COMPARISON RESULTS FOR THE PRECONDITIONED GAUSS-SEIDEL METHODS

JAE HEON YUN

Abstract. In this paper, we provide comparison results of several types of the preconditioned Gauss-Seidel methods for solving a linear system whose coefficient matrix is a $Z$-matrix. Lastly, numerical results are presented to illustrate the theoretical results.

1. Introduction

In this paper, we consider the following linear system

\[(1)\quad Ax = b, \quad x, b \in \mathbb{R}^n,\]

where $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is a nonsingular matrix. Throughout the paper, we always assume that $A = I - L - U$, where $I$ is the identity matrix, $L$ and $U$ are strictly lower and strictly upper triangular matrices, respectively. The basic iterative method for solving the linear system (1) is

\[(2)\quad Mx_{k+1} = Nx_k + b, \quad k = 0, 1, \ldots,\]

where $x_0$ is an initial vector, $A = M - N$ and $M$ is nonsingular. Then (2) can be also written as

\[(3)\quad x_{k+1} = M^{-1}Nx_k + M^{-1}b, \quad k = 0, 1, \ldots,\]

where $M^{-1}N$ is called an iteration matrix of the iterative method (3).

We now transform the original linear system (1) into the preconditioned linear system

\[(4)\quad Pax = Pb,\]

where $P$ is called a preconditioner. If we apply the Gauss-Seidel method to the preconditioned linear systems (4), then we obtain the preconditioned Gauss-Seidel method for solving the linear system (1). The preconditioned Gauss-Seidel method has been studied by many authors [2, 3, 4, 5, 6, 8, 9, 11, 12].

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In 1991, Gunawardena et al. [3] proposed the preconditioner $P_s = I + S$, where

$$S = \begin{pmatrix} 0 & -a_{12} & 0 & \cdots & 0 \\ 0 & 0 & -a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -a_{n-1,n} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

In 2001, Evans et al. [2] proposed the preconditioner $P_1 = I + R_1$, where

$$R_1 = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ -a_{n1} & \cdots & 0 & 0 \end{pmatrix}.$$ 

In 2004, Niki et al. [8] proposed the preconditioner $P_r = I + R$, where

$$R = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{pmatrix}.$$ 

This paper is organized as follows. In Section 2, we present some notation, definitions and preliminary results which we refer to later. In Section 3, we provide comparison results of several types of the preconditioned Gauss-Seidel methods for solving the linear system (1) whose coefficient matrix is a $Z$-matrix. In Section 4, we provide numerical results to illustrate the theoretical results obtained in Section 3.

2. Preliminaries

For a vector $x \in \mathbb{R}^n$, $x \geq 0$ ($x > 0$) denotes that all components of $x$ are nonnegative (positive). For two vectors $x, y \in \mathbb{R}^n$, $x \geq y$ ($x > y$) means that $x - y \geq 0$ ($x - y > 0$). For a vector $x \in \mathbb{R}^n$, $|x|$ denotes the vector whose components are the absolute values of the corresponding components of $x$. These definitions carry immediately over to matrices. A matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is called a $Z$-matrix if $a_{ij} \leq 0$ for $i \neq j$, and $A$ is called an $M$-matrix if $A$ is a $Z$-matrix and $A^{-1} \geq 0$. For a square matrix $A$, $\rho(A)$ denotes the spectral radius of $A$, and $A$ is called irreducible if the directed graph of $A$ is strongly connected [10].

A representation $A = M - N$ is called a splitting of $A$ when $M$ is nonsingular. A splitting $A = M - N$ is called regular if $M^{-1} \geq 0$ and $N \geq 0$, weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$, and an $M$-splitting of $A$ if $M$ is an $M$-matrix and $N \geq 0$. A splitting $A = M - N$ is called the Gauss-Seidel splitting of $A$ if $M$ and $-N$ are lower triangular and strictly upper triangular parts of $A$, respectively. Some useful results which we refer to later are provided below.
Theorem 2.1 ([1]). Let $A \geq 0$ be a matrix. Then the following hold.
(a) If $Ax \geq \beta x$ for a vector $x \geq 0$ and $x \neq 0$, then $\rho(A) \geq \beta$.
(b) If $Ax \leq \gamma x$ for a vector $x > 0$, then $\rho(A) \leq \gamma$. Moreover, if $A$ is irreducible and if $\beta x \leq Ax \leq \gamma x$, equality excluded, for a vector $x \geq 0$ and $x \neq 0$, then $\beta < \rho(A) < \gamma$ and $x > 0$.

Lemma 2.2 ([5]). Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be an irreducible $M$-matrix with $a_{i,i+1} \neq 0$ for $1 \leq i \leq n - 1$, and let $A_s = (I + S)A = M_s - N_s$ be the Gauss-Seidel splitting of $A$. Then $M_s^{-1}N_s$ has a positive Perron vector and $\rho(M_s^{-1}N_s) > 0$.

Lemma 2.3 ([6]). Let $A$ be an $M$-matrix and let $A_s = (I + S)A = M_s - N_s$ be the Gauss-Seidel splitting of $A$. If $\rho(M_s^{-1}N_s) > 0$, then $Ax \geq 0$ for any nonnegative Perron vector of $M_s^{-1}N_s$.

Lemma 2.4 ([7]). Suppose that $A_1 = M_1 - N_1$ and $A_2 = M_2 - N_2$ are weak regular splittings of the monotone matrices $A_1$ and $A_2$, respectively, such that $M_2^{-1} \geq M_1^{-1}$. If there exists a positive vector $x$ such that $0 \leq A_1x \leq A_2x$, then for the monotonic norm associated with $x$

$$\|M_2^{-1}N_2\|_x \leq \|M_1^{-1}N_1\|_x.$$ 

In particular, if $M_1^{-1}N_1$ has a positive Perron vector, then

$$\rho(M_2^{-1}N_2) \leq \rho(M_1^{-1}N_1).$$

3. Comparison results for preconditioned Gauss-Seidel methods

In this section, we provide comparison results of several types of the preconditioned Gauss-Seidel methods for solving the linear system (1). We assume that $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is a $Z$-matrix with $a_{n1} \neq 0$ and $a_{i,i+1} \neq 0$ for $1 \leq i \leq n - 1$. For simplicity of exposition, let

$$P_s = I + S, \quad P_1 = I + R_1, \quad P_{s1} = I + S + R_1, \quad P_r = I + R,$$

$$A_s = P_sA, \quad A_1 = P_1A, \quad A_{s1} = P_{s1}A, \quad A_r = P_rA.$$ 

Let the Gauss-Seidel splittings of $A, A_s, A_1, A_{s1}$ and $A_r$ be defined by

$$A = M - N, \quad A_s = M_s - N_s, \quad A_1 = M_1 - N_1, \quad A_{s1} = M_{s1} - N_{s1}, \quad A_r = M_r - N_r.$$ 

Let $SL = L_0 + E_0, RLU = L_1 + E_1$ and $RU = L_2 + E_2$, where $L_0, L_1$ and $L_2$ are diagonal matrices, and $E_0, E_1$ and $E_2$ are strictly lower triangular matrices. By simple calculation, one obtains

$$M = I - L, \quad N = U,$$

$$M_s = (I - L_0) - (L + E_0), \quad N_s = U - S + SU,$$

$$M_1 = (I - L_1) - (L - R_1 + E_1), \quad N_1 = U,$$

$$M_{s1} = (I - L_0 - L_1) - (L - R_1 + E_0 + E_1), \quad N_{s1} = U - S + SU,$$

$$M_r = (I - L_2) - (L - R + RL + E_2), \quad N_r = U.$$
Notice that $N = N_1 = N_r = U$ and $N_s = N_{s1} = U - S + SU$. Let 
\[ T = M^{-1}N, \quad T_s = M_{s1}^{-1}N_s, \quad T_1 = M_1^{-1}N_1, \quad T_{s1} = M_{s1}^{-1}N_{s1}, \quad T_r = M_r^{-1}N_r. \]
Then $T$ is an iteration matrix of Gauss-Seidel method, and $T_s$, $T_1$, $T_{s1}$ and $T_r$ are iteration matrices of several types of preconditioned Gauss-Seidel methods.

**Theorem 3.1.** Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be a $Z$-matrix. If $a_{1n}a_{n1} < 1$ and $a_{i,i+1}a_{i+1,i} < 1$ for $1 \leq i \leq n - 1$, then
(a) $\rho(T_{s1}) < \rho(T)$ if $\rho(T) < 1$, 
(b) $\rho(T_{s1}) = \rho(T)$ if $\rho(T) = 1$, 
(c) $\rho(T_{s1}) > \rho(T)$ if $\rho(T) > 1$.

**Proof.** Notice that $A_{s1}$ is also a $Z$-matrix. Since $a_{1n}a_{n1} < 1$ and $a_{i,i+1}a_{i+1,i} < 1$ for $1 \leq i \leq n - 1$, $A_{s1} = M_{s1} - N_{s1}$ is an $M$-splitting of $A_{s1}$. Since $a_{n1} \neq 0$ and $a_{i,i+1} \neq 0$ for $1 \leq i \leq n - 1$, $A$ is irreducible. Since $A = M - N$ is an $M$-splitting of $A$ and $N \neq 0$, there exists a positive eigenvector $x$ such that $Tx = \lambda x$, where $\lambda = \rho(T) > 0$. From $Tx = \lambda x$ and $R_1L = 0$, one easily obtains
\[
Ux = \lambda(I - L)x, \\
SUx = \lambda(S - \Lambda_0 - E_0)x, \\
R_1UX = \lambda R_1x.
\]
Using (5) and $R_1U = A_1 + E_1$,
\[
T_{s1}x - \lambda x = M_{s1}^{-1}(U - S + SU - \lambda(I - \Lambda_0 - \Lambda_1) + \lambda(L - R_1 + E_0 + E_1))x \\
= M_{s1}^{-1}((\lambda - 1)S + \lambda(A_1 + E_1) - \lambda R_1)x \\
= M_{s1}^{-1}((\lambda - 1)S + \lambda R_1U - \lambda R_1)x \\
= (\lambda - 1)M_{s1}^{-1}(S + \lambda R_1)x.
\]
If $\lambda < 1$, then from (6) $T_{s1}x < \lambda x$. Since $x > 0$, Theorem 2.1 implies that $\rho(T_{s1}) < \lambda$. For the cases of $\lambda = 1$ and $\lambda > 1$, $T_{s1}x = \lambda x$ and $T_{s1}x > \lambda x$ are obtained from (6), respectively. Hence, the theorem follows from Theorem 2.1. \(\square\)

**Theorem 3.2.** If $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is an $M$-matrix, then
\[ \rho(T_{s1}) \leq \rho(T_s) < 1. \]

**Proof.** Since $A$ is an irreducible $M$-matrix with $a_{i,i+1} \neq 0$, by Lemma 2.2 there exists a positive eigenvector $x$ such that $T_sx = \rho(T_s)x$ and $\rho(T_s) > 0$. Since $N_s = N_{s1}$, $A_{s1} - A_s = R_1A = M_{s1} - M_s$ and thus
\[ M_{s1}^{-1} - M_{s1}^{-1} = M_{s1}^{-1}R_1AM_{s1}^{-1}. \]
From (7), one obtains
\[ T_s - T_{s1} = M_{s1}^{-1}R_1AT_s. \]
Multiplying by $x$ on both sides of (8) gives

\begin{equation}
\rho(T_s)x - T_{s1}x = \rho(T_s)M_{s1}^{-1}R_1Ax.
\end{equation}

Since $\rho(T_s) > 0$, from Lemma 2.3

\begin{equation}
Ax \geq 0.
\end{equation}

From (9) and (10),

\begin{equation}
T_{s1}x \leq \rho(T_s)x.
\end{equation}

From Theorem 2.1 and (11), it follows that $\rho(T_{s1}) \leq \rho(T_s) < 1$. \qed

**Theorem 3.3.** If $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is an $M$-matrix, then

\[ \rho(T_{s1}) \leq \rho(T_1) < 1. \]

**Proof.** We first consider the case where $A_1$ is an irreducible matrix. Since $A$ is an irreducible $M$-matrix and $A = M - N$ is a regular splitting of $A$, there exists a positive eigenvector $x > 0$ such that $Tx = \rho(T)x$ and $\rho(T) > 0$. Since $0 < \rho(T) < 1$, $Ax \geq 0$ and hence

\[ A_{s1}x = (I + S + R_1)Ax \geq (I + R_1)Ax = A_1x \geq 0. \]

It is easy to show that $A_1$ and $A_{s1}$ are $M$-matrices and $M_{s1}^{-1} \geq M_1^{-1}$. Hence, from Lemma 2.4 $\|T_{s1}\|_x \leq \|T_1\|_x$. Since $A_1$ is an irreducible $M$-matrix, $T_1$ has a positive Perron vector. From Lemma 2.4, it also follows that $\rho(T_{s1}) \leq \rho(T_1)$.

We next consider the case where $A_1$ is a reducible matrix. Let $A(\epsilon) = (a_{ij}(\epsilon))$ be defined by

\[ a_{ij}(\epsilon) = \begin{cases} 
    a_{n1} - \epsilon & \text{if } i = n \text{ and } j = 1 \\
    a_{ij} & \text{otherwise},
\end{cases} \]

where $\epsilon > 0$. Let $A_1 = (I + R_1)A = (\hat{a}_{ij})$, $A_1(\epsilon) = (I + R_1)A(\epsilon) = (\hat{a}_{ij}(\epsilon))$, $A_{s1} = (I + S + R_1)A = (\hat{a}_{ij})$ and $A_{s1}(\epsilon) = (I + S + R_1)A(\epsilon) = (\hat{a}_{ij}(\epsilon))$. Then, it can be shown that $\hat{a}_{n1} = \hat{a}_{n1} = 0$.

\[ \hat{a}_{ij}(\epsilon) = \begin{cases} 
    -\epsilon & \text{if } i = n \text{ and } j = 1 \\
    \hat{a}_{ij} & \text{otherwise}
\end{cases} \]

and

\[ \hat{a}_{ij}(\epsilon) = \begin{cases} 
    -\epsilon & \text{if } i = n \text{ and } j = 1 \\
    \hat{a}_{n-1,1} + \epsilon a_{n-1,n} & \text{if } i = n - 1 \text{ and } j = 1 \\
    \hat{a}_{ij} & \text{otherwise}.
\end{cases} \]

Since $A$, $A_1$ and $A_{s1}$ are $M$-matrices, it can be easily shown that $A(\epsilon)$, $A_1(\epsilon)$ and $A_{s1}(\epsilon)$ are also $M$-matrices for any sufficiently small $\epsilon > 0$. Since $A$ is irreducible, $A(\epsilon)$ and $A_1(\epsilon)$ are irreducible matrices for any $\epsilon > 0$. Let $M(\epsilon) = A(\epsilon) + N$, $M_1(\epsilon) = A_1(\epsilon) + N_1$ and $M_{s1}(\epsilon) = A_{s1}(\epsilon) + N_{s1}$. Then $A(\epsilon) = M(\epsilon) - N$, $A_1(\epsilon) = M_1(\epsilon) - N_1$ and $A_{s1}(\epsilon) = M_{s1}(\epsilon) - N_{s1}$ are the Gauss-Seidel splittings of $A(\epsilon)$, $A_1(\epsilon)$ and $A_{s1}(\epsilon)$, respectively. Let $T(\epsilon) = M(\epsilon)^{-1}N$, $T_1(\epsilon) = M_1(\epsilon)^{-1}N_1$ and $T_{s1}(\epsilon) = M_{s1}(\epsilon)^{-1}N_{s1}$. Since $A(\epsilon)$ is irreducible and
$$A(\epsilon) = M(\epsilon) - N$$ is an $M$-splitting of $A(\epsilon)$, there exists a positive eigenvector $x$ such that $T(\epsilon)x = \rho(T(\epsilon))x$ and $\rho(T(\epsilon)) > 0$. Hence, $A(\epsilon)x \geq 0$, which implies that
$$A_{s1}(\epsilon)x \geq A_1(\epsilon)x \geq 0.$$ It is easy to show that $M_{s1}(\epsilon)^{-1} \geq M_1(\epsilon)^{-1}$. Hence, from Lemma 2.4
$$\|T_{s1}(\epsilon)\|_x \leq \|T_1(\epsilon)\|_x.$$ Since $A_1(\epsilon)$ is irreducible and $A_1(\epsilon) = M_1(\epsilon) - N_1$ is an $M$-splitting of $A_1(\epsilon)$, $T_1(\epsilon)$ has a positive Perron vector. From Lemma 2.4, it also follows that
$$\rho(T_{s1}(\epsilon)) \leq \rho(T_1(\epsilon)).$$ If $\epsilon \to 0$, then (12) implies that $\rho(T_{s1}) \leq \rho(T_1)$. Hence, the proof is complete. □

**Theorem 3.4.** If $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is an $M$-matrix, then
$$\rho(T_1) \leq \rho(T) < 1.$$ Proof. Since $A$ is an irreducible $M$-matrix and $A = M - N$ is a regular splitting of $A$, there exists a positive eigenvector $x > 0$ such that $Tx = \rho(T)x$ and $\rho(T) > 0$. Since $N_1 = N$, $M_1 - M = R_1A$ and thus
$$M^{-1} - M_1^{-1} = M_1^{-1}R_1AM^{-1}.$$ From (13), one obtains
$$T - T_1 = M_1^{-1}R_1AT.$$ Multiplying by $x$ on both sides of (14) gives
$$\rho(T)x - T_1x = \rho(T)M_1^{-1}R_1Ax.$$ Since $0 < \rho(T) < 1$, $Ax \geq 0$. From (15),
$$T_1x \leq \rho(T)x.$$ From Theorem 2.1 and (16), it follows that $\rho(T_1) \leq \rho(T) < 1$. □

**Theorem 3.5.** If $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is an $M$-matrix, then
$$\rho(T_r) \leq \rho(T_1) < 1.$$ Proof. We first consider the case where $A_1$ is an irreducible matrix. Since $A_1$ is an $M$-matrix and $A_1 = M_1 - N_1$ is a regular splitting of $A_1$, there exists a positive eigenvector $x > 0$ such that $T_1x = \rho(T_1)x$ and $\rho(T_1) > 0$. Since $N_r = N_1$, $M_r - M_1 = (R - R_1)A$ and thus
$$M_1^{-1} - M_r^{-1} = M_r^{-1}(R - R_1)AM_1^{-1}.$$ From (17), one obtains
$$T_1 - T_r = M_r^{-1}(R - R_1)AT_1.$$ Multiplying by $x$ on both sides of (18) gives
$$\rho(T_1)x - T_rx = \rho(T_1)M_r^{-1}(R - R_1)Ax.$$
Since \((R - R_1)(I + R_1)^{-1} = (R - R_1)\), (19) can be transformed into
\[(20)\]
\[\rho(T_1)x - T_1x = \rho(T_1)M_{\epsilon}^{-1}(R - R_1)A_1x.\]
Since \(0 < \rho(T_1) < 1\), \(A_1x \geq 0\). Since \((R - R_1) \geq 0\), from (20)
\[(21)\]
\[T_1x \leq \rho(T_1)x.\]
From Theorem 2.1 and (21), it follows that \(\rho(T_1) \leq \rho(T_1) < 1\).

We next consider the case where \(A_1\) is a reducible matrix. Let \(A(\epsilon) = (a_{ij}(\epsilon))\) be defined by
\[a_{ij}(\epsilon) = \begin{cases} 
    a_{n1} - \epsilon & \text{if } i = n \text{ and } j = 1 \\
    a_{ij} & \text{otherwise},
\end{cases} \]
where \(\epsilon > 0\). Let \(A_1 = (I + R_1)A = (\tilde{a}_{ij})\), \(A_1(\epsilon) = (I + R_1)A(\epsilon) = (\tilde{a}_{ij}(\epsilon))\), \(A_r = (I + R)A = (\tilde{a}_{ij})\) and \(A_r(\epsilon) = (I + R)A(\epsilon) = (\tilde{a}_{ij}(\epsilon))\). Then \(a_{ij}(\epsilon)\) is defined the same as in the proof of Theorem 3.3, and
\[\tilde{a}_{ij}(\epsilon) = \begin{cases} 
    \tilde{a}_{n1} - \epsilon & \text{if } i = n \text{ and } j = 1 \\
    \tilde{a}_{ij} & \text{otherwise}.
\end{cases} \]
Since \(A\) and \(A_1\) are \(M\)-matrices, \(A(\epsilon)\) and \(A_1(\epsilon)\) are also \(M\)-matrices for any sufficiently small \(\epsilon > 0\). Since \(A\) is irreducible, \(A(\epsilon)\) and \(A_1(\epsilon)\) are also irreducible for any \(\epsilon > 0\). Let \(M_1(\epsilon) = A_1(\epsilon) + N_1\) and \(M_r(\epsilon) = A_r(\epsilon) + N_r\). Then \(A_1(\epsilon) = M_1(\epsilon) - N_1\) and \(A_r(\epsilon) = M_r(\epsilon) - N_r\) are the Gauss-Seidel \(M\)-splittings of \(A_1(\epsilon)\) and \(A_r(\epsilon)\), respectively. Let \(T_1(\epsilon) = M_1(\epsilon)^{-1}N_1\) and \(T_r(\epsilon) = M_r(\epsilon)^{-1}N_r\). In a similar manner as was done in the first case, one can obtain
\[(22)\]
\[\rho(T_r(\epsilon)) \leq \rho(T_1(\epsilon)).\]
If \(\epsilon \to 0\), then (22) implies that \(\rho(T_r) \leq \rho(T_1)\). Hence, the proof is complete.

Combining Theorems 3.2 to 3.5, the following corollary is obtained.

**Corollary 3.6.** If \(A = (a_{ij}) \in \mathbb{R}^{n \times n}\) is an \(M\)-matrix, then
(a) \(\rho(T_r) \leq \rho(T_1) \leq \rho(T) < 1\),
(b) \(\rho(T_{n1}) \leq \rho(T_1) \leq \rho(T) < 1\),
(c) \(\rho(T_{s1}) \leq \rho(T_{s1}) \leq \rho(T) < 1\).

4. Numerical results

In this section, we provide numerical results for the preconditioned Gauss-Seidel methods to illustrate the theoretical results obtained in Section 3. All test matrices \(A = (a_{ij}) \in \mathbb{R}^{n \times n}\) satisfy the assumptions given in Section 3, that is, \(a_{n1} \neq 0, a_{n1+i} \neq 0\) \((1 \leq i \leq n - 1)\) and \(A\) is an \(M\)-matrix. All spectral radii for iteration matrices of preconditioned Gauss-Seidel methods are computed using MATLAB. All notations are defined the same as in Section 3.
Example 4.1. Consider a $5 \times 5$ matrix $A$ of the form

$$A = \begin{pmatrix}
1 & -0.1 & -0.2 & 0 & -0.1 \\
0 & 1 & -0.2 & -0.1 & 0 \\
-0.2 & 0 & 1 & -0.1 & -0.2 \\
-0.1 & -0.2 & 0 & 1 & -0.1 \\
-0.2 & 0 & -0.1 & -0.2 & 1
\end{pmatrix}.$$ 

Spectral radii for iteration matrices of preconditioned Gauss-Seidel methods are listed in Table 1. From Table 1, it can be seen that all comparison results in Section 3 are satisfied. For this matrix $A$, the following holds:

$$\rho(T) > \rho(T_1) > \rho(T_r) > \rho(T_s) > \rho(T_{s1}).$$

<table>
<thead>
<tr>
<th>Table 1. Spectral radii for iteration matrices of preconditioned Gauss-Seidel methods for Example 4.1</th>
</tr>
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<tbody>
<tr>
<td>$\rho(T)$</td>
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<tr>
<td>0.2367</td>
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</table>

Example 4.2. Consider a $5 \times 5$ matrix $A$ of the form

$$A = \begin{pmatrix}
1 & -0.2 & -0.3 & -0.2 & -0.2 \\
-0.1 & 1 & -0.2 & -0.3 & -0.1 \\
0 & 0 & 1 & -0.1 & -0.2 \\
-0.1 & 0 & 0 & 1 & -0.3 \\
-0.3 & 0 & -0.1 & 0 & 1
\end{pmatrix}.$$ 

Spectral radii for iteration matrices of preconditioned Gauss-Seidel methods are listed in Table 2. From Table 2, it can be seen that all comparison results in Section 3 are satisfied. For this matrix $A$, the following holds:

$$\rho(T) > \rho(T_s) > \rho(T_1) > \rho(T_r) > \rho(T_{s1}).$$

<table>
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<th>Table 2. Spectral radii for iteration matrices of preconditioned Gauss-Seidel methods for Example 4.2</th>
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<td>$\rho(T)$</td>
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Example 4.3. Consider a $5 \times 5$ matrix $A$ of the form

$$A = \begin{pmatrix}
1 & -0.1 & -0.4 & -0.2 & -0.2 \\
0 & 1 & -0.1 & -0.4 & -0.2 \\
-0.2 & 0 & 1 & -0.1 & -0.6 \\
0 & -0.1 & 0 & 1 & -0.8 \\
-0.3 & -0.2 & -0.1 & -0.3 & 1
\end{pmatrix}.$$
Spectral radii for iteration matrices of preconditioned Gauss-Seidel methods are listed in Table 3. From Table 3, it can be seen that all comparison results in Section 3 are satisfied. For this matrix \( A \), the following holds:

\[
\rho(T) > \rho(T_s) > \rho(T_1) > \rho(T_{s1}) > \rho(T_r).
\]

Table 3. Spectral radii for iteration matrices of preconditioned Gauss-Seidel methods for Example 4.3.

<table>
<thead>
<tr>
<th>( \rho(T) )</th>
<th>( \rho(T_1) )</th>
<th>( \rho(T_r) )</th>
<th>( \rho(T_s) )</th>
<th>( \rho(T_{s1}) )</th>
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</tbody>
</table>

Notice that \( \rho(T_s) < \rho(T_1) \) and \( \rho(T_s) < \rho(T_r) \) for Example 4.1, but \( \rho(T_1) > \rho(T_s) \) and \( \rho(T_r) > \rho(T_{s1}) \) for Examples 4.2 and 4.3. Also notice that \( \rho(T_{s1}) < \rho(T_r) \) for Examples 4.1 and 4.2, but \( \rho(T_{s1}) > \rho(T_r) \) for Example 4.3. Hence, it can be concluded from Examples 4.1 to 4.3 that there exist no comparison results between \( \rho(T_s) \) and \( \rho(T_1) \), between \( \rho(T_s) \) and \( \rho(T_r) \), and between \( \rho(T_{s1}) \) and \( \rho(T_r) \) under the same assumptions used in Section 3.

References