ON ALMOST SURE CONVERGENCE FOR WEIGHTED SUMS OF $LNQD$ RANDOM VARIABLES

JEONG YEOL CHOI, SO YOUN KIM AND JONG IL BAEK*

Abstract. Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a sequence of $LNQD$ which are dominated randomly by another random variable X . We obtain the complete convergence and almost sure convergence of weighted sums $\sum_{i=1}^n a_{ni}X_{ni}$ for $LNQD$ by using a new exponential inequality, where $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of constants. As corollary, the results of some authors are extended from i.i.d. case to not necessarily identically $LNQD$ case.

1. Introduction

Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a sequence of random variables (r.v.'s) with $EX_n = 0$, and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers. Many authors studied the almost sure convergence of weighted sums $\sum_{i=1}^n a_{ni}X_i$ when $\{X, X_n, n \geq 1\}$ are assumed to be independent and identically distributed (i.i.d.) r.v.'s (see Bai and Cheng (2000), Sung (2001), Cuzick (1995), Chow and Lai (1973) among others). In addition, Bai and Cheng (2000) proved the strong law of large numbers $\sum_{i=1}^n a_{ni}X_i/b_n \rightarrow 0$ a.s. when $\{X, X_n, n \geq 1\}$ is a sequence of i.i.d. r.v.'s with $EX = 0$ and

$$E[\exp(h|X|^\gamma)] < \infty \quad \text{for some } h > 0 \ (\gamma > 0), \quad (1.1)$$

and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of real numbers satisfying

$$A_\alpha = \limsup_{n \rightarrow \infty} A_{\alpha,n} < \infty, \quad A_{\alpha,n}^\alpha = \sum_{i=1}^n |a_{ni}|^\alpha / n \quad (1.2)$$

Received March 29, 2012. Accepted April 23, 2012.

2000 Mathematics Subject Classification. 60F15.

Key words and phrases. Strong law of large numbers, almost sure convergence, arrays, linearly negative quadrant random variables.

*Corresponding Author.

This paper was supported by Wonkwang University in 2010.

for some $1 < \alpha < 2$, where $b_n = n^{1/\alpha}(\log n)^{1/\gamma + \gamma(\alpha-1)/\alpha(1+\gamma)}$.

Sung (2001) extended result of Bai and Cheng (2000) and obtained another almost sure limiting law when condition (1.1) is replaced by stronger condition

$$E[\exp(h|X|^\gamma)] < \infty \quad \text{for any } h > 0 \ (\gamma > 0). \quad (1.3)$$

In this case, $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ if $0 < \gamma \leq 1$.

We extended the result of Sung(2001) by using a new exponential inequality of $LNQD$ r.v.'s with a below concepts. We first recall the definitions and lemmas of negatively associated, negative quadrant dependent and linearly negative quadrant dependent random variables.

Definition 1.1(Alam and Saxena (1981)). A finite sequence of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets A_1, A_2 of $\{1, 2, \dots, n\}$,

$$\text{Cov}\{f(X_i : i \in A_1), g(X_j : j \in A_2)\} \leq 0,$$

whenever f and g are coordinatewise nondecreasing such that this covariance exists. An infinite sequence $\{X_n, n \geq 1\}$ is NA if every finite subcollection is NA .

This definition is introduced by Alam and Saxena (1981). Many authors derived several important properties about NA sequences and also discussed some applications in the area of statistics, probability, reliability and multivariate analysis. Compared to positively associated random variables, the study of NA random variables has received less attention in the literature. Readers may refer to Karlin and Rinott(1980), Joag-Dev and Proschan(1983), Matula(1992) and Roussas(1994) among others. Recently, some authors focussed on the problem of limiting behavior of partial sums of NA sequences(see, Su and Qin(1997), Shao and Su(1999),Liang(2000), Liang et al(2004), and Baek et al(2005)).

Definition 1.2(Lehmann (1966)). Two random variables X and Y are said to be negative quadrant dependent(NQD) if for any $x, y \in \mathbb{R}$,

$$P(X < x, Y < y) \leq P(X < x)P(Y < y).$$

A sequence $\{X_n, n \geq 1\}$ of random variables is said to be pairwise NQD if X_i and X_j are NQD for all $i, j \in \mathbb{N}^+$ and $i \neq j$.

Lemma 1.1(Lehmann (1966)). Let X and Y be NQD random variables, then (a) $EXY \leq EXEY$, (b) $P(X < x, Y < y) \leq P(X <$

$x)P(Y < y)$, and (c) If f and g are both nondecreasing (or both nonincreasing) functions, then $f(X)$ and $g(Y)$ are NQD .

Definition 1.3(Newman (1984)). A sequence $\{X_n, n \geq 1\}$ of random variables is said to be linearly negative quadrant dependent ($LNQD$) if for any disjoint subsets $A, B \subset \mathbb{N}^+$ and positive r_j 's,

$$\sum_{k \in A} r_k X_k \text{ and } \sum_{j \in B} r_j X_j \text{ are } NQD.$$

Lemma 1.2. Let $\{X_n, n \geq 1\}$ be a sequence of $LNQD$ random variables with $EX_n = 0$ for each $n \geq 1$, then for any $t > 0$,

$$Ee^{t \sum_{i=1}^n X_i} \leq \prod_{i=1}^n Ee^{tX_i} \leq e^{t^2/2 \sum_{i=1}^n EX_i^2 e^{t|X_i|}}$$

Proof. Noticing that tX_i and $\sum_{j=i+1}^n tX_j$ are $LNQD$, we know by Definition 1.3, e^{tX_i} and $e^{t \sum_{j=i+1}^n X_j}$ are also $LNQD$ for $i = 1, 2, \dots, n-1$. We will prove the first inequality by mathematical induction that

$$Ee^{t \sum_{i=1}^n X_i} \leq \prod_{i=1}^n Ee^{tX_i}. \quad (1.4)$$

First, we observe that

$$\begin{aligned} Ee^{t(X_1+X_2)} &\leq Ee^{tX_1} Ee^{tX_2} \\ &= \prod_{i=1}^2 Ee^{tX_i}. \end{aligned}$$

Where the inequality follows from Lemma 1.1. Thus, (1.4) is true for $i = 2$. Assume now that the statement is true for $i = k$. We will show that it is true for $i = k + 1$.

$$\begin{aligned} Ee^{t \sum_{i=1}^{k+1} X_i} &= E\left(e^{t \sum_{i=1}^k X_i} e^{tX_{k+1}}\right) \\ &\leq Ee^{t \sum_{i=1}^k X_i} Ee^{tX_{k+1}} \\ &\leq \prod_{i=1}^k Ee^{tX_i} Ee^{tX_{k+1}} \\ &= \prod_{i=1}^{k+1} Ee^{tX_i}. \end{aligned}$$

Finally, we will prove the second inequality that

$$\prod_{i=1}^n Ee^{tX_i} \leq e^{t^2/2 \sum_{i=1}^n EX_i^2 e^{t|X_i|}}.$$

For all $x \in \mathbb{R}$, taking $e^x \leq 1 + x + x^2/2e^{|x|}$ and $EX_i = 0$, we have

$$\begin{aligned} Ee^{tX_i} &\leq 1 + tEX_i + t^2/2EX_i^2 e^{t|X_i|} \\ &= 1 + t^2/2EX_i^2 e^{t|X_i|} \\ &\leq e^{t^2/2EX_i^2 e^{t|X_i|}}, \text{ by } 1 + x \leq e^x. \end{aligned}$$

Thus, we get that

$$\prod_{i=1}^n Ee^{tX_i} \leq e^{t^2/2 \sum_{i=1}^n EX_i^2 e^{t|X_i|}}.$$

From the above definition, it is immediate that NA implies $LNQD$. Newman(1984) introduced the concepts of $LNQD$ r.v.'s and many authors derived several important properties about $LNQD$ r.v.'s and also discussed some applications in several areas(see Newman(1984), Cai and Roussas(1997), Wang and Zhang(2006), Ko et al.(2007) among others).

The main purpose of this paper is to extend results of Sung(2001) for i.i.d. case to not necessarily identically distributed the case of linearly negative quadrant dependent r.v.'s, which contains independent and negatively associated random variables as special cases. First, we shall study the limit properties of weighted sums of $LNQD$ r.v.'s by using a exponential inequalities, which are dominated randomly by another random variable X . In particular, we shall consider the case when $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ are $LNQD$ r.v.'s with $P(|X_{ni}| > x) \leq cP(|X| > x)$ for all i and $x \geq 0$. As corollary, the results of some authors are extended from i.i.d. case to not necessarily identically distributed $LNQD$ setting. Throughout this paper, $a_{ni} = a_{ni}^+ - a_{ni}^-$, where $a_{ni}^+ = \max(a_{ni}, 0)$, $a_{ni}^- = \max(-a_{ni}, 0)$, c denote the positive constant whose values are unimportant and may vary at different place.

2. Main results

We will deal with the complete convergence and almost sure convergence for weighted sums of $LNQD$ r.v.'s by using exponential inequalities in this Chapter.

Theorem 2.1. Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise $LNQD$ random variables with $EX_{ni} = 0$ and $P(|X_{ni}| > x) \leq cP(|X| > x)$ for all $i \geq 1$ and $x \geq 0$. Assume that $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of constants, and that $X_{ni}^2 \sum_{i=1}^n a_{ni}^2 \leq v_n |X_i|^\delta / \log n$ a.s. for some $\delta > 0$ and some sequence $\{v_n\}$ of constants such that $v_n \rightarrow 0$.

- (a) Let $|a_{ni}X_{ni}| \leq u_n |X_i|^\beta / \log n$ a.s. for some $0 < \beta \leq \gamma$ and some sequence $\{u_n\}$ of constants such that $u_n \rightarrow 0$. If $E(e^{h|X|^\gamma}) < \infty$ for some $h > 0$ ($\gamma > 0$), then

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^n a_{ni}X_{ni}\right| > \varepsilon\right) < \infty \text{ for any } \varepsilon > 0. \quad (2.1)$$

- (b) Let $|a_{ni}X_{ni}| \leq c|X_i|^\beta / \log n$ a.s. for some $0 < \beta \leq \gamma$ and some constant $c \geq 0$. If $E(e^{h|X|^\gamma}) < \infty$ for any $h > 0$ ($\gamma > 0$), then (2.1) remains true. Furthermore, both (a) and (b) imply $\sum_{i=1}^n a_{ni}X_{ni} \rightarrow 0$ a.s. as $n \rightarrow \infty$.

Proof. We prove only (a), the proof of (b) is similar to the proof of (a). It suffices to show that

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^n a_{ni}^+ X_{ni}\right| > \varepsilon\right) < \infty \text{ for any } \varepsilon > 0, \quad (2.2)$$

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^n a_{ni}^- X_{ni}\right| > \varepsilon\right) < \infty \text{ for any } \varepsilon > 0. \quad (2.3)$$

We prove only (2.2), the proof of (2.3) is analogous. To prove (2.2), we need only to prove that

$$\sum_{n=1}^{\infty} P\left(\sum_{i=1}^n a_{ni}^+ X_{ni} > \varepsilon\right) < \infty \text{ for any } \varepsilon > 0, \quad (2.4)$$

$$\sum_{n=1}^{\infty} P\left(\sum_{i=1}^n a_{ni}^+ X_{ni} < -\varepsilon\right) < \infty \text{ for any } \varepsilon > 0. \quad (2.5)$$

Note that $\{a_{ni}^+ X_{ni}, 1 \leq i \leq n, n \geq 1\}$ is still an array of rowwise $LNQD$ random variables by Definition 1.3. Thus, by Lemma 1.2 and $|x|^\delta \leq O(e^{(h/2)|x|^\beta})$ for all $x \in R$ and taking $t = M \log n / \varepsilon$, where M is large constant, we have

$$\sum_{n=1}^{\infty} P\left(\sum_{i=1}^n a_{ni}^+ X_{ni} > \varepsilon\right) < \infty.$$

Next, by replacing X_{ni} by $-X_{ni}$ from (2.4) and noticing $\{a_{ni}^+(-X_{ni}), 1 \leq i \leq n, n \geq 1\}$ is still an array of rowwise *LNQD* random variables, we know that

$$\sum_{n=1}^{\infty} P\left(\sum_{i=1}^n a_{ni}^+ X_{ni} < -\varepsilon\right) < \infty \text{ for any } \varepsilon > 0.$$

Hence, the result follows by (2.4) and (2.5). The proof is complete.

Theorem 2.2. Let $0 < p < 2$ and let $\{X_{ni} | 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise *LNQD* random variables with $EX_{ni} = 0$. Assume that $\{a_{ni} | 1 \leq i \leq n, n \geq 1\}$ is an array of real numbers satisfying $\max_{1 \leq i \leq n} |a_{ni}| = O(n^{-1/p})$. If $|X_{ni}| \leq M$, where M is a positive constant, then

$$\sum_{i=1}^n a_{ni} X_{ni} \rightarrow 0 \text{ completely as } n \rightarrow \infty.$$

Proof. As for the proof of Theorem 2.1, it suffices to show that

$$\sum_{n=1}^{\infty} P\left(\sum_{i=1}^n a_{ni}^+ X_{ni} > \varepsilon\right) < \infty \text{ for any } \varepsilon > 0.$$

Without loss of generality, we assume that

$$0 < a_{ni}^+ \leq n^{-1/p}, \text{ for } 1 \leq i \leq n, n \geq 1,$$

We also know that $\{a_{ni}^+ X_{ni} | 1 \leq i \leq n, n \geq 1\}$ is still an array of rowwise *LNQD* random variables and $|a_{ni}^+ X_{ni}| \leq n^{-1/p} M$ and $Ea_{ni}^+ X_{ni} = 0$

Hence, taking $t = 1/\varepsilon \log n$, we have that

$$\begin{aligned}
& \sum_{n=1}^{\infty} P\left(\sum_{i=1}^n a_{ni}^+ X_{ni} > \varepsilon\right) \\
&= \sum_{n=1}^{\infty} P\left(e^{\varepsilon t n^{1/p-1/2}/M} \sum_{i=1}^n a_{ni}^+ X_{ni} > e^{\varepsilon^2 t^2 n^{1/p-1/2}/M}\right) \\
&\leq \sum_{n=1}^{\infty} e^{-\varepsilon^2 t^2 n^{1/p-1/2}/M} \prod_{i=1}^n E e^{\varepsilon t n^{1/p-1/2}/M} (a_{ni}^+ X_{ni}) \\
&\leq \sum_{n=1}^{\infty} e^{-\varepsilon^2 t^2 n^{1/p-1/2}/M} e^{(\varepsilon t n^{1/p-1/2}/2M)^2} \sum_{i=1}^n E (X_{ni} a_{ni}^+)^2 e^{\varepsilon t n^{1/p-1/2}/M} |a_{ni}^+ X_{ni}| \\
&\leq \sum_{n=1}^{\infty} e^{-\varepsilon^2 t^2 n^{1/p-1/2}/M} e^{(\varepsilon t n^{1/p-1/2}/2M)^2} n^{1-2/p} M^2 e^{(\varepsilon t n^{1/p-1/2}/M)(n^{-1/p} M)} \\
&\leq c \sum_{n=1}^{\infty} e^{-\varepsilon^2 t^2 n^{1/p-1/2}/M} e^{c \varepsilon^2 t^2} \\
&\leq c \sum_{n=1}^{\infty} e^{-(\log n)^2 n^{1/p-1/2}/M} < \infty,
\end{aligned}$$

since $0 < p < 2$ and $1/p - 1/2 > 0$. The proof is complete.

Theorem 2.3. Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise $LNQD$ random variables with $EX_{ni} = 0$ and $P(|X_{ni}| > x) \leq cP(|X| > x)$ for all $i \geq 1$ and $x \geq 0$. Assume that $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of constants satisfying (1.2).

(a) If $E(e^{h|X|^\gamma}) < \infty$ for any $h > 0$ and $0 < \gamma \leq 1$, then

$$\sum_{i=1}^n a_{ni} X_{ni} / n^{1/\alpha} (\log n)^{1/\gamma} \rightarrow 0 \text{ a.s. as } n \rightarrow \infty.$$

(b) If $E(e^{h|X|^\gamma}) < \infty$ for some $h > 0$ and $\gamma > 0$, then

$$\sum_{i=1}^n a_{ni} X_{ni} / n^{1/\alpha} (\log n)^{(1/\gamma)+\delta} \rightarrow 0 \text{ a.s. as } n \rightarrow \infty,$$

where $\delta = 1 - 1/\gamma - (\gamma - 1)/(1 + \alpha\gamma - \alpha)$.

Proof. We prove only (a), the proof of (b) is similar to the proof of (a). Let $X_{ni} = X'_{ni} + X''_{ni}$ with $X'_{ni} = X_{ni} I(X_{ni} \leq (\log n)^{1/\gamma}) + (\log n)^{1/\gamma} I(X_{ni} > (\log n)^{1/\gamma})$. Then $\{a_{ni} X'_{ni} | 1 \leq i \leq n, n \geq 1\}$ and $\{a_{ni} X''_{ni} | 1 \leq i \leq n, n \geq 1\}$ are still an array of rowwise $LNQD$ sequences

by the Definition 1.3 of X'_{ni} and X''_{ni} , respectively, and noticing that $EX'_{ni} + EX''_{ni} = 0$, we have that

$$\begin{aligned}
 & \sum_{i=1}^n a_{ni} X_{ni} / n^{1/\alpha} (\log n)^{1/\gamma} \\
 = & \sum_{i=1}^n a_{ni} (X'_{ni} - EX'_{ni}) / n^{1/\alpha} (\log n)^{1/\gamma} \\
 & + \sum_{i=1}^n a_{ni} (X''_{ni} - EX''_{ni}) / n^{1/\alpha} (\log n)^{1/\gamma} \\
 =: & I_1 + I_2
 \end{aligned}$$

As to I_2 , according to Markov's inequality and conditions of Theorem 2.3, it follows that

$$\begin{aligned}
 & P(I_2 > \varepsilon) \\
 = & P\left(\sum_{i=1}^n a_{ni} (X''_{ni} - EX''_{ni}) > \varepsilon n^{1/\alpha} (\log n)^{1/\gamma}\right) \\
 \leq & \sum_{i=1}^n a_{ni}^2 E(X''_{ni})^2 / n^{1/\alpha} (\log n)^{2/\gamma} \\
 \leq & \sum_{i=1}^n a_{ni}^2 (E(X''_{ni})^2 I(X_{ni} > (\log n)^{1/\gamma}) + \\
 & (\log n)^{2/\gamma} P(X_{ni} > (\log n)^{1/\gamma})) / n^{2/\alpha} (\log n)^{2/\gamma} \\
 \leq & cA_{\alpha,n}^2 EX^2 / (\log n)^{2/\gamma} \rightarrow 0 \text{ as } n \rightarrow \infty,
 \end{aligned}$$

$$\text{and } \sum_{n=1}^{\infty} P\left(\sum_{i=1}^n a_{ni} (X''_{ni} - EX''_{ni}) > \varepsilon n^{1/\alpha} (\log n)^{1/\gamma}\right) < \infty.$$

Thus, by the Borel-Cantelli Lemma, we obtain that

$$\sup \sum_{i=1}^n a_{ni} (X''_{ni} - EX''_{ni}) / n^{1/\alpha} (\log n)^{1/\gamma} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (2.6)$$

Next, as to I_1 ,

$$\begin{aligned}
 P(I_1 > \varepsilon) & \leq e^{-\varepsilon t} E^t \sum_{i=1}^n a_{ni} (X'_{ni} - EX'_{ni}) / n^{1/\alpha} (\log n)^{1/\gamma} \\
 & \leq e^{-\varepsilon t} \prod_{i=1}^n E e^{ta_{ni} (X'_{ni} - EX'_{ni}) / n^{1/\alpha} (\log n)^{1/\gamma}}. \quad (2.7)
 \end{aligned}$$

Since $EX'_{ni} + EX''_{ni} = 0$ and $X''_{ni} \geq 0, EX'_{ni} \leq 0$ and

$$\begin{aligned} -EX'_{ni} &= EX_{ni}I(X_{ni} > (\log n)^{1/\gamma}) - (\log n)^{1/\gamma}P(X_{ni} > (\log n)^{1/\gamma}) \\ &\leq cE|X|I(|X| > (\log n)^{1/\gamma}) \\ &\leq ce^{-\log n}E(|X|e^{|X|^\gamma}) \\ &\leq cn^{-1}. \end{aligned}$$

Hence, by Lemma 1.2 and conditions of Theorem 2.3, we get that

$$\begin{aligned} &\prod_{i=1}^n Ee^{ta_{ni}(X'_{ni}-EX'_{ni})/n^{1/\alpha}(\log n)^{1/\gamma}} \\ &= \prod_{i=1}^n Ee^{ta_{ni}(X'_{ni})/n^{1/\alpha}(\log n)^{1/\gamma}} e^{ta_{ni}(-EX'_{ni})/n^{1/\alpha}(\log n)^{1/\gamma}} \\ &\leq \prod_{i=1}^n Ee^{ta_{ni}(X'_{ni})/n^{1/\alpha}(\log n)^{1/\gamma}} e^{ctA_{\alpha,n}/n^{1/\alpha}(\log n)^{1/\gamma}} \\ &\leq c \prod_{i=1}^n Ee^{ta_{ni}(X'_{ni})/n^{1/\alpha}(\log n)^{1/\gamma}} \\ &\leq ce^{t^2/2 \sum_{i=1}^n a_{ni}^2 E(X_{ni})^2 e^{t|a_{ni}X_{ni}|/n^{1/\alpha}(\log n)^{1/\gamma}}/n^{2/\alpha}(\log n)^{2/\gamma}} \\ &\leq c \left(e^{t^2/2 \sum_{i=1}^n a_{ni}^2 EX_{ni}^2 I(0 \leq X_{ni} \leq (\log n)^{1/\gamma})} \right. \\ &\quad \left. e^{t|a_{ni}X_{ni}|I(0 \leq X_{ni} \leq (\log n)^{1/\gamma})/n^{1/\alpha}(\log n)^{1/\gamma}}/n^{2/\alpha}(\log n)^{2/\gamma} \right. \\ &\quad \left. + e^{t^2/2 \sum_{i=1}^n a_{ni}^2 (\log n)^{2/\gamma} P(X_{ni} > (\log n)^{1/\gamma})} e^{t|a_{ni}|I(X_{ni} > (\log n)^{1/\gamma})/n^{1/\alpha}(\log n)^{1/\gamma}}/n^{2/\alpha}(\log n)^{2/\gamma} \right) \\ &\leq ce^{t^2/2 \sum_{i=1}^n a_{ni}^2 EX_{ni}^2 e^{ta_{ni}|X_{ni}|^\gamma(\log n)^{1/\gamma-1}/n^{1/\alpha}(\log n)^{1/\gamma}}/n^{2/\alpha}(\log n)^{2/\gamma}} \\ &\leq ce^{t^2/2 \sum_{i=1}^n a_{ni}^2 EX_{ni}^2 e^{tA_{\alpha,n}|X_{ni}|^\gamma/(\log n)}/n^{2/\alpha}(\log n)^{2/\gamma}} \end{aligned}$$

From the inequality above and $E(e^{t|X|^\gamma}) < \infty$, we have that

$$\prod_{i=1}^n Ee^{ta_{ni}(X'_{ni}-EX'_{ni})/n^{1/\alpha}(\log n)^{1/\gamma}} \leq ce^{ct^2 A_{\alpha,n}^2/(\log n)^{2/\gamma}}. \quad (2.8)$$

Hence, by (2.7) and (2.8) and taking $t = \log n/\varepsilon$, we have that

$$\begin{aligned} P(I_1 > \varepsilon) &\leq e^{-\varepsilon t} Ee^{t \sum_{i=1}^n a_{ni}(X'_{ni}-EX'_{ni})/n^{1/\alpha}(\log n)^{1/\gamma}} \\ &\leq e^{-\varepsilon t} \prod_{i=1}^n Ee^{ta_{ni}(X'_{ni}-EX'_{ni})/n^{1/\alpha}(\log n)^{1/\gamma}} \\ &\leq ce^{-\varepsilon t} e^{Ct^2 A_{\alpha,n}^2/(\log n)^{2/\gamma}} \\ &\leq cn^{-1} e^{cA_{\alpha,n}^2/\varepsilon^2 (\log n)^{2(1-\gamma)/\gamma}} \rightarrow 0 \text{ a.s as } n \rightarrow \infty. \end{aligned} \quad (2.9)$$

Thus, by (2.6) and (2.9), the proof is complete.

We extended results of Sung(2001), and Chow and Lai(1973) from i.i.d. case to not necessary identically distributed *LNQD* case.

Corollary 2.1. Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a sequence of *LNQD* random variables with $EX_{ni} = 0$ and $P(|X_{ni}| > x) \leq cP(|X| > x)$ for all $i \geq 1$ and $x \geq 0$.

- (a) Assume that $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of constants satisfying (1.2) for $1 < \alpha \leq 2$. If $E(e^{h|X|^\gamma}) < \infty$ for some $h > 0$ and $b_n = n^{1/\alpha}(\log n)^{1/\gamma + \alpha + \beta}$ for $\gamma > 1$ and $\beta > 0$, then

$$\sum_{i=1}^n a_{ni} X_{ni} / b_n \rightarrow 0 \text{ a.s. as } n \rightarrow \infty.$$

- (b) Assume that $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of constants satisfying $\limsup_{n \rightarrow \infty} \sum_{i=1}^n a_{ni}^2 < \infty$. If $E(e^{h|X|}) < \infty$ for all $h > 0$ and $b_n = \log n$, then

$$\sum_{i=1}^n a_{ni} X_{ni} / b_n \rightarrow 0 \text{ a.s. as } n \rightarrow \infty.$$

Proof of Corollary 2.1. When weights a_{ni} satisfy (1.2), taking for $1 < \alpha \leq 2$, $|a_{ni}|/n^{1/\alpha} \leq (\sum_{i=1}^n |a_{ni}|^\alpha)^{1/\alpha}/n^{1/\alpha} = A_{\alpha,n}^\alpha$ and $\sum_{i=1}^n a_{ni}^2/n^{2/\alpha} \leq (\sum_{i=1}^n |a_{ni}|^\alpha)^{2/\alpha}/n^{2/\alpha} = A_{\alpha,n}^2$, we can obtain the result of Corollary 2.1 by using Theorem 2.3, and the proof of (b) is similar to the proof (a).

References

- [1] Alam, K., and Saxena, K. M. L. (1981). Positive dependence in multivariate distributions. *Commun. Statist. Theor. Meth*, **A10**, 1183-1196.
- [2] Baek, J. I., Niu, S. L., Lim, P. K., Ahn, Y. Y. and Chung, S. M. (2005). Almost sure convergence for weighted sums of NA random variables. *Journal of the Korean Statistical Society*, **32**(4), 263-272.
- [3] Baek, J. I., Park, S. T., Chung, S. M., Liang, H. Y. and Lee, C. Y. (2005). On the complete convergence of weighted sums for dependent random variables. *Journal of the Korean Statistical Society*, **34**(1), 21-33.
- [4] Baek, J. I., Park, S. T., Chung, S. M., and Seo, H. Y. (2005). On the almost sure convergence of weighted sums of negatively associated random variables. *Comm. Kor. Math. Soci* **20**(3), 539-546.
- [5] Bai, Z. D. and Cheng, P. E. (2000). Marcinkiewicz strong laws for linear statistics. *Statist. & Probab. Lett.* **46**, 105-112.

- [6] Cai, Z. and Roussas, G. G.(1997). Smooth estimate of quantiles under association. *Stat.& Probab.Lett.* **36**, 275-287.
- [7] Chow, Y. S. and Lai, T. L.(1973). Limit behaviour of weighted sums of independent random variables. *Ann. Probab.* **1**, 810-824.
- [8] Cuzick, J.(1995). A strong law for weighted sums of i.i.d. random variables. *J. Theoret. Probab.* **8**, 625-641.
- [9] Joag-Dev, K. and Proschan, F.(1983). Negative association of random variables with applications. *Ann. Statist.* **11**, 286-295.
- [10] Karlin, S., Rinott, R.(1980). Classes of ordering measures and related correlation inequalities. II. Multivariate reverse rule distributions. *Journal of Multivariate Analy.* **10**, 499-516.
- [11] Ko, M. H., Ryu, D. H., & Kim, T. S.(2007). Limiting behaviors of weighted sums for linearly negative quadrant dependent random variables. *Taiwanese Journal of Mathematics*, **11(2)**, 511-522.
- [12] Liang, H. Y., Zhang, D. X., and Baek, J.L.(2004). Convergence of weighted sums for dependent random variables. *Jour.Kor.Math.Soc.*, **41(5)**, 883-894.
- [13] Liang, H. Y.(2000). Complete convergence for weighted sums of negatively associated random variables. *Statist.&Probab.Lett.* **48**, 317-325.
- [14] Lehmann, E. L.(1966). Some concepts of dependence. *The Annals of Mathematical Statistics*, **37**, 1137-1153.
- [15] Matula, P.(1992). A note on the almost sure convergence of sums of negatively dependent random variables. *Statist. Probab. Lett.* **15**, 209-213.
- [16] Newman, C. M.(1984). Asymptotic independence and limit theorems for positively and negatively dependent random variables. In Y. L. Tong(Ed.), *Statistics and probability*, vol. **5**(pp. 127-140). Hayward, CA: Inst. Math. Statist.
- [17] Roussas, G. G.(1994). Asymptotic normality of random fields of positively or negatively associated processes. *J. Multiv. Analysis*, **50**, 152-173.
- [18] Shao, Q. M., Su, C.(1999). The law of the iterated logarithm for negatively associated random variables. *Stochastic Process Appl.* **83**, 139-148.
- [19] Su, C. Qin, Y. S.(1997). Limit theorems for negatively associated sequences. *Chinese Science Bulletin*, **42**, 243-246.
- [20] Sung, S.(2001). Strong laws for weighted sums of i.i.d.random variables. *Statist. & Probab. Lett.* **52**, 413-419.
- [21] Wang, J., & Zhang, L.(2006). A Berry-Esseen theorem for weakly negatively dependent random variables and its applications. *Acta Mathematica Hungarica*, **110(4)**, 293-308.

Jeong Yeol Choi

School of mathematical Science and Institute of Basic Natural Science,
Wonkwang University,
Iksan 570-749, Korea.
E-mail: jychoi@wku.ac.kr

So Youn Kim

School of mathematical Science and Institute of Basic Natural Science,
Wonkwang University,
Iksan 570-749, Korea.

E-mail: rhqnrlidywid@nate.com

Jong Il Baek

School of mathematical Science and Institute of Basic Natural Science,
Wonkwang University,
Iksan 570-749, Korea.

E-mail: jibaek@wonkwang.ac.kr