ON THE RECURRENCE SEQUENCES

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Abstract. We investigate relations of polynomials $x^n = \sum_{i=0}^{n-1} x^i$ and recurrence sequences. We consider certain variations of the Fibonacci sequence and investigate explicit ways to compute the general n th term.

1. Introduction

The Fibonacci sequence $F_n = F_{n-1} + F_{n-2}$ ($n \geq 2$) with $F_0 = 0$ and $F_1 = 1$ is well connected to $f(x) = x^2 - x - 1$. A variation of the Fibonacci sequence that begins with $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ ($n \geq 2$) is called the Lucas sequence. The ratio L_n/L_{n-1} satisfies $x^2 - x - 1 = 0$ for large enough n , and it meets $W_n = AW_{n-1}$ with $W_n = \begin{bmatrix} L_n \\ L_{n-1} \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Since the characteristic polynomial $x^2 - x - 1$ of A has two roots ϕ_i with eigenvectors $\begin{bmatrix} \phi_i \\ 1 \end{bmatrix}$, A is diagonalized to $D = \begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix}$ with $P = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 1 \end{bmatrix}$, so $L_n = \phi_1^n + \phi_2^n = \phi_1^n + (-\phi_1)^{-n}$. This shows that both Fibonacci and Lucas sequences are represented by $f(x) = x^2 - x - 1$, and their n th terms are determined by diagonalization of matrix whose eigenvalues are the zeros of $f(x)$. The purpose of this work is to investigate interrelationships of polynomials and recurrence sequences. We will study certain variations of Fibonacci sequence and find concrete methods to calculate general n th terms.

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2. Variations of Fibonacci sequence

The relations of L_n and ϕ_i can also be easily obtained by iteration of recurrence.

Lemma 1. $\phi_1^n - \phi_2^n = \sqrt{5}F_n$ and $\phi_1^n + \phi_2^n = L_n$ for all $n \in \mathbb{Z}$.

A generalization of Fibonacci sequence is as follows.

Theorem 2. Let $M_n = c_1M_{n-1} + c_2M_{n-2}$ ($n \geq 2$, $c_i \in \mathbb{R}$) with $M_0 = a$, $M_1 = b$. If λ is a root of $x^2 - c_1x - c_2$ and $c_1^2 + 4c_2 \neq 0$, then the n th term M_n is

$$M_n = \frac{b(\lambda^n - (-c_2\lambda^{-1})^n) + ac_2(\lambda^{n-1} - (-c_2\lambda^{-1})^{n-1})}{\sqrt{c_1^2 + 4c_2}}.$$

Proof. Let $W_n = \begin{bmatrix} M_n \\ M_{n-1} \end{bmatrix}$. Then $W_n = AW_{n-1} = A^{n-1} \begin{bmatrix} b \\ a \end{bmatrix}$ with $A = \begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix}$. The characteristic polynomial of A is $x^2 - c_1x - c_2$ having two roots λ_i , whose eigenvectors are $\begin{bmatrix} \lambda_i \\ 1 \end{bmatrix}$. So A is diagonalized to $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ with $P = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$, and

$$\begin{aligned} \begin{bmatrix} M_n \\ M_{n-1} \end{bmatrix} = W_n &= \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} b \\ a \end{bmatrix} \\ &= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} b(\lambda_1^n - \lambda_2^n) - a(\lambda_1^n \lambda_2 - \lambda_1 \lambda_2^n) \\ b(\lambda_1^{n-1} - \lambda_2^{n-1}) - a(\lambda_1^{n-1} \lambda_2 - \lambda_1 \lambda_2^{n-1}) \end{bmatrix}. \end{aligned}$$

Since $\lambda_1 + \lambda_2 = c_1$, $\lambda_1 \lambda_2 = -c_2$ and $\lambda_1 - \lambda_2 = \sqrt{c_1^2 + 4c_2}$, we have

$$\begin{aligned} M_n &= b \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} - a \lambda_1 \lambda_2 \frac{\lambda_1^{n-1} - \lambda_2^{n-1}}{\lambda_1 - \lambda_2} = b \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} + ac_2 \frac{\lambda_1^{n-1} - \lambda_2^{n-1}}{\lambda_1 - \lambda_2} \\ &= \frac{b(\lambda^n - (-c_2\lambda^{-1})^n) + ac_2(\lambda^{n-1} - (-c_2\lambda^{-1})^{n-1})}{\sqrt{c_1^2 + 4c_2}}. \quad \square \end{aligned}$$

We briefly review the resultant and discriminant ([3]). Let r_i ($1 \leq i \leq n$) and s_j ($1 \leq j \leq m$) be roots of $f(x) = \sum_{i=0}^n a_i x^i$, $g(x) = \sum_{j=0}^m b_j x^j$ respectively in a splitting field. The resultant $R(f, g)$ and the discriminant

$\Delta(f)$ are defined by

$$R(f, g) = a_n^m b_m^n \prod_{i=1}^n \prod_{j=1}^m (r_i - s_j) \quad \text{and} \quad \Delta(f) = a_n^{2n-2} \prod_{1 \leq i < j \leq n} (r_i - r_j)^2.$$

Moreover $R(f, g) = \begin{vmatrix} a_n & a_{n-1} & \cdots & a_0 & \cdots & 0 \\ & \ddots & \cdots & & \ddots & \\ 0 & \cdots & a_n & a_{n-1} & \cdots & a_0 \\ b_m & b_{m-1} & b_0 & 0 & \cdots & \\ & \ddots & \cdots & & \ddots & \\ 0 & 0 & b_m & b_{m-1} & \cdots & b_0 \end{vmatrix}$, the determinant of

$(n+m) \times (n+m)$ Sylvester matrix. And $\Delta(f) = (-1)^{\frac{n(n-1)}{2}} \frac{1}{a_n} R(f, f')$ where f' is the derivative of f , so $\Delta(f) \neq 0$ if and only if $f(x)$ has all distinct roots. Thus $\Delta(f)$ provide a criteria if f has a multiple root, while $R(f, g)$ is used to verify if f, g have a common root. Let A_n be a sequence with t predetermined terms $A_0 = \dots = A_{t-2} = 0$, $A_{t-1} = 1$ and each term is the sum of preceding t terms

$$A_n = A_{n-1} + A_{n-2} + \cdots + A_{n-t} \quad (n \geq t).$$

Theorem 3. The A_{n+1}/A_n satisfies the characteristic equation $f(x)$ of $t \times t$ matrix $\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$. If λ_i are roots of $f(x)$ and $\mu =$

$\prod_{1 \leq i < j \leq t} (\lambda_i - \lambda_j)$ then

$$A_n = \frac{1}{\mu} \sum_{i=1}^t \lambda_i^n \prod_{\substack{1 \leq j < k \leq t \\ j, k \neq i}} (\lambda_j - \lambda_k) = \sum_{i=1}^t \frac{\lambda_i^n}{\prod_{i \neq j=1}^t (\lambda_i - \lambda_j)}.$$

Proof. We first assume $t = 3$. As n gets larger the ratio

$$\frac{A_n}{A_{n-1}} = \frac{A_{n-1} + A_{n-2} + A_{n-3}}{A_{n-1}} = 1 + \frac{A_{n-2}}{A_{n-1}} + \frac{A_{n-3}/A_{n-2}}{A_{n-1}/A_{n-2}}$$

satisfies $x = 1 + \frac{1}{x} + \frac{1}{x^2}$, i.e., $f(x) = x^3 - x^2 - x - 1$. Let $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

and $Y_n = \begin{bmatrix} A_n \\ A_{n-1} \\ A_{n-2} \end{bmatrix}$. Then $Y_n = M Y_{n-1} = M^{n-2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $f(x)$ is the

characteristic polynomial of M having all distinct roots λ_i , because

$$\Delta(f) = (-1)^{\frac{3-2}{2}} \text{Res}(f, f') = - \begin{vmatrix} 1 & -1 & -1 & -1 & 0 \\ 0 & 1 & -1 & -1 & -1 \\ 3 & -2 & -1 & 0 & 0 \\ 0 & 3 & -2 & -1 & 0 \\ 0 & 0 & 3 & -2 & -1 \end{vmatrix} = -44 \neq 0.$$

Each λ_i is an eigenvalue of M with corresponding eigenvector $\begin{bmatrix} \lambda_i^2 \\ \lambda_i \\ 1 \end{bmatrix}$, so

$$M \text{ can be diagonalized to } D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \text{ with } P = \begin{bmatrix} \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & 1 & 1 \end{bmatrix}.$$

Hence

$$\begin{aligned} \begin{bmatrix} A_n \\ A_{n-1} \\ A_{n-2} \end{bmatrix} &= \begin{bmatrix} \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^{n-2} & 0 & 0 \\ 0 & \lambda_2^{n-2} & 0 \\ 0 & 0 & \lambda_3^{n-2} \end{bmatrix} \begin{bmatrix} \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \frac{1}{\mu} \begin{bmatrix} \lambda_1^n(\lambda_2 - \lambda_3) - \lambda_2^n(\lambda_1 - \lambda_3) + \lambda_3^n(\lambda_1 - \lambda_2) \\ \lambda_1^{n-1}(\lambda_2 - \lambda_3) - \lambda_2^{n-1}(\lambda_1 - \lambda_3) + \lambda_3^{n-1}(\lambda_1 - \lambda_2) \\ \lambda_1^{n-2}(\lambda_2 - \lambda_3) - \lambda_2^{n-2}(\lambda_1 - \lambda_3) + \lambda_3^{n-2}(\lambda_1 - \lambda_2) \end{bmatrix} \end{aligned}$$

where $\mu = \prod_{1 \leq i < j \leq 3} (\lambda_i - \lambda_j) = \det(P)$, the determinant of P due to Vandermonde. We note that $\Delta \neq 0$ so $\mu \neq 0$. Thus

$$A_n = \frac{1}{\mu} (\lambda_1^n(\lambda_2 - \lambda_3) - \lambda_2^n(\lambda_1 - \lambda_3) + \lambda_3^n(\lambda_1 - \lambda_2)) = \sum_{i=1}^3 \frac{\lambda_i^n}{\prod_{i \neq j=1}^3 (\lambda_i - \lambda_j)}.$$

Similarly if $t = 4$, then for the large enough n , $\frac{A_{n+1}}{A_n} = 1 + \frac{A_{n-1}}{A_n} + \frac{A_{n-2}/A_{n-1}}{A_n/A_{n-1}} + \frac{(A_{n-3}/A_{n-2})/(A_{n-1}/A_{n-2})}{A_n/A_{n-1}}$ satisfies $x = 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$, so the ratio is a root of $f(x) = x^4 - x^3 - x^2 - x - 1$. Moreover

$$Y_n = MY_{n-1} = M^{n-3}Y_3 \quad \text{with } M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, Y_n = \begin{bmatrix} B_n \\ B_{n-1} \\ B_{n-2} \\ B_{n-3} \end{bmatrix},$$

and $f(x)$ is the characteristic polynomial of M with all distinct roots λ_i , for

$$\Delta(f) = (-1)^{\frac{4.3}{2}} \text{Res}(f, f') = \begin{vmatrix} 1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 & -1 \\ 4 & -3 & -2 & -1 & 0 & 0 & 0 \\ 0 & 4 & -3 & -2 & -1 & 0 & 0 \\ 0 & 0 & 4 & -3 & -2 & -1 & 0 \\ 0 & 0 & 0 & 4 & -3 & -2 & -1 \end{vmatrix} = -563 \neq 0.$$

Each λ_i ($1 \leq i \leq 4$) corresponds to eigenvector $\begin{bmatrix} \lambda_i^3 \\ \lambda_i^2 \\ \lambda_i \\ 1 \end{bmatrix}$, so M can be diagonalized to $D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$ with $P = \begin{bmatrix} \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. Hence

$$Y_n = M^{n-3} Y_3 = \frac{1}{\mu} \begin{bmatrix} \sum_{k=1}^4 \lambda_k^n \prod_{i,j \neq k}^{i < j} (\lambda_i - \lambda_j) \\ \sum_{k=1}^4 \lambda_k^{n-1} \prod_{i,j \neq k}^{i < j} (\lambda_i - \lambda_j) \\ \sum_{k=1}^4 \lambda_k^{n-2} \prod_{i,j \neq k}^{i < j} (\lambda_i - \lambda_j) \\ \sum_{k=1}^4 \lambda_k^{n-3} \prod_{i,j \neq k}^{i < j} (\lambda_i - \lambda_j) \end{bmatrix}$$

with $\mu = \prod_{1 \leq i < j \leq 4} (\lambda_i - \lambda_j) = \det(P)$. Therefore it follows that

$$A_n = \frac{1}{\mu} \sum_{i=1}^4 \lambda_i^n \prod_{i \neq j,k}^{1 \leq j < k \leq 4} (\lambda_j - \lambda_k) = \sum_{i=1}^4 \frac{\lambda_i^n}{\prod_{i \neq j=1}^4 (\lambda_i - \lambda_j)}.$$

Assume $t > 4$. It is not hard to see the ratio $\frac{A_{n+1}}{A_n}$ satisfies $x^t - x^{t-1} - \dots - 1 = 0$, and the sequence A_n yields the $t \times t$ matrix $M_t = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ 0 & \ddots & 1 & 0 \end{bmatrix}$.

For characteristic polynomial $\text{char}(M_t)$ of M_t , claim that

$$\text{char}(M_t) = |\lambda I - M_t| = \lambda^t - \lambda^{t-1} - \dots - \lambda - 1.$$

Obviously, $\text{char}(M_2) = |\lambda I - M_2| = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - \lambda - 1$ and

$$\begin{aligned} \text{char}(M_3) &= |\lambda I - M_3| = \begin{vmatrix} \lambda - 1 & -1 & -1 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda \end{vmatrix} = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & 0 \end{vmatrix} + \lambda \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{vmatrix} \\ &= -1 + \lambda |\lambda I - M_2| = \lambda^3 - \lambda^2 - \lambda - 1. \end{aligned}$$

Assume $\text{char}(M_i) = \lambda^i - \dots - \lambda - 1$ for all $i \leq t-1$. Then $\text{char}(M_t)$ equals

$$\begin{vmatrix} \lambda-1 & -1 & \cdots & -1 & -1 \\ -1 & \lambda & 0 & 0 & 0 \\ 0 & -1 & \lambda & \cdots & 0 \\ 0 & 0 & \ddots & -1 & \lambda \end{vmatrix} = \begin{vmatrix} \lambda-1 & -1 & \cdots & -1 \\ -1 & \lambda & \cdots & 0 \\ 0 & -1 & \ddots & 0 \\ 0 & \cdots & -1 & 0 \end{vmatrix} + \lambda \begin{vmatrix} \lambda-1 & -1 & \cdots & -1 & -1 \\ -1 & \lambda & \cdots & 0 & 0 \\ 0 & -1 & \lambda & \ddots & 0 \\ 0 & \cdots & 0 & -1 & \lambda \end{vmatrix}$$

where the former one of the last equation equals $\begin{vmatrix} \lambda-1 & -1 \\ -1 & 0 \end{vmatrix} = -1$ while the latter one is $|\lambda I - M_{t-1}| = \lambda^{t-1} - \cdots - \lambda - 1$. Hence

$$\text{char}(M_t) = -1 + \lambda(\lambda^{t-1} - \cdots - \lambda - 1) = \lambda^t - \lambda^{t-1} - \cdots - \lambda - 1,$$

has distinct roots λ_i ($1 \leq i \leq t$). Thus for $\mu = \prod_{1 \leq i < j \leq t} (\lambda_i - \lambda_j)$, A_n equals

$$A_n = \frac{1}{\mu} \sum_{i=1}^t \lambda_i^n \prod_{\substack{j < k \\ i \neq j, k}} (\lambda_j - \lambda_k) = \sum_{i=1}^t \frac{\lambda_i^n}{\prod_{i \neq j=1}^t (\lambda_i - \lambda_j)}. \quad \square$$

3. General linear homogeneous recurrence sequence

We have seen that the sequence $A_n = A_{n-1} + A_{n-2} + \cdots + A_{n-t}$ ($n \geq t$) with $A_0 = \cdots = A_{t-2} = 0$, $A_{t-1} = 1$ is related to $f(x) = x^t - x^{t-1} - \cdots - 1$, and the n th A_n is represented by diagonalization of $t \times t$ matrix. Practically, however it is not useful for large t . For $c_i \in \mathbb{R}$, set $f(x) = x^t - c_1 x^{t-1} - \cdots - c_t$ with roots r_i ($1 \leq i \leq t$) and consider a general linear homogeneous recurrence relation

$$\{A_n\} : A_n = c_1 A_{n-1} + c_2 A_{n-2} + \cdots + c_t A_{n-t} \quad (n \geq t).$$

Lemma 4. [4] *With the above $\{A_n\}$ and $f(x)$, if $f(x)$ has all distinct roots r_i then A_n is a linear combination $\sum_{i=1}^t \alpha_i r_i^n$ ($\alpha_i \in \mathbb{R}$) of r_i .*

The $f(x)$ is called the characteristic polynomial of $\{A_n\}$. Lemma 4 yields Lemma 1 immediately that the characteristic polynomial of $F_n = F_{n-1} + F_{n-2}$ is $f(x) = x^2 - x - 1$. If λ_1, λ_2 are two real roots of $f(x)$, then F_n can be written as $F_n = \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n$. Since $0 = F_0 = \alpha_1 + \alpha_2$ and $1 = F_1 = \alpha_1 \lambda_1 + \alpha_2 \lambda_2$, we have $\alpha_1 = 1/(\lambda_1 - \lambda_2)$ and $\alpha_2 = -1/(\lambda_1 - \lambda_2)$, so $F_n = \frac{1}{\lambda_1 - \lambda_2} (\lambda_1^n - \lambda_2^n)$.

Instead of finding roots of high degree polynomial, it might be at least useful to determine whether polynomial has distinct roots.

Lemma 5. *The polynomials $f(x) = x^3 - x^2 - x - 1$, $g(x) = x^4 - x^3 - x^2 - x - 1$ and $h(x) = x^5 - x^4 - x^3 - x^2 - x - 1$ have no multiple roots.*

Proof. $\Delta(f) = -44 \neq 0$, so the roots of $f(x)$ are all distinct by Lemma 3. Similarly

$$\begin{aligned} \Delta(g) &= \begin{vmatrix} 1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 & -1 \\ 4 & -3 & -2 & -1 & 0 & 0 & 0 \\ 0 & 4 & -3 & -2 & -1 & 0 & 0 \\ 0 & 0 & 4 & -3 & -2 & -1 & 0 \\ 0 & 0 & 0 & 4 & -3 & -2 & -1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 7 & 8 & 4 & 3 \\ 0 & 0 & 0 & 3 & 4 & 5 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 4 & -3 & -2 & -1 \end{vmatrix} \\ &= \begin{vmatrix} 7 & 8 & 4 & 3 \\ 3 & 4 & 5 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & -3 & -2 & -1 \end{vmatrix} = -563 \neq 0 \end{aligned}$$

and

$$\begin{aligned} \Delta(h) &= \begin{vmatrix} 1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 15 & 16 & 11 & 10 & 7 \\ 0 & 0 & 0 & 0 & 7 & 8 & 9 & 4 & 3 \\ 0 & 0 & 0 & 0 & 3 & 4 & 5 & 6 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 & -4 & -3 & -2 & -1 \end{vmatrix} \\ &= \begin{vmatrix} 15 & 16 & 11 & 10 & 7 \\ 7 & 8 & 9 & 4 & 3 \\ 3 & 4 & 5 & 6 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 5 & -4 & -3 & -2 & -1 \end{vmatrix} = 9584 \neq 0 \end{aligned}$$

, so $g(x)$ and $h(x)$ have all distinct roots. $\square$

Theorem 6. Let $\{A_n\}$: $A_n = A_{n-1} + A_{n-2} + A_{n-3}$ with $A_0 = A_1 = 0$ and $A_2 = 1$. Let λ_i ($i = 1, 2, 3$) be the roots of characteristic polynomial

$f(x)$ of $\{A_n\}$. Then

$$A_n = \sum_{i=1}^3 \frac{\lambda_i^n}{\prod_{i \neq j=1}^3 (\lambda_i - \lambda_j)} = \frac{1}{3^n} \frac{(\tau + \frac{4}{\tau} + 1)^{n+1}}{(\tau + \frac{4}{\tau} + 2)(\tau + \frac{4}{\tau} - 2)}$$

for $\tau = \sqrt[3]{19 + 3\sqrt{33}}$. Moreover $A_n \sim \lambda_1^n / (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)$ asymptotically.

Proof. Due to Lemma 4, the characteristic polynomial of $\{A_n\}$ is $f(x) = x^3 - x^2 - x - 1$, and $A_n = \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \alpha_3 \lambda_3^n$ with $\alpha_i \in \mathbb{R}$. The α_i is determined by initial values $0 = A_0 = A_1$ and $1 = A_2$ so we have

$$\begin{bmatrix} 1 & 1 & 1 & | & 0 \\ \lambda_1 & \lambda_2 & \lambda_3 & | & 0 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & | & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 0 & \lambda_2 - \lambda_1 & \lambda_3 - \lambda_1 & | & 0 \\ 0 & 0 & (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2) & | & 1 \end{bmatrix},$$

hence $\alpha_i = \frac{1}{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)}$, thus

$$\begin{aligned} A_n &= \frac{\lambda_1^n(\lambda_2 - \lambda_3) - \lambda_2^n(\lambda_1 - \lambda_3) + \lambda_3^n(\lambda_1 - \lambda_2)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \\ &= \frac{1}{\prod_{1 \leq i < j \leq 3} (\lambda_i - \lambda_j)} \sum_{k=1}^3 \lambda_k^n \cdot \prod_{i < j, i, j \neq k} (\lambda_i - \lambda_j) = \sum_{i=1}^3 \frac{\lambda_i^n}{\prod_{i \neq j=1}^3 (\lambda_i - \lambda_j)}. \end{aligned}$$

Due to the famous Cardano's method, the cubic equation $f(x)$ has roots

$$\begin{cases} \lambda_1 &= \frac{1}{3}(1 + \tau + \frac{4}{\tau}) = \frac{1}{3}(1 + X) \\ \lambda_2 &= \frac{1}{6}(2 - (\tau + \frac{4}{\tau}) + i\sqrt{3}(\tau - \frac{4}{\tau})) = \frac{1}{6}(2 - X + i\sqrt{3}Y) \\ \lambda_3 &= \frac{1}{6}(2 - (\tau + \frac{4}{\tau}) - i\sqrt{3}(\tau - \frac{4}{\tau})) = \frac{1}{6}(2 - X - i\sqrt{3}Y) \end{cases}$$

where $\tau = \sqrt[3]{19 + 3\sqrt{33}}$, $\tau + \frac{4}{\tau} = X$ and $\tau - \frac{4}{\tau} = Y$. Then

$$\begin{aligned} (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) &= \frac{1}{3}((\tau + \frac{4}{\tau})^2 - 4) = \frac{1}{3}(X + 2)(X - 2) \\ (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3) &= \frac{-1}{6}(\tau - \frac{4}{\tau})((\tau - \frac{4}{\tau}) + i\sqrt{3}(\tau + \frac{4}{\tau})) = \frac{Y}{6}(Y + i\sqrt{3}X) \\ (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) &= \frac{-1}{6}(\tau - \frac{4}{\tau})((\tau - \frac{4}{\tau}) - i\sqrt{3}(\tau + \frac{4}{\tau})) = -\frac{Y}{6}(Y - i\sqrt{3}X) \end{aligned}$$

so

$$(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) = i\frac{\sqrt{3}}{9}\left(\tau^3 - (\frac{4}{\tau})^3\right) = i\frac{\sqrt{3}}{9}6\sqrt{33} = i2\sqrt{11}.$$

Thus

$$A_n = \frac{-i}{2\sqrt{11}} \left(\lambda_1^n \frac{i\sqrt{3}}{3} Y - \lambda_2^n \frac{3X + i\sqrt{3}Y}{6} + \lambda_3^n \frac{3X - i\sqrt{3}Y}{6} \right)$$

$$= \frac{\sqrt{3}}{12\sqrt{11}} (2\lambda_1^n - \lambda_2^n - \lambda_3^n) \left(\tau - \frac{4}{\tau}\right) + \frac{i}{4\sqrt{11}} (\lambda_2^n - \lambda_3^n) \left(\tau + \frac{4}{\tau}\right).$$

Moreover since

$$X^3 = \left(\tau + \frac{4}{\tau}\right)^3 = \tau^3 + \left(\frac{4}{\tau}\right)^3 + 12\left(\tau + \frac{4}{\tau}\right) = 38 + 12\left(\tau + \frac{4}{\tau}\right) = 38 + 12X$$

and $Y^3 = \left(\tau - \frac{4}{\tau}\right)^3 = 6\sqrt{33} - 12\left(\tau - \frac{4}{\tau}\right)$, X and Y are real roots of $x^3 - 12x - 38 = 0$ and $x^3 + 12x - 6\sqrt{33} = 0$ respectively, so $X \sim 4.51$ and $Y \sim 2.10$. Thus $\lambda_1 = \frac{1}{3}(1 + X) \sim 1.84$ is a real root of $f(x)$ and the other λ_2, λ_3 are complex roots

$$\frac{1}{6}(2 - X \pm i\sqrt{3}Y) \sim \frac{1}{6}(-2.51 \pm i2.10\sqrt{3}) \sim -0.41 \pm 0.60i.$$

Hence as n gets larger, $\lambda_2^n = \lambda_3^n$ goes to 0, so

$$A_n \sim \frac{\lambda_1^n}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} = \frac{\left(\frac{1}{3}\left(\tau + \frac{4}{\tau} + 1\right)\right)^n}{\frac{1}{3}\left(\tau + \frac{4}{\tau} + 2\right)\left(\tau + \frac{4}{\tau} - 2\right)} = \frac{(X + 1)^n}{3^n(X + 2)(X - 2)}. \quad \square$$

Theorem 7. Let $B_n = B_{n-1} + \dots + B_{n-t}$ ($n \geq t$) with $B_0 = \dots = B_{t-2} = 0$, $B_{t-1} = 1$. Then $B_n = \sum_{i=1}^t \frac{\lambda_i^n}{\prod_{i \neq j=1}^t (\lambda_i - \lambda_j)}$.

Proof. Let $t = 4$. Then the characteristic polynomial $f(x) = x^4 - x^3 - x^2 - x - 1$ of recurrence $\{B_n\}$ has 4 distinct roots λ_i ($1 \leq i \leq 4$), two of them are real and the others are conjugate complex roots ([1]). So $B_n = \alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \alpha_3\lambda_3^n + \alpha_4\lambda_4^n$ with $\alpha_i \in \mathbb{R}$ are determined by

initial conditions $\begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, thus due to the Gauss

elimination we have $\alpha_i = \prod_{i \neq j=1}^4 \frac{1}{(\lambda_i - \lambda_j)}$. Hence

$$B_n = \sum_{i=1}^4 \frac{\lambda_i^n}{\prod_{i \neq j=1}^4 (\lambda_i - \lambda_j)} = \sum_{i=1}^4 \frac{\lambda_i^n \prod_{1 \leq j < k \leq 4, i \neq j} (\lambda_j - \lambda_k)}{\prod_{1 \leq i < j \leq 4} (\lambda_i - \lambda_j)}.$$

From the above formula of B_n for $t = 4$ as well as Theorem 6 for $t = 3$, it is not hard to see the recurrence relation with t predetermined

terms forms $B_n = \sum_{i=1}^t \frac{\lambda_i^n}{\prod_{i \neq j=1}^t (\lambda_i - \lambda_j)}$. $\square$

4. Matrix formula distribution

The formulas of the n th term A_n discussed above is related to roots of polynomials. In this section we investigate more concrete method to compute A_n . Let us begin to distribute the Fibonacci sequence $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, \dots$ in rectangle form. May refer to [2] for Lucas sequence. We begin with 3 or 4 columns rectangle distribution.

3 columns			4 columns				
1	1	2		1	1	2	3
3	5	8		5	8	13	21
13	21	34		34	55	89	144
55	89	144		233	377	610	987
233	377	...		1597	2584	4181	...

In the 3 columns distribution we find $233 = (2 \cdot 2)55 + 13$ and $377 = (2 \cdot 2)89 + 21$. This means that

$$F_{13} = 2F_3F_{10} + F_7 \quad \text{and} \quad F_{14} = 2F_3F_{11} + F_8.$$

Similarly from the 4 columns rectangle we have $4181 = (2 \cdot 3 + 1)610 - 89$, i.e.,

$$F_{19} = (2F_4 + F_1)F_{15} + (-1)^3F_{11}.$$

Lemma 8. Any n th Fibonacci number F_n satisfies $2F_3F_n + F_{n-3} = F_{n+3}$ and $(2F_4 + F_1)F_n + (-1)^3F_{n-4} = F_{n+4}$.

Proof. We first assume that $n \geq 4$. If $n = 4$ then $2F_3F_4 + F_1 = 13 = F_7$. Assume $2F_3F_m + F_{m-3} = F_{m+3}$ is true for all $m \leq n - 1$. Then

$$\begin{aligned}
 2F_3F_n + F_{n-3} &= 2F_3(F_{n-1} + F_{n-2}) + (F_{n-4} + F_{n-5}) \\
 &= (2F_3F_{n-1} + F_{n-4}) + (2F_3F_{n-2} + F_{n-5}) \\
 &= F_{n+2} + F_{n+1} \\
 &= F_{n+3}.
 \end{aligned}$$

Moreover by considering F_n for negative n , such as $F_{-1} = 1, F_{-2} = -1, F_{-3} = 2, F_{-4} = -3, \dots$, it is easy to see that the equality holds for all $n \in \mathbb{Z}$. Similarly the second identity also follows by induction that

$$\begin{aligned}
 (2F_4 + F_1)F_n + (-1)^3F_{n-4} &= (2F_4 + F_1)(F_{n-1} + F_{n-2}) + (-1)^3(F_{n-5} + F_{n-6}) \\
 &= ((2F_4 + F_1)F_{n-1} + (-1)^3F_{n-5}) + ((2F_4 + F_1)F_{n-2} + (-1)^3F_{n-6}) \\
 &= F_{n+3} + F_{n+2} = F_{n+4}. \quad \square
 \end{aligned}$$

For instance, the 25th Fibonacci F_{25} is, by the case of 3 columns rectangle,

$$F_{25} = 4F_{22} + F_{19} = 17F_{19} + 4F_{16} = 72F_{16} + 17F_{13} = 305F_{13} + 72F_{10}$$

$$= 1292F_{10} + 305F_7 = 5473F_7 + 1292F_4 = 23184F_4 + 5473F_1 = 75025.$$

On the other hand, by considering 4 column rectangle, F_{25} is

$$\begin{aligned} F_{25} &= 7F_{21} - F_{17} = 48F_{17} - 7F_{13} = 329F_{13} - 48F_9 = 2255F_9 - 329F_5 \\ &= 15456F_5 - 2255F_1 = 75025. \end{aligned}$$

We continue to consider 5 or 6 columns rectangle. The table with $k = 5$

1	1	2	3	5
8	13	21	34	55
89	144	233	377	610
987	1597	2584	4181	6765
10946	17711	28657	46386	...

shows that $(2 \cdot 5 + 1)(987) + 89 = 10946$, i.e. $(2F_5 + F_2)F_{16} + F_{11} = F_{21}$. And 6 columns distribution provides that $(2 \cdot 8 + 2)28657 - 1597 = 514229$, i.e., $(2F_6 + F_3)F_{23} - F_{17} = F_{29}$. These observations yield the following theorem.

Theorem 9. *Let $n \geq 1$ and $k \geq 3$ be given integers. Then*

$$F_{n+k} = (2F_k + F_{k-3})F_n + (-1)^{k-1}F_{n-k}.$$

Moreover by dividing n by k , let $n = Qk + r$ ($Q, r \in \mathbb{Z}$) with $1 \leq r \leq k$. Then

$$F_n = F_{Qk+r} = (2F_k + F_{k-3})F_{(Q-1)k+r} + (-1)^{k-1}F_{(Q-2)k+r}.$$

Proof. Suppose $\{F_n\}$ is distributed in k columns rectangle

$$\begin{array}{cccccc} F_1 & F_2 & F_3 & \cdots & F_{k-1} & F_k \\ F_{k+1} & F_{k+2} & F_{k+3} & \cdots & F_{k+(k-1)} & F_{2k} \\ & \ddots & & \cdots & \ddots & \\ F_{tk+1} & F_{tk+2} & F_{tk+3} & \cdots & F_{tk+(k-1)} & F_{(t+1)k} \end{array}$$

It was proved that $F_{n+k} = (2F_k + F_{k-3})F_n + (-1)^{k-1}F_{n-k}$ for $k = 3, 4$ in Lemma 8. Assume this is true up to $k - 1$. Then due to induction, we have

$$\begin{aligned} F_{n+k} &= F_{n+(k-1)} + F_{n+(k-2)} \\ &= (2(F_{k-1} + F_{k-2}) + F_{k-4} + F_{k-5})F_n + (-1)^{k-1}(-F_{n-(k-1)} + F_{n-(k-2)}) \\ &= (2F_k + F_{k-3})F_n + (-1)^{k-1}F_{n-k}. \end{aligned}$$

Moreover if we write $n = Qk + r$ then

$$\begin{aligned} F_{Qk+r} &= F_n = F_{(n-k)+k} = (2F_k + F_{k-3})F_{n-k} + (-1)^{k-1}F_{n-2k} \\ &= (2F_k + F_{k-3})F_{(Q-1)k+r} + (-1)^{k-1}F_{(Q-2)k+r}. \quad \square \end{aligned}$$

It shows $F_n = F_{Qk+r}$ is a combination of $F_{(Q-1)k+r}$ and $F_{(Q-2)k+r}$. The following theorem will prove that F_n is expressed by only F_k , F_{k-3} , F_r and F_{k+r} .

Theorem 10. *Under the same context above, if $n = Qk + r$ ($1 \leq r \leq k$) then F_n is a linear combination of F_k , F_{k-3} , F_r and F_{k+r} .*

Proof. Write $n = Qk + r$ with $Q, r \in \mathbb{Z}$, $1 \leq r \leq k$. By Theorem 9,

$$F_{Qk+r} = \mu F_{(Q-1)k+r} + (-1)^{k-1} F_{(Q-2)k+r}$$

where $\mu = (2F_k + F_{k-3})$. For convenience we first assume k is odd. Then

$$F_{3k+r} = \mu F_{2k+r} + F_{k+r} = \mu(\mu F_{k+r} + F_r) + F_{k+r} = (\mu^2 + 1)F_{k+r} + \mu F_r,$$

$$F_{4k+r} = \mu F_{3k+r} + F_{2k+r} = (\mu^3 + 2\mu)F_{k+r} + (\mu^2 + 1)\mu F_r.$$

Suppose it is true for each $t \leq Q - 1$. Then the t th stage can be written by

$$F_{tk+r} = \alpha_t F_{k+r} + \beta_t F_r$$

where α_t is μ times the first coefficient in the $(t-1)$ th stage added to the second coefficient in $t-1$ th stage, and β_t is the first coefficient in $t-1$ th stage, that is,

$$\alpha_t = \mu\alpha_{t-1} + \beta_{t-1} \quad \text{and} \quad \beta_t = \alpha_{t-1}.$$

We will prove that $F_{Qk+r} = F_{(t+1)k+r} = (\mu\alpha_t + \beta_t)F_{k+r} + \alpha_t F_r$. Due to hypothesis we have $F_{(t-1)k+r} = \alpha_{t-1}F_{k+r} + \beta_{t-1}F_r$. Hence by Theorem 9,

$$\begin{aligned} F_{Qk+r} &= F_{(t+1)k+r} = (2F_k + F_{k-3})F_{tk+r} + F_{(t-1)k+r} \\ &= \mu(\alpha_t F_{k+r} + \beta_t F_r) + \alpha_{t-1}F_{k+r} + \beta_{t-1}F_r \\ &= (\mu\alpha_t + \alpha_{t-1})F_{k+r} + (\mu\beta_t + \beta_{t-1})F_r \\ &= (\mu\alpha_t + \beta_t)F_{k+r} + (\mu\alpha_{t-1} + \beta_{t-1})F_r = (\mu\alpha_t + \beta_t)F_{k+r} + \alpha_t F_r \end{aligned}$$

as is desired. Since $\mu = (2F_k + F_{k-3})$, F_n is a combination of F_k , F_{k-3} , F_r and F_{k+r} . The case for even k also follows immediately. $\square$

Theorem 10 provides a good method to get F_n by knowing a little information. Since r and k can be regarded as very smaller ones than n , F_n can be obtained by only relatively small four Fibonacci numbers F_k , F_{k-3} , F_r and F_{k+r} .

Example 1. For the F_{50} , we may take any k , say for instance $k = 7$. Then

$$\begin{aligned} F_{50} &= F_{7 \cdot 7 + 1} = \mu F_{6 \cdot 7 + 1} + F_{5 \cdot 7 + 1} \\ &= (\mu^2 + 1)F_{5 \cdot 7 + 1} + \mu F_{4 \cdot 7 + 1} \\ &= (\mu(\mu^2 + 1) + \mu)F_{4 \cdot 7 + 1} + (\mu^2 + 1)F_{3 \cdot 7 + 1} \\ &= (\mu(\mu^3 + 2\mu) + \mu^2 + 1)F_{3 \cdot 7 + 1} + (\mu^3 + 2\mu)F_{2 \cdot 7 + 1} \\ &= (\mu(\mu^4 + 3\mu^2 + 1) + \mu^3 + 2\mu)F_{2 \cdot 7 + 1} + (\mu^4 + 3\mu^2 + 1)F_{7 + 1} \end{aligned}$$

$$= (\mu(\mu^5 + 4\mu^3 + 3\mu) + \mu^4 + 3\mu^2 + 1)F_{7+1} + (\mu^5 + 4\mu^3 + 3\mu)F_1 = 12,586,269,025$$

because $F_7 = 13$, $F_4 = 3$, $F_1 = 1$, $F_8 = 21$ and $\mu = 2F_7 + F_4 = 29$.

Regarding the above k columns rectangle as a matrix with k columns, we may consider, for instance, F_{k+3} as the entry $\mathcal{E}_{(2,3)}$ at the 2nd row and 3rd column, and in general we may consider $F_{(Q-1)k+r} = \mathcal{E}_{(Q,r)}$ as the entry at Q th row and r th column. Then the above theorem implies that, when $n = Qk + r$

$$F_n = \mathcal{E}_{(Q+1,r)} = \mu\mathcal{E}_{(Q,r)} + (-1)^{k-1}\mathcal{E}_{(Q-1,r)}$$

with $\mu = 2\mathcal{E}_{(1,k)} + \mathcal{E}_{(1,k-3)}$. It shows that F_n is a linear combination of two preceding entries $\mathcal{E}_{(Q,r)}$ and $\mathcal{E}_{(Q-1,r)}$ located in the same column that F_n belongs.

Corollary 11. F_n is a linear combination of three entries $\mathcal{E}_{(1,k)}$, $\mathcal{E}_{(1,k-3)}$ and $\mathcal{E}_{(1,r)}$ in the first row, and one entry $\mathcal{E}_{(1,k+r)}$ in the second row.

We shall extend this investigation to the 3-step tribonacci sequence T_n such that $T_1 = 0, T_2 = 1, T_3 = 1, T_4 = 2, T_5 = 4, T_6 = 7, \dots$. We begin with 4 or 5 columns.

4 columns				5 columns					
1	1	2	4		1	1	2	4	7
7	13	24	44		13	24	44	81	149
81	149	274	504		274	504	927	1705	3136
927	1705	3136	5768		5768	10609	19513	35890	66012
10609	19513	35890	...		121415	223317	410744	...	

It is clear that, in 4 columns rectangle

$$T_{16} = 5768 = 11 \cdot 504 + 5 \cdot 44 + 4 = (3T_4 - 1)T_{12} + (T_4 + 1)T_8 + T_4,$$

$$T_{19} = 35890 = 11 \cdot 3136 + 5 \cdot 274 + 24 = (3T_4 - 1)T_{15} + (T_4 + 1)T_{11} + T_7.$$

$$\text{Also in 5 columns, } T_{17} = 10690 = 21 \cdot 504 + 24 + 1 = (3T_5)T_{12} + T_7 + T_2,$$

and

$$T_{23} = 410744 = 21 \cdot 19513 + 927 + 44 = (3T_5)T_{18} + T_{13} + T_6.$$

Moreover considering the 6 columns rectangle we can find

$$T_{22} = 223317 = 39 \cdot 5768 - (13 - 2)149 + 4 = (3T_6)T_{16} - (T_6 - 2)T_{10} + T_4.$$

These considerations would yield the following theorem.

Theorem 12. Let $n = kQ + r$ with $1 \leq r \leq k$. Then for $4 \leq k \leq 9$

$$T_{kQ+r} = \mu_1 T_{k(Q-1)+r} + \mu_2 T_{k(Q-2)+r} + \mu_3 T_{k(Q-3)+r}$$

where the coefficients (μ_1, μ_2, μ_3) are determined as follows.

k	(μ_1, μ_2, μ_3)	k	(μ_1, μ_2, μ_3)	k	(μ_1, μ_2, μ_3)
4	$(3T_4 - 1, T_4 + 1, 1)$	5	$(3T_5, 1, 1)$	6	$(3T_6, -T_6 + 2, 1)$
7	$(3T_7 - 1, 15, 1)$	8	$(3T_8 - 1, -1, 1)$	9	$(3T_9 - 2, -23, 1)$

Proof. The above observations prove this for $4 \leq k \leq 6$. When $k = 7$,

1	1	2	4	7	13	24
44	81	149	274	504	927	1705
3136	5768	10609	19513	35890	66012	121415
223317	410744	755476	1389537	2555757	4700770	...

it is easy to see that

$$T_{22} = 223317 = 71 \cdot 3136 + 15 \cdot 44 + 1 = (3T_7 - 1)T_{15} + 15T_8 + T_1.$$

Similarly for $k = 8, 9$ and 10 , we see that

$$T_{25} = 1389537 = 131 \cdot 10609 - 3 \cdot 81 + 1 = (3T_8 - 1)T_{17} - 3T_9 + T_1,$$

$$T_{33} = 241 \cdot 755476 - 23 \cdot 3136 + 13 = (3T_9 - 2)T_{24} - (23)T_{15} + T_6,$$

$$T_{31} = 443 \cdot 121415 + 41 \cdot 274 + 1 = (3T_{10} - 4)T_{21} + (41)T_{11} + T_1$$

Based on these observation, the proof follows by induction. For instance the observation $(3T_9 - 2)T_{24} - (23)T_{15} + T_6T_{33}$ shows

$$(3T_k - 2)T_m - 23T_{m-k} + T_{m-2k} = T_{m+k} \quad \text{with } k = 9, m = 24.$$

We assume the identity is true for each $m \leq n$. Then

$$\begin{aligned} & (3T_k - 2)T_{n+1} - 23T_{(n+1)-k} + T_{(n+1)-2k} \\ &= (3T_k - 2)(T_n + T_{n-1} + T_{n-2}) - 23(T_{n-k} + T_{n-k-1} + T_{n-k-2}) \\ & \quad + (T_{n-2k} + T_{n-2k-1} + T_{n-2k-2}) \\ &= ((3T_k - 2)T_n - 23T_{n-k} + T_{n-2k}) + ((3T_k - 2)T_{n-1} - 23T_{(n-1)-k} + T_{(n-1)-2k}) \\ & \quad + ((3T_k - 2)T_{n-2} - 23T_{(n-2)-k} + T_{(n-2)-2k}) \\ &= T_{n+k} + T_{(n-1)+k} + T_{(n-2)+k} = T_{n+k+1} = T_{(n+1)+k}. \end{aligned}$$

Therefore for $k = 9$ and $n = kQ + r$ with $1 \leq r \leq k$, it follows that

$$\begin{aligned} T_{kQ+r} &= T_n = T_{(n-k)+k} \\ &= (3T_k - 2)T_{n-k} - 23T_{(n-k)-k} + T_{(n-k)-2k} \\ &= (3T_k - 2)T_{k(Q-1)+r} - 23T_{k(Q-2)+r} + T_{k(Q-3)+r} \end{aligned}$$

as is desired. For the other k , the proof follows similarly. $\square$

The formula for the coefficients that represent F_n as linear combination in Theorem 9 are the same no matter we choose any k , but the form for the coefficient that represent T_n in Theorem 12 vary depending on k .

Regarding T_{Qk+r} as an entry $\mathcal{E}_{(Q+1,r)}$ in k -columns rectangle, we can write

$$T_n = \mathcal{E}_{(Q+1,r)} = \mu_1 \mathcal{E}_{(Q,r)} + \mu_2 \mathcal{E}_{(Q-1,r)} + \mu_3 \mathcal{E}_{(Q-2,r)}$$

where the coefficient (μ_1, μ_2, μ_3) forms as follows:

k	(μ_1, μ_2, μ_3)	k	(μ_1, μ_2, μ_3)	k	(μ_1, μ_2, μ_3)
4	$(3\mathcal{E}_{(1,4)} - 1, 3\mathcal{E}_{(1,4)} + 1, 1)$	5	$(3\mathcal{E}_{(1,5)}, 1, 1)$	6	$(3\mathcal{E}_{(1,6)}, -\mathcal{E}_{(1,6)} + 2, 1)$
7	$(3\mathcal{E}_{(1,7)} - 1, 15, 1)$	8	$(3\mathcal{E}_{(1,8)} - 1, -1, 1)$	9	$(3\mathcal{E}_{(1,9)} - 2, -23, 1)$

As an analogue of Theorem 10 or Corollary 11, we are able to write $T_n = T_{Qk+r}$ as a combination of $\mathcal{E}_{(3,r)}$, $\mathcal{E}_{(2,r)}$ and $\mathcal{E}_{(1,r)}$. In fact

$$\begin{aligned} T_{Qk+r} &= \mathcal{E}_{(Q+1,r)} \\ &= \mu_1 \mathcal{E}_{(Q,r)} + \mu_2 \mathcal{E}_{(Q-1,r)} + \mu_3 \mathcal{E}_{(Q-2,r)} \\ &= \mu_1 (\mu_1 \mathcal{E}_{(Q-1,r)} + \mu_2 \mathcal{E}_{(Q-2,r)} + \mu_3 \mathcal{E}_{(Q-3,r)}) + \mu_2 \mathcal{E}_{(Q-1,r)} + \mu_3 \mathcal{E}_{(Q-2,r)} \\ &= (\mu_1^2 + \mu_2) \mathcal{E}_{(Q-1,r)} + (\mu_1 \mu_2 + \mu_3) \mathcal{E}_{(Q-2,r)} + \mu_1 \mathcal{E}_{(Q-3,r)} \\ &= (\mu_1^2 + \mu_2) (\mu_1 \mathcal{E}_{(Q-2,r)} + \mu_2 \mathcal{E}_{(Q-3,r)} + \mu_3 \mathcal{E}_{(Q-4,r)}) + (\mu_1 \mu_2 + \mu_3) \mathcal{E}_{(Q-2,r)} + \mu_1 \mathcal{E}_{(Q-3,r)} \\ &= (\mu_1 (\mu_1^2 + \mu_2) + (\mu_1 \mu_2 + \mu_3)) \mathcal{E}_{(Q-2,r)} + (\mu_2 (\mu_1^2 + \mu_2) + \mu_3) \mathcal{E}_{(Q-3,r)} + (\mu_1^2 + \mu_2) \mu_3 \mathcal{E}_{(Q-4,r)}. \end{aligned}$$

Hence after some stages, if we write

$$T_{Qk+r} = \theta_1 \mathcal{E}_{(Q-s,r)} + \theta_2 \mathcal{E}_{(Q-(s+1),r)} + \theta_3 \mathcal{E}_{(Q-(s+2),r)}$$

then T_{Qk+r} in the next step can be written as

$$T_{Qk+r} = (\mu_1 \theta_1 + \theta_2) \mathcal{E}_{(Q-(s+1),r)} + (\mu_2 \theta_1 + \theta_3) \mathcal{E}_{(Q-(s+2),r)} + \mu_3 \theta_1 \mathcal{E}_{(Q-(s+3),r)}.$$

The k could be any integer. But if k is large then the r in $Qk + r$ could be also large, so that T_k and T_r themselves are too big to be used for T_n .

Example 2. For T_{50} , if we take $k = 7$ then with the coefficient $(\mu_1, \mu_2, \mu_3) = (3\mathcal{E}_{(1,7)} - 1, 15, 1) = (71, 15, 1)$, it follows inductively that

$$\begin{aligned} T_{50} &= \mathcal{E}_{(8,1)} = 71 \mathcal{E}_{(7,1)} + 15 \mathcal{E}_{(6,1)} + \mathcal{E}_{(5,1)} \\ &= (71 \cdot 71 + 15) \mathcal{E}_{(6,1)} + (15 \cdot 71 + 1) \mathcal{E}_{(5,1)} + 71 \mathcal{E}_{(4,1)} \\ &= 360042 \mathcal{E}_{(5,1)} + 75911 \mathcal{E}_{(4,1)} + 5056 \mathcal{E}_{(3,1)} \\ &= 25638893 \mathcal{E}_{(4,1)} + 5405686 \mathcal{E}_{(3,1)} + 360042 \mathcal{E}_{(2,1)} \\ &= 1825767089 \mathcal{E}_{(3,1)} + 384943437 \mathcal{E}_{(2,1)} + 25638893 \mathcal{E}_{(1,1)} \\ &= 5,742,568,741,225 \end{aligned}$$

because $\mathcal{E}_{(3,1)} = 3136$, $\mathcal{E}_{(2,1)} = 44$ and $\mathcal{E}_{(1,1)} = 1$ in 7 columns distribution.

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