{aligned}
 (31) \quad & \frac{\Gamma(m+3)}{\Gamma(b)} \sum_{n=1}^{\infty} \frac{n(a)_n \Gamma(b+kn)}{\Gamma(m+3+kn)} \frac{z^n}{(n-1)!} \\
 &= z^2 {}_2\ddot{R}_1(a, b; m+3; k; z) + z {}_2\dot{R}_1(a, b; m+3; k; z).
 \end{aligned}$$

Now applying (30) and (31) to (24), we get

$$\begin{aligned}
 (32) \quad S = & \left\{ \frac{k^2}{(m+2)} \right\} z^2 {}_2\ddot{R}_1(a, b; m+3; k; z) \\
 & + \left\{ \frac{k^2 + k + \alpha}{(m+2)} \right\} z {}_2\dot{R}_1(a, b; m+3; k; z) \\
 & + \frac{(m+\beta)}{(m+2)} {}_2R_1(a, b; m+3; k; z).
 \end{aligned}$$

From (22), (23) and (32), we arrive at

$$\begin{aligned} & (m+1) {}_2R_1(a, b; m+1; k; z) - {}_2R_1(a, b; m+2; k; z) \\ &= \left\{ \frac{k^2}{(m+2)} \right\} z^2 {}_2\ddot{R}_1(a, b; m+3; k; z) + \frac{k}{(m+2)} \{k+2(m+1)\} \\ & \quad \times z {}_2\dot{R}_1(a, b; m+3; k; z) + m {}_2R_1(a, b; m+3; k; z). \end{aligned} \quad \square$$

5. Some integral representations of the generalized hypergeometric function

Theorem 5.1. For $k > 0$, $\Re(a) > 0$, $\Re(b) > 0$, $\Re(m) > 0$, $\Re(m-a-b) > 0$ we get

$$(33) \quad \int_0^1 t^m {}_2R_1(a, b; m; k; t^k) dt = \frac{{}_2R_1(a, b; m+1; k; 1)}{(m)} - \frac{{}_2R_1(a, b; m+2; k; 1)}{(m)(m+1)}.$$

Proof. On putting $z = 1$ in (23), it yields

$$\begin{aligned} (34) \quad & \frac{\Gamma(m)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(b+kn)}{(m+1+kn) \Gamma(m+kn) n!} \\ &= \frac{{}_2R_1(a, b; m+1; k; 1)}{m} - \frac{{}_2R_1(a, b; m+2; k; 1)}{(m+1)m}. \end{aligned}$$

Let

$$\begin{aligned} (35) \quad & \int_0^z t^m {}_2R_1(a, b; m; k; t^k) dt = \int_0^z t^m \left\{ \frac{\Gamma(m)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(b+kn)}{\Gamma(m+kn)} \frac{t^{kn}}{n!} \right\} dt \\ &= \frac{\Gamma(m)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(b+kn)}{(m+1+kn) \Gamma(m+kn) n!} z^{m+1+kn}. \end{aligned}$$

Now, comparing (34) and (35) after setting $z = 1$ in (35), we have

$$\int_0^1 t^m {}_2R_1(a, b; m; k; t^k) dt = \frac{{}_2R_1(a, b; m+1; k; 1)}{m} - \frac{{}_2R_1(a, b; m+2; k; 1)}{(m)(m+1)}.$$

This is the proof of Theorem 5.1. $\square$

Theorem 5.2. If $a, b, c, \tau \in \mathbb{C}$ such that $\Re(a) > 0$, $\Re(b) > 0$, $\Re(c) > 0$, $\Re(c-\tau) > 0$, $k > 0$ and $|z| < 1$, then

$$\begin{aligned} & {}_2R_1(a, b; c; \tau; z) \\ &= k z^{\tau-c} \int_0^{\infty} \exp\left(-\frac{t^k}{z^k}\right) t^{c-\tau-1} \left(\sum_{n=0}^{\infty} \frac{\Gamma(b+\tau n) \Gamma(c) (a)_n t^n}{\Gamma(b) \Gamma(c+\tau n) \Gamma\left(\frac{c-\tau+n}{k}\right)} \right) dt. \end{aligned}$$

Proof. Consider, $\int_0^\infty \exp\left(-\frac{t^k}{z^k}\right) t^{c-\tau-1} \left(\sum_{n=0}^\infty \frac{\Gamma(b+\tau n)\Gamma(c)(a)_n t^n}{\Gamma(b)\Gamma(c+\tau n)\Gamma\left(\frac{c-\tau+n}{k}\right)}\right) dt$ and substituting $\frac{t^k}{z^k} = u$, we get

$$\begin{aligned} &= \int_0^\infty e^{-u} z^{c-\tau-1} u^{\frac{c-\tau-1}{k}} \left(\sum_{n=0}^\infty \frac{\Gamma(b+\tau n)\Gamma(c) z^n u^{\frac{n}{k}} (a)_n}{\Gamma(b)\Gamma(c+\tau n) n! \Gamma\left(\frac{c-\tau+n}{k}\right)}\right) \frac{z}{k} u^{\frac{1-k}{k}} du \\ &= \frac{z^{c-\tau}}{k} \sum_{n=0}^\infty \frac{\Gamma(b+\tau n)\Gamma(c) z^n (a)_n}{\Gamma(b)\Gamma(c+\tau n) n! \Gamma\left(\frac{c-\tau+n}{k}\right)} \Gamma\left(\frac{c-\tau+n}{k}\right) \end{aligned}$$

which, on further simplification, gives

$$\begin{aligned} &\int_0^\infty \exp\left(-\frac{t^k}{z^k}\right) t^{c-\tau-1} \left(\sum_{n=0}^\infty \frac{\Gamma(b+\tau n)\Gamma(c)(a)_n t^n}{\Gamma(b)\Gamma(c+\tau n)\Gamma\left(\frac{c-\tau+n}{k}\right)}\right) dt \\ &= \frac{z^{c-\tau}}{k} {}_2R_1(a, b; c; \tau; z). \end{aligned}$$

This is the proof of Theorem 5.2. $\square$

Theorem 5.3. If $a, b, c, \tau \in \mathbb{C}$ such that $\Re(a) > 0, \Re(b) > 0, \Re(c) > 0, \Re(c-\tau) > 0$ and $|z| < 1$. Then

$${}_2R_1(a, b; c; \tau; z) = \frac{\Gamma(c)}{\tau\Gamma(\tau)\Gamma(c-\tau)} \int_0^1 \left(1 - t\frac{1}{\tau}\right)^{c-\tau-1} {}_2R_1(a, b; \tau; \tau; tz) dt.$$

Proof. Let

$$\begin{aligned} &\int_0^1 \left(1 - t\frac{1}{\tau}\right)^{c-\tau-1} {}_2R_1(a, b; \tau; \tau; tz) dt \\ &= \int_0^1 \left(1 - t\frac{1}{\tau}\right)^{c-\tau-1} \left(\sum_{n=0}^\infty \frac{(a)_n \Gamma(b+\tau n) \Gamma(\tau) (tz)^n}{\Gamma(b)\Gamma(\tau+\tau n) n!}\right) dt. \end{aligned}$$

On substituting $t\frac{1}{\tau} = u$, we get

$$\begin{aligned} &\int_0^1 \left(1 - t\frac{1}{\tau}\right)^{c-\tau-1} {}_2R_1(a, b; \tau; \tau; tz) dt \\ &= \sum_{n=0}^\infty \frac{(a)_n \Gamma(b+\tau n) \Gamma(\tau) (z)^n}{\Gamma(b)\Gamma(\tau+\tau n) n!} \tau \beta(c-\tau, \tau+\tau n) \\ &= \frac{\tau\Gamma(\tau)\Gamma(c-\tau)}{\Gamma(c)} {}_2R_1(a, b; c; \tau; z). \end{aligned}$$

This can easily be written as

$${}_2R_1(a, b; c; \tau; z) = \frac{\Gamma(c)}{\tau\Gamma(\tau)\Gamma(c-\tau)} \int_0^1 \left(1-t^{\frac{1}{\tau}}\right)^{c-\tau-1} {}_2R_1(a, b; \tau; \tau; tz) dt.$$

This is the proof of Theorem 5.3. $\square$

Theorem 5.4. *If $a, b, c, \tau \in \mathbb{C}$ such that $\Re(a) > 0, \Re(b) > 0, \Re(c) > 0, \Re(c-\tau) > 0$ and $|z| < 1$, then*

$$\begin{aligned} & {}_2R_1(a, b; c; \tau; z) \\ &= \frac{\Gamma(c)}{\Gamma(\tau)\Gamma(c-\tau)} \int_0^1 t^{\tau-1} (1-t)^{c-\tau-1} {}_2R_1(a, b; c-\tau; \tau; z(1-t)^\tau) dt. \end{aligned}$$

Proof.

$$\begin{aligned} & \int_0^1 t^{\tau-1} (1-t)^{c-\tau-1} {}_2R_1(a, b; c-\tau; \tau; z(1-t)^\tau) dt \\ &= \int_0^1 t^{\tau-1} (1-t)^{c-\tau-1} \left(\frac{\Gamma(c-\tau)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(b+\tau n)}{\Gamma(c-\tau+\tau n)} \frac{z^n (1-t)^{n\tau}}{n!} \right) dt \\ &= \frac{\Gamma(c-\tau)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(b+\tau n)}{\Gamma(c-\tau+\tau n)} \frac{z^n}{n!} \left(\int_0^1 t^{\tau-1} (1-t)^{c-\tau+\tau n-1} \right) dt \\ &= \frac{\Gamma(\tau)\Gamma(c-\tau)}{\Gamma(c)} \left\{ \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!} \right\} \\ &= \frac{\Gamma(\tau)\Gamma(c-\tau)}{\Gamma(c)} {}_2R_1(a, b; c; \tau; z). \end{aligned}$$

This gives

$$\begin{aligned} & {}_2R_1(a, b; c; \tau; z) \\ &= \frac{\Gamma(c)}{\Gamma(\tau)\Gamma(c-\tau)} \int_0^1 t^{\tau-1} (1-t)^{c-\tau-1} {}_2R_1(a, b; c-\tau; \tau; z(1-t)^\tau) dt, \end{aligned}$$

which proves Theorem 5.4. $\square$

Appendix: A

Proposition. *For $\tau > 0$, and $\Re(a) > 0, \Re(b) > 0, \Re(c) > 0$; on $|z| = 1$, the function ${}_2R_1(a, b; c; \tau; z)$ is defined provided $\Re(c-a-b) > 0$.*

Proof. For $\tau > 0$, and $\Re(a) > 0, \Re(b) > 0, \Re(c) > 0$ the function ${}_2R_1(a, b; c; \tau; z)$ is

$$(36) \quad {}_2R_1^-(z) = {}_2R_1(a, b; c; \tau; z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(b+\tau n)}{\Gamma(c+\tau n) n!} z^n.$$

We have to prove its convergence condition for $|z| = 1$ by using the comparison test.

For $|z| = 1$, $\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \left| \frac{\Gamma(c)}{\Gamma(b)} \frac{(a)_n \Gamma(b+\tau n)}{\Gamma(c+\tau n) n!} z^n \right| = 1 + \sum_{n=1}^{\infty} \left| \frac{\Gamma(c)}{\Gamma(b)} \frac{(a)_n \Gamma(b+\tau n)}{\Gamma(c+\tau n) n!} \right|$
and for $\Re(c-a-b) > 0$, $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}}$; ($\delta = \frac{1}{2}\Re(c-a-b) > 0$),

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(\frac{u_n}{v_n} \right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{\left| \frac{\Gamma(c)}{\Gamma(b)} \frac{(a)_n \Gamma(b+\tau n)}{\Gamma(c+\tau n) n!} \right|}{\frac{1}{n^{1+\delta}}} \right) = \lim_{n \rightarrow \infty} \left| n^{1+\delta} \frac{\Gamma(c)}{\Gamma(b)} \frac{(a)_n \Gamma(b+\tau n)}{\Gamma(c+\tau n) n!} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(a)_n}{(n-1)! n^a} \frac{\Gamma(b+\tau n)}{\Gamma(b) (n-1)! n^b} \frac{\Gamma(c) (n-1)! n^c}{\Gamma(c+\tau n)} \frac{(n-1)! n^{1+\delta}}{n! n^{c-a-b}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{1}{\Gamma(a)} \frac{\Gamma(b+\tau n)}{\Gamma(b) (n-1)! n^b} \frac{\Gamma(c) (n-1)! n^c}{\Gamma(c+\tau n)} \right| \lim_{n \rightarrow \infty} \left| \frac{1}{n^{c-a-b-\delta}} \right| \\ &= (\text{Finite Number}) \cdot \lim_{n \rightarrow \infty} \left| \frac{1}{n^{c-a-b-\delta}} \right| \quad (\text{Explanation is given below in Section I}) \\ &= 0, \text{ because } \Re(c-a-b-\delta) = 2\delta - \delta > 0. \end{aligned}$$

Therefore $\sum_{n=0}^{\infty} u_n = 1 + \sum_{n=1}^{\infty} \left| \frac{\Gamma(c)}{\Gamma(b)} \frac{(a)_n \Gamma(b+\tau n)}{\Gamma(c+\tau n) n!} \right|$ is convergent, since $\sum_{n=1}^{\infty} v_n$ is convergent.

Thus the series in (36) is absolutely convergent on $|z| = 1$ when $\Re(c-a-b) > 0$.

Hence, ${}_2R_1^-(z) = {}_2R_1(a, b; c; \tau; z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(b+\tau n)}{\Gamma(c+\tau n) n!} z^n$ is convergent for $|z| = 1$ when $\Re(c-a-b) > 0$. $\square$

Section I : For $\Re(a) > 0, n \in \mathbb{N}, \tau \geq 1, 0 < |\Gamma(a+n)| \leq |\Gamma(a+\tau n)|$. Therefore, $0 < \left| \frac{1}{\Gamma(a+\tau n)} \right| \leq \left| \frac{1}{\Gamma(a+n)} \right|$.

Thus

$$(37) \quad 0 < \lim_{n \rightarrow \infty} \left| \frac{(n-1)! n^a}{\Gamma(a+\tau n)} \right| \leq \lim_{n \rightarrow \infty} \left| \frac{(n-1)! n^a}{\Gamma(a+n)} \right| \Rightarrow 0 < \lim_{n \rightarrow \infty} \left| \frac{(n-1)! n^a}{\Gamma(a+\tau n)} \right| \leq 1.$$

Also, for $\Re(a) > 0, n \in \mathbb{N}, 0 < \tau \leq 1, 0 < |\Gamma(a+\tau n)| \leq |\Gamma(a+n)|$.

Which implies that

$$\begin{aligned} (38) \quad 0 < \left| \frac{(n-1)! n^a}{\Gamma(a+n)} \right| &\leq \left| \frac{(n-1)! n^a}{\Gamma(a+\tau n)} \right| \Rightarrow 0 < \lim_{n \rightarrow \infty} \left| \frac{\Gamma(a+\tau n)}{(n-1)! n^a} \right| \leq \lim_{n \rightarrow \infty} \left| \frac{\Gamma(a+n)}{(n-1)! n^a} \right| \\ &\Rightarrow 0 < \lim_{n \rightarrow \infty} \left| \frac{\Gamma(a+\tau n)}{(n-1)! n^a} \right| \leq 1. \end{aligned}$$

From (37) and (38), afterwards taking the limit $n \rightarrow \infty$ as given below,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left| \frac{1}{\Gamma(a)} \frac{\Gamma(b + \tau n)}{\Gamma(b)(n-1)!n^b} \frac{\Gamma(c)(n-1)!n^c}{\Gamma(c + \tau n)} \right| \\ &= \left| \frac{1}{\Gamma(a)} \right| \left| \lim_{n \rightarrow \infty} \frac{\Gamma(b + \tau n)}{\Gamma(b)(n-1)!n^b} \right| \left| \lim_{n \rightarrow \infty} \frac{\Gamma(c)(n-1)!n^c}{\Gamma(c + \tau n)} \right|. \end{aligned}$$

This is finite for all $\tau > 0$.

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SNEHAL B. RAO
DEPARTMENT OF APPLIED MATHEMATICS
THE M.S. UNIVERSITY OF BARODA
VADODARA-390 001, INDIA
E-mail address: sbr_msub@yahoo.com

AMIT D. PATEL
DEPARTMENT OF APPLIED MATHEMATICS AND HUMANITIES
S.V. NATIONAL INSTITUTE OF TECHNOLOGY
SURAT-395 007, INDIA
E-mail address: ptlmit83@gmail.com

JYOTINDRA C. PRAJAPATI
DEPARTMENT OF MATHEMATICS
CHAROTAR INSTITUTE OF TECHNOLOGY
CHANGA, ANAND-380 421, INDIA
E-mail address: jyotindra18@rediffmail.com

AJAY K. SHUKLA
DEPARTMENT OF APPLIED MATHEMATICS AND HUMANITIES
S.V. NATIONAL INSTITUTE OF TECHNOLOGY
SURAT-395 007, INDIA
E-mail address: ajayshukla2@rediffmail.com