A NOTE ON THE q -EULER NUMBERS AND POLYNOMIALS WITH WEIGHT (α, w)

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Abstract. The main purpose of this paper is to introduce a new type of q -Euler numbers and polynomials with weak weight (α, w) : $\tilde{E}_{n,q}^{(\alpha,w)}$ and $\tilde{E}_{n,q}^{(\alpha,w)}(x)$, respectively. By using the fermionic p -adic q -integral on $\mathbb{Z}_p$, we can obtain some results and derive some recurrence identities for the q -Euler numbers and polynomials with weak weight (α, w) .

1. Introduction

In many areas of mathematics and physics, the Euler numbers and polynomials possess many interesting properties. Recently, many mathematicians have studied the q -Euler polynomials and related topics (see [1-15]). In this paper, we construct a new type of the q -Euler numbers $\tilde{E}_{n,q}^{(\alpha,w)}$ and polynomials $\tilde{E}_{n,q}^{(\alpha,w)}(x)$ with weak weight (α, w) . By using some interesting relationships of the fermionic p -adic q -integral on $\mathbb{Z}_p$, we obtain some recurrence identities for q -Euler numbers and polynomials with weak weight (α, w) .

Let p be a fixed odd prime number. Throughout this paper, the symbols $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and $\mathbb{C}_p$ will denote the ring of rational integers, the ring of p -adic integers, the field of p -adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$.

When one talks of q -extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally

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assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Throughout this paper, we use the notation :

$$(1.1) \quad [x]_q = \frac{1 - q^x}{1 - q} \text{ and } [x]_{-q} = \frac{1 - (-q)^x}{1 + q} \text{ (see [1 - 11], [15 - 19]).}$$

Hence, $\lim_{q \rightarrow 1} [x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the fermionic p -adic q -integral on $\mathbb{Z}_p$ is defined by Kim as follows :

(1.2)

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x) (-q)^x \text{ (see [3 - 22]).}$$

If we take $f_1(x) = f(x+1)$ in (1.2), then we easily see that

$$(1.3) \quad qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0).$$

For $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$.

Then we can derive the following equation from (1.2):

(1.4)

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-l-1} q^l f(l). \text{ (see [3 - 6]).}$$

For $n \in \mathbb{N}$, let T_p be the p -adic locally constant space defined by

$$(1.5) \quad T_p = \bigcup_{n \geq 1} C_{p^n} = \lim_{n \rightarrow \infty} C_{p^n} = C_{p^\infty},$$

where $C_{p^n} = \{\zeta \in \mathbb{C}_p \mid \zeta^{p^n} = 1 \text{ for some } n \geq 0\}$ is the cyclic group of order p^n .

As well known definition, the Euler polynomials are defined by

$$(1.6) \quad \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}$$

and $E_n(0) = E_n$ are called the Euler numbers (see [1-11]).

The purpose of this paper is to define the q -Euler numbers $\tilde{E}_{n,q}^{(\alpha,w)}$ and polynomials $\tilde{E}_{n,q}^{(\alpha,w)}(x)$ with weak weight (α, w) . We investigate some properties which are related to the q -Euler numbers $\tilde{E}_{n,q}^{(\alpha,w)}$ and polynomials $\tilde{E}_{n,q}^{(\alpha,w)}(x)$ with weak weight (α, w) . We also derive the existence of specific interpolation functions which interpolate the q -Euler numbers

$\tilde{E}_{n,q}^{(\alpha,w)}$ and polynomials $\tilde{E}_{n,q}^{(\alpha,w)}(x)$ with weak weight (α, w) at negative integers. Finally, we obtain some recurrence identities for q -Euler numbers and polynomials with weak weight (α, w) .

2. On the q -Euler polynomials and numbers with weak weight (α, w)

In this section, we define the q -Euler numbers $\tilde{E}_{n,q}^{(\alpha,w)}$ and polynomials $\tilde{E}_{n,q}^{(\alpha,w)}(x)$ with weak weight (α, w) . We also find generating functions of the q -Euler numbers $\tilde{E}_{n,q}^{(\alpha,w)}$ and polynomials $\tilde{E}_{n,q}^{(\alpha,w)}(x)$ with weak weight (α, w) .

For $\alpha \in \mathbb{Z}$, $w \in T_p$ and $q \in \mathbb{C}_1$ with $|q - 1|_p \leq 1$, let us consider the q -Euler numbers with weak weight (α, w) as

$$(2.1) \quad \tilde{E}_{n,q}^{(\alpha,w)} = \int_{\mathbb{Z}_p} w^x [x]_q^n d\mu_{-q^\alpha}(x).$$

By using p -adic q -integral on $\mathbb{Z}_p$, we note that

$$(2.2) \quad \begin{aligned} \int_{\mathbb{Z}_p} w^x [x]_q^n d\mu_{-q^\alpha}(x) &= \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q^\alpha}} \sum_{x=0}^{p^N-1} w^x [x]_q^n (-q^\alpha)^x \\ &= \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q^\alpha}} \sum_{x=0}^{p^N-1} (-wq^\alpha)^x \left\{ \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \right\} \frac{1}{(1-q)^n} \\ &= \frac{[2]_{q^\alpha}}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{1+wq^{\alpha+l}}. \end{aligned}$$

Therefore we obtain the following theorem.

Theorem 1 For $n \in \mathbb{Z}_+$, we have

$$(2.3) \quad \tilde{E}_{n,q}^{(\alpha,w)} = \frac{[2]_{q^\alpha}}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{1+wq^{\alpha+l}}.$$

Let $F_q^{(\alpha,w)}(t)$ be the generating function of $\tilde{E}_{n,q}^{(\alpha,w)}$ with

$$(2.4) \quad F_q^{(\alpha,w)}(t) = \sum_{n=0}^{\infty} \tilde{E}_{n,q}^{(\alpha,w)} \frac{t^n}{n!}.$$

Then we have

(2.5)

$$\begin{aligned}
F_q^{(\alpha, w)}(t) &= \sum_{n=0}^{\infty} \tilde{E}_{n, q}^{(\alpha, w)} \frac{t^n}{n!} = \int_{\mathbb{Z}_p} w^x e^{[x]_q t} d\mu_{-q^\alpha}(x) \\
&= \sum_{n=0}^{\infty} \left\{ \int_{\mathbb{Z}_p} w^x [x]_q^n d\mu_{-q^\alpha}(x) \right\} \frac{t^n}{n!} \\
&= [2]_{q^\alpha} \sum_{n=0}^{\infty} \left\{ \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{1+wq^{\alpha+l}} \right\} \frac{t^n}{n!} \\
&= [2]_{q^\alpha} \sum_{n=0}^{\infty} \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \left\{ \sum_{m=0}^{\infty} (-w)^m q^{(\alpha+l)m} \right\} \frac{t^n}{n!} \\
&= [2]_{q^\alpha} \sum_{m=0}^{\infty} (-w)^m q^{\alpha m} \sum_{n=0}^{\infty} [m]_q^n \frac{t^n}{n!} \\
&= [2]_{q^\alpha} \sum_{m=0}^{\infty} (-w)^m q^{\alpha m} e^{[m]_q t}.
\end{aligned}$$

We assume that $q \in \mathbb{C}$ with $|q| < 1$.

From (2.5), we can observe that

$$\begin{aligned}
(2.6) \quad \tilde{E}_{n, q}^{(\alpha, w)} &= \frac{d^n F_q^{(\alpha, w)}(t)}{dt^n} \Big|_{t=0} \\
&= [2]_{q^\alpha} \sum_{m=1}^{\infty} (-w)^m q^{\alpha m} [m]_q^n.
\end{aligned}$$

For $s \in \mathbb{C}$, we define the q -Euler zeta function as follows:

Definition 2 For $s \in \mathbb{C}$,

$$(2.7) \quad \zeta_q^{(\alpha, w)}(s) = [2]_{q^\alpha} \sum_{m=1}^{\infty} \frac{(-wq^\alpha)^m}{[m]_q^s}.$$

Now we consider the q -Euler polynomials with weak weight (α, w) as follows. We define the q -Euler polynomials with weak weight (α, w) as

$$(2.8) \quad \tilde{E}_{n, q}^{(\alpha, w)}(x) = \int_{\mathbb{Z}_p} w^y [x+y]_q^n d\mu_{-q^\alpha}(y).$$

From (2.8), we can easily derive that

$$(2.9) \quad \tilde{E}_{n,q}^{(\alpha,w)}(x) = \frac{[2]_{q^\alpha}}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \frac{1}{1+wq^{\alpha+l}}.$$

Let $F_q^{(\alpha,w)}(t, x) = \sum_{n=0}^{\infty} \tilde{E}_{n,q}^{(\alpha,w)}(x) \frac{t^n}{n!}$. Then we have

$$(2.10)$$

$$\begin{aligned} F_q^{(\alpha,w)}(t, x) &= \sum_{n=0}^{\infty} \tilde{E}_{n,q}^{(\alpha,w)}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} w^y e^{[x+y]_q t} d\mu_{-q^\alpha}(y) \\ &= \sum_{n=0}^{\infty} \frac{[2]_{q^\alpha}}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \sum_{m=0}^{\infty} (-w)^m q^{(\alpha+l)m} \frac{t^n}{n!} \\ &= [2]_{q^\alpha} \sum_{m=0}^{\infty} (-w)^m q^{\alpha m} e^{[x+m]_q t}. \end{aligned}$$

From (2.10), we obtain

$$(2.11) \quad \begin{aligned} \tilde{E}_{n,q}^{(\alpha,w)}(x) &= \frac{d^n F_q^{(\alpha,w)}(t, x)}{dt^n} \Big|_{t=0} \\ &= [2]_{q^\alpha} \sum_{m=0}^{\infty} (-w)^m q^{\alpha m} [x+m]_q^n. \end{aligned}$$

For $s \in \mathbb{C}$, we define the Hurwitz q -Euler zeta function as follows:

Definition 3 For $s \in \mathbb{C}$,

$$(2.12) \quad \zeta_q^{(\alpha,w)}(s, x) = [2]_{q^\alpha} \sum_{m=0}^{\infty} \frac{(-wq^\alpha)^m}{[x+m]_q^s}$$

By (2.6), (2.7), (2.11) and (2.12), we obtain the following theorem.

Theorem 4 For $k \in \mathbb{N}$, we have

$$(2.13) \quad \zeta_q^{(\alpha,w)}(-k) = \tilde{E}_{n,q}^{(\alpha,w)} \text{ and } \zeta_q^{(\alpha,w)}(-k, x) = \tilde{E}_{n,q}^{(\alpha,w)}(x).$$

In Theorem 4, we can observe that $\zeta_q^{(\alpha,w)}(-k)$ interpolates $\tilde{E}_{n,q}^{(\alpha,w)}$ and $\zeta_q^{(\alpha,w)}(-k, x)$ interpolates $\tilde{E}_{n,q}^{(\alpha,w)}(x)$ at negative integers, respectively.

By using the fermionic p -adic q -integral on $\mathbb{Z}_p$, we can easily see that

$$(2.14) \quad q^\alpha I_{-q^\alpha}(f_1) + I_{-q^\alpha}(f) = [2]_{q^\alpha} f(0),$$

where $f_1(x) = f(x+1)$.

If we take $f(x) = w^x[x]_q^n$ in (2.14), we have

$$(2.15) \quad wq^\alpha \int_{\mathbb{Z}_p} w^x[x+1]_q^n d\mu_{-q^\alpha}(x) + \int_{\mathbb{Z}_p} w^x[x]_q^n d\mu_{-q^\alpha}(x) = \begin{cases} [2]_{q^\alpha} & \text{if } n = 0 \\ 0 & \text{if } n > 0. \end{cases}$$

Therefore, we obtain the following theorem.

Theorem 5 For $n \in \mathbb{Z}_+$, we have

$$(2.16) \quad wq^\alpha \tilde{E}_{n,q}^{(\alpha,w)}(1) + \tilde{E}_{n,q}^{(\alpha,w)}(x) = \begin{cases} [2]_{q^\alpha} & \text{if } n = 0 \\ 0 & \text{if } n > 0. \end{cases}$$

By using (2.8), we can have

$$(2.17) \quad \begin{aligned} \tilde{E}_{n,q}^{(\alpha,w)}(x) &= \int_{\mathbb{Z}_p} w^y[x+y]_q^n d\mu_{-q^\alpha}(y) \\ &= \sum_{l=0}^n \binom{n}{l} [x]_q^{n-l} q^{lx} \int_{\mathbb{Z}_p} w^y[y]_q^l d\mu_{-q^\alpha}(y) \\ &= ([x]_q + q^x \tilde{E}_q^{(\alpha,w)})^n \end{aligned}$$

with the usual convention of replacing $(E_q^{(\alpha,w)})^l$ by $E_{l,q}^{(\alpha,w)}$.

By Theorem 5 and (2.17), we obtain the following corollary.

Corollary 6 For $n \in \mathbb{Z}_+$. Then we have

$$(2.18) \quad wq^\alpha (1 + q\tilde{E}_q^{(\alpha,w)})^n + \tilde{E}_{n,q}^{(\alpha,w)} = \begin{cases} [2]_{q^\alpha} & \text{if } n = 0 \\ 0 & \text{if } n > 0. \end{cases}$$

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