SOME STRONG FORMS OF (g, g') -CONTINUITY ON GENERALIZED TOPOLOGICAL SPACES

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Abstract. We introduce and investigate the notions of super (g, g') -continuous functions and strongly $\theta(g, g')$ -continuous functions on generalized topological spaces, which are strong forms of (g, g') -continuous functions. We also investigate relationships among such the functions, (g, g') -continuity and (δ, δ') -continuity.

1. Introduction

Császár [2] introduced the notion of generalized topological spaces. He also introduced the notions of continuous functions and associated interior and closure operators on generalized topological spaces. Characterizations for the generalized continuous ($= (g, g')$ -continuous) function were investigated in [2,4]. In this paper, we introduce and investigate the notions of super (g, g') -continuous functions and strongly $\theta(g, g')$ -continuous functions on generalized topological spaces, which are strong forms of (g, g') -continuous functions. We also investigate relationships among such the functions, (g, g') -continuity and (δ, δ') -continuity.

2. Preliminaries

We recall some notions and notations defined in [2]. Let X be a nonempty set and g be a collection of subsets of X . Then g is called a *generalized topology* (simply GT) on X iff $\emptyset \in g$ and $G_i \in g$ for $i \in I \neq \emptyset$ implies $G = \cup_{i \in I} G_i \in g$. We call the pair (X, g) a *generalized topological space* on X . A generalized topology g on X is called *strong* [3] if $X \in g$.

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The elements of g are called g -open sets and the complements are called g -closed sets. The generalized-closure of a subset S of X , denoted by $c_g(S)$, is the intersection of generalized closed sets including S . And the interior of S , denoted by $i_g(S)$, the union of generalized open sets included in S . Let (X, g) be a generalized topological space and $A \subseteq X$. Then A is said to be gr -open [3] if $A = i_g(c_g(A))$. The complement of gr -open is called a gr -closed set. Let us define $\delta(g) = \delta \subseteq \exp(X)$ by $A \in \delta$ iff $A \subseteq X$ and, if $x \in A$, then there is a g -closed set Q such that $x \in i_g(Q) \subseteq A$ [3].

Theorem 2.1 ([3, 5]). Let (X, g) be a generalized topological space and $A \subseteq X$.

- (1) $x \in c_\delta(A)$ iff $A \cap R \neq \emptyset$ for every gr -open set R containing x .
- (2) $x \in i_\delta(A)$ iff there exists a gr -open set R containing x such that $R \subseteq A$.
- (3) $A = c_\delta(A)$ iff A is δ -closed, $A = i_\delta(A)$ iff A is δ -open.
- (4) $i_\delta(A) = X - c_\delta(X - A)$.
- (5) $c_\delta(A) = X - i_\delta(X - A)$.

Let g and g' be generalized topologies on X and Y , respectively. Then a function $f : (X, g) \rightarrow (Y, g')$ is said to be

- (1) (g, g') -continuous [2] if $G' \in g'$ implies that $f^{-1}(G') \in g$;
- (2) (δ, δ') -continuous [5] if for $x \in X$ and each g' -open set V containing $f(x)$, there is a g -open set U containing x such that $f(i_g(c_g(V))) \subseteq i_{g'}(c_{g'}(U))$.

3. Main Results

Definition 3.1. Let (X, g) and (Y, g') be generalized topological spaces. Then a function $f : X \rightarrow Y$ is said to be $super (g, g')$ -continuous at $x \in X$ if for each g -open set V containing $f(x)$, there exists a g -open set U containing x such that $f(i_g(c_g(U))) \subseteq V$. The function f is said to be $super (g, g')$ -continuous if it is $super (g, g')$ -continuous at every point of X .

Remark 3.2. Let $f : (X, g) \rightarrow (Y, g')$ be a function on between generalized topological spaces (X, g) and (Y, g') . If the generalized topological spaces (X, g) and (Y, g') are topological spaces, then the $super (g, g')$ -continuous function is super continuous [1, 6].

Theorem 3.3. Let $f : X \rightarrow Y$ be a function on generalized topological spaces (X, g) and (Y, g') . Then the following are equivalent:

- (1) f is super (g, g') -continuous at $x \in X$.
- (2) $x \in i_\delta(f^{-1}(V))$ for every g' -open subset V containing $f(x)$.
- (3) For a g' -open set V containing $f(x)$, there exists a gr -open set U containing x such that $f(U) \subseteq V$.

Proof. (1) $\Rightarrow$ (2) Let V be any g' -open subset of Y containing $f(x)$. From definition of super (g, g') -continuity, there exists a g -open subset U of X containing x such that $f(i_g(c_g(U))) \subseteq V$. Since $x \in i_g(c_g(U)) \subseteq f^{-1}(V)$ and $i_g(c_g(U))$ is gr -open, by Theorem 2.1 (2), $x \in i_\delta(f^{-1}(V))$.

(2) $\Rightarrow$ (3) Let V be any g' -open subset of Y containing $f(x)$. Then from (2) and Theorem 2.1, there exists a gr -open set U containing x such that $U \subseteq f^{-1}(V)$. Hence (3) is obtained.

(3) $\Rightarrow$ (1) For $x \in X$ and any g' -open subset V of Y containing $f(x)$, there is a gr -open set U containing x such that $f(U) \subseteq V$. Since $U = i_g(c_g(U))$, we have $x \in i_g(c_g(U)) \subseteq f^{-1}(V)$, and hence f is super (g, g') -continuous at x . $\square$

Theorem 3.4. Let $f : (X, g) \rightarrow (Y, g')$ be a function on generalized topological spaces (X, g) and (Y, g') . Then the following are equivalent:

- (1) f is super (g, g') -continuous.
- (2) For every g' -open subset V in Y , $f^{-1}(V)$ is δ -open.
- (3) For every g' -closed subset F in Y , $f^{-1}(F)$ is δ -closed.
- (4) $c_\delta(f^{-1}(B)) \subseteq f^{-1}(c_{g'}(B))$ for every subset B in Y .
- (5) $f^{-1}(i_{g'}(B)) \subseteq i_\delta(f^{-1}(B))$ for every subset B in Y .
- (6) $f(c_\delta(A)) \subseteq c_{g'}(f(A))$ for every subset A in X .

Proof. (1) $\Rightarrow$ (2) Let V be a g' -open set in Y and $x \in f^{-1}(V)$. Then by Theorem 3.3 (3), there exists a gr -open set U of X containing x such that $U \subseteq f^{-1}(V)$. This implies $f^{-1}(V) = i_\delta(f^{-1}(V))$, and hence from Theorem 2.1 (3), $f^{-1}(V)$ is δ -open.

(2) $\Rightarrow$ (1) For $x \in X$, let V be any g' -open set of Y with $f(x) \in V$, then $f^{-1}(V)$ is a δ -open set. So there exists a g -closed set Q such that $x \in i_g(Q) \subseteq f^{-1}(V)$. Put $U = i_g(Q)$. Then U is a g -open set containing x and

$$U \subseteq i_g(c_g(U)) \subseteq i_g(c_g(i_g(Q))) \subseteq i_g(Q) \subseteq f^{-1}(V).$$

Thus f is super (g, g') -continuous.

(2) $\Leftrightarrow$ (3) Obvious.

(3) $\Rightarrow$ (4) It follows $c_\delta(f^{-1}(B)) \subseteq c_\delta(f^{-1}(c_{g'}(B))) = f^{-1}(c_{g'}(B))$, from Theorem 2.1. So (4) is easily obtained.

(3) $\Rightarrow$ (4) Obvious.

(4) $\Leftrightarrow$ (5) It follows from Theorem 2.1.

(4) $\Rightarrow$ (6) For $A \subseteq X$, by (4), $c_\delta(f^{-1}(f(A))) \subseteq f^{-1}(c_{g'}(f(A)))$. Finally we have $f(c_\delta(A)) \subseteq c_{g'}(f(A))$.

Similarly, it is proved that (6) $\Rightarrow$ (4). □

Let g be a GT on a nonempty set X and $P(X)$ the power set of X . Let us define the collection $\theta(g) \subseteq P(X)$ by $A \in \theta(g)$ iff for each $x \in A$, there is $M \in g$ such that $c_g(M) \subseteq A$ [3]. Then $\theta(g)$ is also a GT included in g [3]. The elements of $\theta(g)$ are called θ -open sets and the complements are called θ -closed sets. Simply, $\theta(g)$ is denoted by θ .

Definition 3.5. Let (X, g) and (Y, g') be generalized topological spaces. Then a function $f : X \rightarrow Y$ is said to be *strongly $\theta(g, g')$ -continuous* if for each $x \in X$ and each g' -open set V of $f(x)$, there exists a g -open set U of x such that $f(c_g(U)) \subseteq V$.

Remark 3.6. Let $f : X \rightarrow Y$ be a function on between generalized topological spaces (X, g) and (Y, g') . If the generalized topologies g and g' are topologies, then the strongly $\theta(g, g')$ -continuous function is strongly θ -continuous [7].

Theorem 3.7. Let $f : X \rightarrow Y$ be a function on generalized topological spaces (X, g) and (Y, g') . Then the following are equivalent:

- (1) f is strongly $\theta(g, g')$ -continuous.
- (2) For every g' -open subset V in Y , $f^{-1}(V)$ is θ -open.
- (3) For every g' -closed subset F in Y , $f^{-1}(F)$ is θ -closed.

Proof. (1) $\Rightarrow$ (2) Let V be a g' -open set in Y and $x \in f^{-1}(V)$. Then by hypothesis, there exists a g -open set U of X containing x such that $U \subseteq c_g(U) \subseteq f^{-1}(V)$. Hence $f^{-1}(V)$ is θ -open.

(2) $\Rightarrow$ (1) Let $x \in X$ and V any g' -open set V of $f(x)$. Then $f^{-1}(V)$ is a θ -open set containing x , and so there exists a g -open set U such that $x \in U \subseteq c_g(U) \subseteq f^{-1}(V)$. It implies f is strongly $\theta(g, g')$ -continuous.

(2) $\Leftrightarrow$ (3) Obvious. □

Remark 3.8. The following implications are obtained but the reverse relations may not be true in general.

Example 3.9. Let $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3\}$. Consider generalized topologies $g = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, X\}$ on X and $g' = \{\emptyset, \{1, 2\}\}$ on Y . Then $\{\emptyset, \{a\}, \{b, c\}, X\}$ is the collection of all gr -open sets in X and

$$\delta = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}; \quad \theta = \{\emptyset, X\}.$$

(1) Let $f : (X, g) \rightarrow (Y, g')$ be a function defined as follows $f(a) = 1$, $f(b) = f(d) = 2$, and $f(c) = 3$. Then f is (g, g') -continuous but not super (g, g') -continuous.

(2) Let $g : (X, g) \rightarrow (Y, g')$ be a function defined as follows $g(a) = 1$, $g(b) = g(c) = 2$, and $g(d) = 3$. Then g is super (g, g') -continuous but not strongly $\theta(g, g')$ -continuous.

(3) Let $h : (X, g) \rightarrow (X, g)$ be the identity function. Then h is (δ, δ') -continuous but not super (g, g') -continuous.

Let (X, g) be a generalized topological space. Then X is said to be *relative G -regular* (simply, *G -regular*) [5] on $\mathcal{M}_g$ if for $x \in \mathcal{M}_g$ and a g -closed set F with $x \notin F$, there exist $U, V \in g$ such that $x \in U$, $F \cap \mathcal{M}_g \subseteq V$ and $U \cap V = \emptyset$. If X is a topological space, then the G -regularity on X is the regularity.

Lemma 3.10 ([Theorem 3.13 of [5]]). Let (X, g) be a generalized topological space. If X is G -regular, every g -open set is δ -open.

Theorem 3.11. Let $f : (X, g) \rightarrow (Y, g')$ be a (g, g') -continuous function on generalized topological spaces (X, g) and (Y, g') . Then if X is G -regular, then f is super (g, g') -continuous.

Proof. It is obvious from Lemma 3.10. □

Theorem 3.12. Let $f : (X, g) \rightarrow (Y, g')$ be a (g, g') -continuous function on generalized topological spaces (X, g) and (Y, g') . Then if Y is G -regular and strong, then f is strongly $\theta(g, g')$ -continuous.

Proof. Let $x \in X$ and V any g' -open set containing $f(x)$. From G -regularity of Y , there exists a g' -open set W in Y such that $f(x) \in W \subseteq c_{g'}(W) \cap \mathcal{M}_{g'} \subseteq V$. Since Y is strong, $\mathcal{M}_{g'} = Y$ and so $f(x) \in W \subseteq$

$c_{g'}(W) \subseteq V$. Since f is (g, g') -continuous, $x \in f^{-1}(W) \subseteq c_g(f^{-1}(W)) \subseteq f^{-1}(c_{g'}(W)) \subseteq f^{-1}(V)$. Put $U = f^{-1}(W)$; then U is g -open and $c_g(U) \subseteq f^{-1}(V)$. So f is strongly $\theta(g, g')$ -continuous. $\square$

Corollary 3.13. Let $f : (X, g) \rightarrow (Y, g')$ be a function on generalized topological spaces (X, g) and (Y, g') . If Y is G -regular and strong, then the following are equivalent:

- (1) (g, g') -continuity.
- (2) super (g, g') -continuity.
- (3) strongly $\theta(g, g')$ -continuity.

Corollary 3.14. Let $f : X \rightarrow Y$ be a function on topological spaces X and Y . If Y is regular, then the following are equivalent:

- (1) continuity.
- (2) super continuity.
- (3) strongly θ -continuity.

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