FGK INEQUALITY ON THE ANALOGUE OF WIENER SPACE

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Abstract. This work aims at proving the FKG inequalities for the analogue of Wiener functionals with special orders on the analogue of Wiener space.

1. Introduction

In 1971, Fortuin, Kastelyn and Ginibre proved an inequality as following; If Γ is a finite distributive lattice, μ is a positive measure on Γ such that $\mu(x \vee y)\mu(x \wedge y) \geq \mu(x)\mu(y)$ and f and g are both increasing functions on Γ then $\langle fg \rangle - \langle f \rangle \langle g \rangle \geq 0$ where $\langle f \rangle = (\sum_{x \in \Gamma} f(x)\mu(x))/(\sum_{x \in \Gamma} \mu(x))$ in [3]. In the research of correlation functions for the Ising and other lattice models in statistical mechanics this inequality, so-called the FKG inequality, played a fundamental role and was applied to various subject, like fields of optimal transportation problems and cellular automata. In 2005, Barbato introduced an ordering on Wiener space and proved a FKG inequality for Wiener functionals [1]. In 2002, the author and Dr. Im presented the definition of analogue of Wiener space, a kind of the generalization of the concrete Wiener space, and its properties [5]. The purpose of this work is to study orders and FKG inequalities to the analogue of Wiener functionals.

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2. Orders on the Analogue of Wiener Space

In this section, we introduce some definitions of four partially orders on the analogue of Wiener space $C[0, T]$ and investigate their properties.

Throughout in this note, let T be a positive real number, let $C[0, T]$ be the space of all continuous functions on a closed interval $[0, T]$ with the supremum norm $\|x\|_\infty = \sup_{t \in [0, T]} |x(t)|$, let ϕ be a probability Borel measure on $\mathbb{R}$ and m_ϕ be the analogue of Wiener measure on $C[0, T]$ [5]. When ϕ is a Dirac measure δ_0 at the origin in $\mathbb{R}$, m_ϕ is the concrete Wiener measure, this measure is denoted by m_w .

Definition 2.1. For x, y in $C[0, T]$, we say that $x \leq_w y$ if for all t in $[0, T]$, $x(t) \leq y(t)$. For $\vec{t} = (t_0, t_1, t_2, \dots, t_n)$ with $0 \leq t_0 < t_1 < t_2 < \dots < t_n \leq T$ and for x, y in $C[0, T]$, we say that $x \leq_{w, \vec{t}} y$ if for all $j = 0, 1, \dots, n$, $x(t_j) \leq y(t_j)$ and we say that $x \leq_{s, \vec{t}} y$ if $x(0) \leq y(0)$ and for all $j = 1, 2, \dots, n$, $x(t_j) - x(t_{j-1}) \leq y(t_j) - y(t_{j-1})$. For x, y in $C[0, T]$, we say that $x \leq_s y$ if $x(0) \leq y(0)$ and for all t_1, t_2 in $[0, T]$ with $0 \leq t_1 < t_2 \leq T$, $x(t_2) - x(t_1) \leq y(t_2) - y(t_1)$.

Remark 2.2. 1) $\leq_w, \leq_{w, \vec{t}}, \leq_{s, \vec{t}}$ and $\leq_s$ are all partially orders on $C[0, T]$.

2) On $C[0, T]$, $\leq_{w, \vec{t}}$ is finer than $\leq_w$, that is, $x \leq_w y$ implies $x \leq_{w, \vec{t}} y$, clearly.

3) On $C[0, T]$, $\leq_w$ is finer than $\leq_s$ because if $x \leq_s y$ then for all t in $[0, T]$, $x(t) - x(0) \leq y(t) - y(0)$, so for all t in $[0, T]$, $y(t) - x(t) \leq y(0) - x(0) \leq 0$.

4) On $C[0, T]$, $\leq_{s, \vec{t}}$ is finer than $\leq_s$, that is, $x \leq_s y$ implies $x \leq_{s, \vec{t}} y$, clearly.

5) $\leq_w$ and $\leq_{s, \vec{t}}$ are incomparable and $\leq_{w, \vec{t}}$ and $\leq_{s, \vec{t}}$ are also incomparable.

6) For x, y in $C[0, T]$, $x \leq_w y$ if and only if $x \leq_{w, \vec{t}} y$ for all $\vec{t}$ and, $x \leq_s y$ if and only if $x \leq_{s, \vec{t}} y$ for all $\vec{t}$.

Theorem 2.3. The inequalities $\leq_w, \leq_{w, \vec{t}}, \leq_{s, \vec{t}}$ and $\leq_s$ are all closed relations on $C[0, T]$.

Proof. Let $\vec{t} = (t_0, t_1, t_2, \dots, t_k)$ with $0 = t_0 < t_1 < t_2 < \dots < t_k \leq T$ and let $C[\leq_{w, \vec{t}}] = \{(x, y) \in C[0, T] \times C[0, T] \mid x \leq_{w, \vec{t}} y\}$. If a sequence $\langle (x_n, y_n) \rangle$ in $C[\leq_{w, \vec{t}}]$ converges to (x, y) in the norm $\|(x, y)\| = \max\{\|x\|_\infty, \|y\|_\infty\}$ sense then for all $0 \leq j \leq k$, $x_n(t_j) \leq y_n(t_j)$, then for all $0 \leq j \leq k$, $x(t_j) \leq y(t_j)$, so (x, y) is in $C[\leq_{w, \vec{t}}]$, that is, $C[\leq_{w, \vec{t}}]$ is

closed which implies that $\leq_{w,\vec{t}}$ is a closed relation. From 6) in Remark 2.2, letting $C[\leq_w] = \{(x, y) \in C[0, T] \times C[0, T] \mid x \leq_w y\}$, $C[\leq_w] = \cap_{\vec{t}} C[\leq_{w,\vec{t}}]$, so $C[\leq_w]$ is closed which implies that $\leq_w$ is a closed relation. Using the essentially similar process, we know that $\leq_{s,\vec{t}}$ and $\leq_s$ are all closed relations on $C[0, T]$. $\square$

Definition 2.4. A Borel subset A of $C[0, T]$ is called $\leq_w$ increasing ($\leq_{w,\vec{t}}$ increasing, $\leq_{s,\vec{t}}$ increasing, $\leq_s$ increasing, respectively) if x is in A and $x \leq_w y$ ($x \leq_{w,\vec{t}} y$, $x \leq_{s,\vec{t}} y$, $x \leq_s y$, respectively) implies y is in A . Let $\mathcal{I}_w(\mathcal{I}_{w,\vec{t}}, \mathcal{I}_{s,\vec{t}}, \mathcal{I}_s, \text{ respectively})$ be the set of all $\leq_w$ increasing ($\leq_{w,\vec{t}}$ increasing, $\leq_{s,\vec{t}}$ increasing, $\leq_s$ increasing, respectively) subsets of $C[0, T]$. The $\leq_w$ decreasing ($\leq_{w,\vec{t}}$ decreasing, $\leq_{s,\vec{t}}$ decreasing, $\leq_s$ decreasing, respectively) subsets $\mathcal{D}_w(\mathcal{D}_{w,\vec{t}}, \mathcal{D}_{s,\vec{t}}, \mathcal{D}_s, \text{ respectively})$ of $C[0, T]$ are defined analogously.

Remark 2.5. 1) Let $\vec{t} = (t_0, t_1, t_2, \dots, t_k)$ with $0 = t_0 < t_1 < t_2 < \dots < t_k \leq T$. Then A is in $\mathcal{I}_{w,\vec{t}}$ if and only if there are intervals $I_1, I_2, \dots, I_k$ having an end point $+\infty$ such that $A = \{x \in C[0, T] \mid x(t_j) \in I_j \text{ for } j = 0, 1, 2, \dots, k\}$ and B is in $\mathcal{I}_{s,\vec{t}}$ if and only if there are intervals $I_1, I_2, \dots, I_k$ having an end point $+\infty$ such that $B = \{x \in C[0, T] \mid x(t_j) - x(t_{j-1}) \in I_j \text{ for } j = 1, 2, \dots, k\}$.
 2) Let $C = \{x \in C[0, T] \mid \text{for all } t \in [0, T], x(t) \geq 0\}$. Then C is in $\mathcal{I}_w$ and is in $\mathcal{I}_s$ because if x is in C and $x \leq_s y$ then $0 \leq x(0) \leq y(0)$ and for all $t \in [0, T]$, $x(t) - x(0) \leq y(t) - y(0)$, so $0 \leq y(0) - x(0) \leq y(t) - x(t)$. But C is neither in $\mathcal{I}_{w,\vec{t}}$ for each $\vec{t}$ nor in $\mathcal{I}_{s,\vec{t}}$ for each $\vec{t}$.
 3) Let $\vec{t} = (0, T)$ and $D = \{x \in C[0, T] \mid x(0) \geq 0 \text{ and } x(T) - x(0) \geq 0\}$. Then D is in $\mathcal{I}_{s,\vec{t}}$ and in $\mathcal{I}_s$ but D is neither in $\mathcal{I}_w$ nor in $\mathcal{I}_{w,\vec{t}}$.
 4) Let $E = \{x \in C[0, T] \mid x(0) \geq 0 \text{ and } x \text{ is increasing}\}$. Then E is in $\mathcal{I}_s$ but is neither in $\mathcal{I}_w$ nor in $\mathcal{I}_{w,\vec{t}}$ for all $\vec{t}$ nor in $\mathcal{I}_{s,\vec{t}}$.
 5) It is a conclusion that $\mathcal{I}_s \subsetneq \mathcal{I}_{s,\vec{t}}, \mathcal{I}_s \subsetneq \mathcal{I}_w \subsetneq \mathcal{I}_{w,\vec{t}}$ for $\vec{t}$.

Definition 2.6. An ordered space $C[0, T]$ is normally ordered if, given disjoint closed subsets F_1, F_2 with F_1 increasing, F_2 decreasing, there exist disjoint open sets G_1, G_2 such that G_1 is increasing, G_2 is decreasing and $F_1 \subset G_1, F_2 \subset G_2$.

From [4], one can find the following lemma.

Lemma 2.7. Let X be a commutative group with an invariant partial order $\leq$, that is, $x \leq y$ implies $x + z \leq y + z$ for all $z \in X$ and an invariant metric d , that is, $d(x, y) = d(x + z, y + z)$ for all $z \in X$. Then X is normally ordered.

By Lemma 2.7, directly we have the following theorem.

Theorem 2.8. *The inequalities $\leq_w$, $\leq_{w,\vec{t}}$, $\leq_{s,\vec{t}}$ and $\leq_s$ are all normally orders on $C[0, T]$.*

Proof. Clearly $(C[0, T], +)$ is a commutative group where $+$ is an usual addition for functions and $x \leq_w y$ ($x \leq_{w,\vec{t}} y$, $x \leq_{s,\vec{t}} y$, $x \leq_s y$, respectively) implies $x + z \leq_w y + z$ ($x + z \leq_{w,\vec{t}} y + z$, $x + z \leq_{s,\vec{t}} y + z$, $x + z \leq_s y + z$, respectively) for all $z \in C[0, T]$. Since $d(x, y) = \|x - y\|_\infty$ is an invariant metric on $C[0, T]$. So, the inequalities $\leq_w$, $\leq_{w,\vec{t}}$, $\leq_{s,\vec{t}}$ and $\leq_s$ are all normally orders on $C[0, T]$. $\square$

Definition 2.9. A measure μ on $C[0, T]$ said to satisfy the FKG condition with respect to $\leq_w$ ($\leq_{w,\vec{t}}$, $\leq_{s,\vec{t}}$, $\leq_s$, respectively) if for any two $\leq_w$ ($\leq_{w,\vec{t}}$, $\leq_{s,\vec{t}}$, $\leq_s$, respectively) increasing sets A, B , $\mu(A \cup B)\mu(A \cap B) \geq \mu(A)\mu(B)$ holds. A measure μ on $C[0, T]$ is called associated $\leq_w$ ($\leq_{w,\vec{t}}$, $\leq_{s,\vec{t}}$, $\leq_s$, respectively) if for any two $\leq_w$ ($\leq_{w,\vec{t}}$, $\leq_{s,\vec{t}}$, $\leq_s$, respectively) increasing sets A, B , $\mu(A \cap B) \geq \mu(A)\mu(B)$ holds.

Remark 2.10. 1) Let $T > 2$ and let $(C_0[0, T], m_w)$ be the concrete Wiener measure space. Put $A = \{x \in C_0[0, T] | x(1) \geq 0\}$ and $B = \{x \in C_0[0, T] | x(2) \geq 0\}$. Then A and B are both $\leq_w$ increasing sets and $m_w(A) = m_w(B) = \frac{1}{2}$. By the change of variable $u_2 - u_1 = v$,

$$\begin{aligned} m_w(A \cap B) &= \frac{1}{2\pi} \int_0^\infty \left(\int_0^\infty \exp\left\{-\frac{1}{2}[u_1^2 + (u_2 - u_1)^2]\right\} du_2 \right) du_1 \\ &= \frac{1}{2\pi} \int_0^\infty \exp\left\{-\frac{u_1^2}{2}\right\} \left(\int_0^\infty \exp\left\{-\frac{v^2}{2}\right\} dv + \int_{-u_1}^0 \exp\left\{-\frac{v^2}{2}\right\} dv \right) du_1 \\ &> \frac{1}{4}. \end{aligned}$$

By the essentially same method, we have $m_w(A^c) = m_w(B^c) = \frac{1}{2}$, $m_w(A^c \cap B^c) > \frac{1}{4}$. From now on, we assume that m_w satisfies the FKG condition with respect to $\leq_w$. Then $m_w(A \cup B)m_w(A \cap B) \geq m_w(A)m_w(B)$ holds. So $m_w(A \cup B)m_w(A \cap B) - m_w(A)m_w(B) = -(m_w(A \cap B) - \frac{1}{2})^2 \geq 0$ which implies that $m_w(A \cap B) = \frac{1}{2}$. Hence $m_w(A \setminus (A \cap B)) = m_w(A \cap B^c) = 0$, a contradiction. Therefore, m_w doesn't satisfy the FKG condition with respect to $\leq_w$. Thus, it is not true that the analogue of Wiener measure m_ϕ satisfies the FKG condition with respect to $\leq_w$ for all ϕ .

2) Let $T > 2$ and let $(C[0, T], m_\phi)$ be the analogue of Wiener measure space. Put $C = \{x \in C[0, T] | x(1) - x(0) \geq 0\}$, $D = \{x \in$

$C[0, T][x(2) - x(1) \geq 0]$ and $\vec{t} = (0, 1, 2)$. Then C and D are both $\leq_{\vec{t}}$ increasing sets, are stochastic independent [5] and $m_\phi(C) = m_\phi(D) = \frac{1}{2}$. We assume that m_ϕ satisfies the FKG condition with respect to $\leq_{\vec{t}}$. Then $m_\phi(C \cup D)m_\phi(C \cap D) = m_\phi(C \cup D)m_\phi(C)m_\phi(D) \geq m_\phi(C)m_\phi(D)$ holds. So $m_\phi(C \cup D) \geq 1$, a contradiction. Therefore, it is not true that m_ϕ satisfies the FKG condition with respect to $\leq_{\vec{t}}$ for all ϕ .

3) Let $0 < t_1 < t_2 \leq T$ be given and let α, β be two positive real numbers. Let $A = \{x \in C[0, T] | x(t_1) \geq \alpha\}$ and $B = \{x \in C[0, T] | x(t_2) \geq \beta\}$. Then $m_w(A) = \frac{1}{2\pi t_1} \int_\alpha^\infty \exp\{-\frac{u^2}{2t_1}\} du$ and $m_w(B) = \frac{1}{2\pi t_2} \int_\beta^\infty \exp\{-\frac{u^2}{2t_2}\} du$. By the change of variable $\frac{\sqrt{t_2}(u_2 - u_1)}{\sqrt{t_2 - t_1}} = v_1$,

$$\begin{aligned} m_w(A \cap B) &= \frac{1}{2\pi\sqrt{t_1(t_2 - t_1)}} \int_\alpha^\infty \int_\beta^\infty \exp\{-\frac{1}{2}[\frac{u_1^2}{t_1} + \frac{(u_2 - u_1)^2}{t_2 - t_1}]\} du_2 du_1 \\ &= \frac{1}{2\pi\sqrt{t_1 t_2}} \int_\alpha^\infty \exp\{-\frac{u_1^2}{2t_1}\} (\int_{\frac{\sqrt{t_2}(\beta - u_1)}{\sqrt{t_2 - t_1}}}^\infty \exp\{-\frac{v_1^2}{2t_2}\} dv_1) du_1 \\ &= \frac{1}{2\pi\sqrt{t_1 t_2}} \int_\alpha^\infty \exp\{-\frac{u_1^2}{2t_1}\} (\int_\beta^\infty \exp\{-\frac{v_1^2}{2t_2}\} dv_1 \\ &\quad + \int_{\frac{\sqrt{t_2}(\beta - u_1)}{\sqrt{t_2 - t_1}}}^\beta \exp\{-\frac{v_1^2}{2t_2}\} dv_1) du_1 \\ &= m_w(A)m_w(B) + \frac{1}{2\pi\sqrt{t_1 t_2}} \int_\alpha^\infty \exp\{-\frac{u_1^2}{2t_1}\} \times (\int_{\frac{\sqrt{t_2}(\beta - u_1)}{\sqrt{t_2 - t_1}}}^\beta \exp\{-\frac{v_1^2}{2t_2}\} dv_1) du_1. \end{aligned}$$

Now we assume that for all α in $\mathbb{R}$, $m_w(A \cap B) \geq m_w(A)m_w(B)$. Then $U \equiv \frac{1}{2\pi\sqrt{t_1 t_2}} \int_\alpha^\infty \exp\{-\frac{u_1^2}{2t_1}\} (\int_{\frac{\sqrt{t_2}(\beta - u_1)}{\sqrt{t_2 - t_1}}}^\beta \exp\{-\frac{v_1^2}{2t_2}\} dv_1) du_1 \geq 0$. Putting $\beta = 1, t_1 = 1, t_2 = 2$,

$$\begin{aligned} U &\equiv \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty \exp\{-\frac{u^2}{2}\} (\int_{1-u}^{\frac{1}{\sqrt{2}}} \exp\{-\frac{v^2}{2}\} dv) du \\ &= \frac{1}{2\pi} (\int_{\frac{2-\sqrt{2}}{2}}^\infty [\int_{1-u}^{\frac{1}{\sqrt{2}}} \exp\{-\frac{u^2 + v^2}{2}\} du] dv - \int_{-\infty}^{\frac{2-\sqrt{2}}{2}} [\int_{\frac{1}{\sqrt{2}}}^{1-u} \exp\{-\frac{u^2 + v^2}{2}\} du] dv) \\ &= \frac{1}{2\pi} (\int_{-\infty}^{-\frac{2-\sqrt{2}}{2}} [\int_{-\frac{1}{\sqrt{2}}}^{-1-u} \exp\{-\frac{u^2 + v^2}{2}\} du] dv - \int_{-\infty}^{\frac{2-\sqrt{2}}{2}} [\int_{\frac{1}{\sqrt{2}}}^{1-u} \exp\{-\frac{u^2 + v^2}{2}\} du] \\ &\quad dv) < 0 \end{aligned}$$

contradicted by the assumption. Hence the concrete Wiener measure m_w is not associated $\leq_{w, \vec{t}}$ where $\vec{t} = (1, 2)$, so the concrete Wiener

measure m_w is not associated $\leq_w$. Thus, it is not true that the analogue of Wiener measure m_ϕ is associated $\leq_w$ for all ϕ .

3. FGK inequality on the Analogue of Wiener Space

In this section, we study the FGK inequalities on the analogue of Wiener space.

Theorem 3.1. *The analogue of Wiener measure m_ϕ is associated $\leq_{s,\vec{t}}$ for all $\vec{t}$ and for all ϕ , and the analogue of Wiener measure m_ϕ is associated $\leq_s$ for all ϕ .*

Proof. Without loss of generality we let $\vec{t} = (t_0, t_1, \dots, t_n)$ with $0 \leq t_0 < t_1 < \dots < t_n \leq T$, let $A = \{x \in C[0, T] \mid \text{for } 1 \leq j \leq n, x(t_j) - x(t_{j-1}) > a_j, x(0) > a_0\}$ and let $B = \{x \in C[0, T] \mid \text{for } 1 \leq j \leq n, x(t_j) - x(t_{j-1}) > b_j, x(0) > b_0\}$ where a_j and b_j ($1 \leq j \leq n$) are all extended real numbers. Putting $a \vee b = \max\{a, b\}$, for any positive real number α , $\frac{1}{\sqrt{2\pi\alpha}} \int_{a \vee b}^\infty \exp\{-\frac{u^2}{2\alpha}\} du \geq \frac{1}{2\pi\alpha} [\int_a^\infty \exp\{-\frac{u^2}{2\alpha}\} du]$ $[\int_b^\infty \exp\{-\frac{u^2}{2\alpha}\} du]$ and $\phi((a \vee b, +\infty)) \geq \phi((a, +\infty)) \phi((b, +\infty))$. Hence, by the change of variables,

$$\begin{aligned} & m_\phi(A \cap B) \\ &= \frac{1}{\sqrt{(2\pi)^n}} \int_{a_0 \vee b_0}^\infty \int_{a_1 \vee b_1}^\infty \cdots \int_{a_n \vee b_n}^\infty \exp\{-\frac{1}{2} \sum_{j=1}^n u_j^2\} du_n du_{n-1} \cdots du_1 d\phi(u_0) \\ &\geq \frac{1}{(2\pi)^n} \left(\int_{a_0}^\infty \int_{a_1}^\infty \cdots \int_{a_n}^\infty \exp\{-\frac{1}{2} \sum_{j=1}^n u_j^2\} du_n du_{n-1} \cdots du_1 d\phi(u_0) \right) \\ &\quad \left(\int_{b_0}^\infty \int_{b_1}^\infty \cdots \int_{b_n}^\infty \exp\{-\frac{1}{2} \sum_{j=1}^n u_j^2\} du_n du_{n-1} \cdots du_1 d\phi(u_0) \right) \\ &= m_\phi(A) m_\phi(B), \end{aligned}$$

so the analogue of Wiener measure m_ϕ is associated $\leq_{s,\vec{t}}$ for all $\vec{t}$. If A and B are both $\leq_s$ increasing sets then A and B are both $\leq_{s,\vec{t}}$ increasing sets. So, by the previous part in this proof, we have $m_\phi(A \cap B) \geq m_\phi(A) m_\phi(B)$, so the analogue of Wiener measure m_ϕ is associated $\leq_s$. $\square$

Definition 3.2. A function $f : C[0, T] \rightarrow \mathbb{R}$ said to be $\leq_w$ increasing ($\leq_{w,\vec{t}}$ increasing, $\leq_{s,\vec{t}}$ increasing, $\leq_s$ increasing, respectively) if for all $x \leq_w y$ ($x \leq_{w,\vec{t}} y$, $x \leq_{s,\vec{t}} y$, $x \leq_s y$, respectively) in $C[0, T]$, $f(x) \leq f(y)$.

- Remark 3.3.** 1) Letting $f(x) = \max\{x(t) | t \in [0, T]\}$ on $C[0, T]$, f is $\leq_w$ increasing.
 2) Letting $\vec{t} = (0, T)$ and $f(x) = x(T)$ on $C[0, T]$, f is $\leq_{w, \vec{t}}$ increasing.
 3) Letting $\vec{t} = (0, T)$ and $f(x) = x(T) - x(0)$ on $C[0, T]$, f is $\leq_{s, \vec{t}}$ increasing.
 4) Letting $f(x) = \min\{x(t) - x(s) | s \text{ and } t \text{ are in } [0, T] \text{ with } s < t\}$ on $C[0, T]$, f is $\leq_s$ increasing.

We can find the following lemma in [2].

Lemma 3.4. *Let X be a Polish space and let $\leq$ be a closed partially order on X . Let P be a Borel probability measure on X . Then the following are equivalent.*

- (1) P is associated.
- (2) $\int_X f(x)g(x)dP(x) \geq \int_X f(x)dP(x) \int_X g(x)dP(x)$ for all real-valued bounded increasing measurable functions f, g on X .
- (3) $P(C_1 \cap C_2) \geq P(C_1)P(C_2)$ for all closed increasing subsets C_1, C_2 of X .
- (4) $P(C_1 \cap C_2) \geq P(C_1)P(C_2)$ for all compact generated increasing subsets C_1, C_2 of X .

Moreover, if X is normally ordered, then each of the above conditions is equivalent to :

- (5) $\int_X f(x)g(x)dP(x) \geq \int_X f(x)dP(x) \int_X g(x)dP(x)$ for all real-valued bounded increasing continuous functions f, g on X .

So, we obtain the following theorems directly from the above lemma.

- Theorem 3.5.** (1) $\int_{C[0, T]} f(x)g(x)dm_w(x) \leq \int_{C[0, T]} f(x)dm_w(x) \int_{C[0, T]} g(x)dm_w(x)$ holds for some real-valued $\leq_w$ increasing (or $\leq_{w, \vec{t}}$ increasing) measurable functions f, g on $C[0, T]$.
 (2) $m_w(C_1 \cap C_2) \leq m_w(C_1)m_w(C_2)$ for some closed $\leq_w$ increasing (or $\leq_{w, \vec{t}}$ increasing) subsets C_1, C_2 of $C[0, T]$.
 (3) $m_w(C_1 \cap C_2) \leq m_w(C_1)m_w(C_2)$ for some compact generated $\leq_w$ increasing (or $\leq_{w, \vec{t}}$ increasing) subsets C_1, C_2 of $C[0, T]$.
 (4) $\int_{C[0, T]} f(x)g(x)dm_w(x) \leq \int_{C[0, T]} f(x)dm_w(x) \int_{C[0, T]} g(x)dm_w(x)$ holds for some real-valued $\leq_w$ increasing (or $\leq_{w, \vec{t}}$ increasing) continuous functions f, g on $C[0, T]$.

- Theorem 3.6.** (1) $\int_{C[0, T]} f(x)g(x)dm_\phi(x) \geq \int_{C[0, T]} f(x)dm_\phi(x) \int_{C[0, T]} g(x)dm_\phi(x)$ holds for all real-valued bounded $\leq_s$ increasing (or $\leq_{s, \vec{t}}$ increasing) measurable functions f, g on $C[0, T]$ and for all Borel probability measure ϕ .
 (2) $m_\phi(C_1 \cap C_2) \geq m_\phi(C_1)m_\phi(C_2)$ for all closed $\leq_s$ increasing (or $\leq_{s, \vec{t}}$ increasing) subsets C_1, C_2 of $C[0, T]$ and for all Borel probability measure ϕ .

(3) $m_\phi(C_1 \cap C_2) \geq m_\phi(C_1)m_\phi(C_2)$ for all compact generated $\leq_s$ increasing (or $\leq_{s,\bar{t}}$ increasing) subsets C_1, C_2 of $C[0, T]$ and for all Borel probability measure ϕ .

(4) $\int_{C[0, T]} f(x)g(x)dm_\phi(x) \geq \int_{C[0, T]} f(x)dm_\phi(x) \int_{C[0, T]} g(x)dm_\phi(x)$ holds for all real-valued bounded $\leq_s$ increasing (or $\leq_{s,\bar{t}}$ increasing) continuous functions f, g on $C[0, T]$ and for all Borel probability measure ϕ .

Definition 3.7. An ordered measure space $(C[0, T], \mathcal{B}(C[0, T]), \mu, \leq)$ is said to be satisfied the FGK inequality if for all $\leq$ increasing functions f, g in $L^2(C[0, T], \mu)$ $\int_{C[0, T]} f(x)g(x)d\mu(x) \geq \int_{C[0, T]} f(x)d\mu(x) \int_{C[0, T]} g(x)d\mu(x)$ holds.

Theorem 3.8. The ordered measure spaces $(C[0, T], \mathcal{B}(C[0, T]), \mu, \leq)$, $\leq_{s,\bar{t}}$ and $(C[0, T], \mathcal{B}(C[0, T]), \mu, \leq_s)$ satisfy the FGK inequality.

Proof. By Theorem 3.6 in above, $\int_{C[0, T]} f(x)g(x)dm_\phi(x) \geq \int_{C[0, T]} f(x)dm_\phi(x) \int_{C[0, T]} g(x)dm_\phi(x)$ holds for all real-valued bounded $\leq_{s,\bar{t}}$ increasing measurable functions f, g . For a real-valued $\leq_{s,\bar{t}}$ increasing measurable functions f, g , let

$$f_n(x) = \begin{cases} n & , n < f(x) \\ f(x) & , |f(x)| \leq n \\ -n & , f(x) < -n \end{cases}, \quad g_n(x) = \begin{cases} n & , n < g(x) \\ g(x) & , |g(x)| \leq n \\ -n & , g(x) < -n \end{cases}.$$

and for a function $h : C[0, T] \rightarrow \mathbb{R}$, let $h^{(1)}(x) = \max\{h(x), 0\}$ and $h^{(2)}(x) = \max\{-h(x), 0\}$. Since f and g are in $L^2(C[0, T], m_\phi)$, by the dominated convergence theorem, for $i, j = 1, 2$,

$$\lim_{n \rightarrow \infty} \int_{C[0, T]} f_n^{(i)}(x)dm_\phi(x) = \int_{C[0, T]} f^{(i)}(x)dm_\phi(x),$$

$$\lim_{n \rightarrow \infty} \int_{C[0, T]} g_n^{(i)}(x)dm_\phi(x) = \int_{C[0, T]} g^{(i)}(x)dm_\phi(x),$$

$$\lim_{n \rightarrow \infty} \int_{C[0, T]} f_n^{(i)}(x)g_n^{(j)}(x)dm_\phi(x) = \int_{C[0, T]} f^{(i)}(x)g^{(j)}(x)dm_\phi(x),$$

and

$$\lim_{n \rightarrow \infty} \int_{C[0, T]} f^{(i)}(x)g_n^{(j)}(x)dm_\phi(x) = \int_{C[0, T]} f^{(i)}(x)g^{(j)}(x)dm_\phi(x).$$

So the limits $\lim_{n \rightarrow \infty} \int_{C[0, T]} f_n(x)dm_\phi(x)$, $\lim_{n \rightarrow \infty} \int_{C[0, T]} g_n(x)dm_\phi(x)$ and $\lim_{n \rightarrow \infty} \int_{C[0, T]} f_n(x)g_n(x)dm_\phi(x)$ exist. Since for natural number

n ,

$$\int_{C[0,T]} f_n(x)g_n(x)dm_\phi(x) \geq \int_{C[0,T]} f_n(x)dm_\phi(x) \int_{C[0,T]} g_n(x)dm_\phi(x),$$

we have $\int_{C[0,T]} f(x)g(x)dm_\phi(x) \geq \int_{C[0,T]} f(x)dm_\phi(x) \int_{C[0,T]} g(x)dm_\phi(x)$ holds. $\square$

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