APPLICATION OF THE OPERATOR $H(\alpha, \beta)$ TO THE SARAN FUNCTION F_E AND SOME OTHER RESULTS

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Abstract. The enormous success of the theory of hypergeometric series in a single variable has stimulated the development of a corresponding theory in two and more variables. A wide variety of investigations in the theory of several variable hypergeometric functions have been essentially motivated by the fact that solutions of many applied problems involving partial differential equations are obtainable with the help of such hypergeometric functions. Here, in this trend, we aim at presenting further decomposition formulas for Saran function F_E , which are used to give some integral representations of the function F_E . We also present a system of partial differential equations for the Saran function F_E .

1. Introduction and Preliminaries

In addition to the remarkable success of the theory of hypergeometric series in a single variable, a great interest in the theory of hypergeometric functions in several variables is stimulated, essentially, by the fact that the solutions of many applied problems involving (for example) partial differential equations are obtainable with the help of such hypergeometric functions (see, for details, [26, p. 47]; see also the recent works [8, 10, 11, 12, 20, 21] and the references cited therein). For instance, the energy absorbed by some no ferromagnetic conductor sphere included in an internal magnetic field can be calculated with the aid of such functions [16, 19]. Hypergeometric functions of several variables are also used in physical and quantum chemical applications (see [18, 24, 25]).

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Especially, many problems in gas dynamics lead to solutions of degenerate second-order partial differential equations, which are then solvable in terms of multiple hypergeometric functions. Among examples, we can cite the problem of adiabatic flat-parallel gas flow without whirlwind, the flow problem of supersonic current from vessel with flat walls, and a number of other problems connected with gas flow (see [2, 9]). We note that Riemann's functions and fundamental solutions of the degenerate second-order partial differential equations are expressible by means of hypergeometric functions of several variables (see [10, 11, 12]). In particular, the Saran [23, p. 68, Eq. (34)] (see also [26]) initiated a systematic study of these ten triple hypergeometric functions of Lauricella's set and defined the function F_E by the following triple series:

$$(1.1) \quad F_E(\alpha; \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+n+p}(\beta_1)_m(\beta_2)_{n+p}}{(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p m!n!p!} x^m y^n z^p$$

$$(|x| < r, |y| < s, |z| < t, r + s = rs, s = t),$$

where $(\alpha)_m$ denotes the Pochhammer symbol defined by $(\alpha)_m := \Gamma(\alpha + m)/\Gamma(\alpha)$, and Γ being the familiar Gamma function.

Over six decades ago, Burchnall and Chaundy [3, 4] and Chaundy [5] systematically presented a number of expansions and decomposition formulas for some double hypergeometric functions in series of simpler hypergeometric functions. Their method is based upon the following inverse pairs of symbolic operators:

$$\nabla_{xy}(h) := \frac{\Gamma(h)\Gamma(\delta_1 + \delta_2 + h)}{\Gamma(\delta_1 + h)\Gamma(\delta_2 + h)} = \sum_{k=0}^{\infty} \frac{(-\delta_1)_k(-\delta_2)_k}{(h)_k k!}, \quad (1.2)$$

$$\begin{aligned} \Delta_{xy}(h) &:= \frac{\Gamma(\delta_1 + h)\Gamma(\delta_2 + h)}{\Gamma(h)\Gamma(\delta_1 + \delta_2 + h)} = \sum_{k=0}^{\infty} \frac{(-\delta_1)_k(-\delta_2)_k}{(1-h-\delta_1-\delta_2)_k k!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (h)_{2k} (-\delta_1)_k (-\delta_2)_k}{(h+k-1)_k (h+\delta_1)_k (h+\delta_2)_k k!}, \end{aligned} \quad (1.3)$$

and

$$\begin{aligned}\nabla_{xy}(h)\Delta_{xy}(g) &:= \frac{\Gamma(h)\Gamma(\delta_1+\delta_2+h)\Gamma(\delta_1+g)\Gamma(\delta_2+g)}{\Gamma(\delta_1+h)\Gamma(\delta_2+h)\Gamma(g)\Gamma(\delta_1+\delta_2+g)} \\ &= \sum_{k=0}^{\infty} \frac{(g-h)_k(g)_{2k}(-\delta_1)_k(-\delta_2)_k}{(g+k-1)_k(g+\delta_1)_k(g+\delta_2)_k k!} \\ &= \sum_{k=0}^{\infty} \frac{(h-g)_k(-\delta_1)_k(-\delta_2)_k}{(h)_k(1-g-\delta_1-\delta_2)_k k!} \quad \left(\delta_1 := x \frac{\partial}{\partial x}; \delta_2 := y \frac{\partial}{\partial y} \right).\end{aligned}$$

For various multivariable hypergeometric functions including the Lauricella multivariable functions $F_A^{(r)}$, $F_B^{(r)}$, $F_C^{(r)}$ and $F_D^{(r)}$, Hasanov and Srivastava [13, 14] presented a number of decompositions formulas in terms of such simpler hypergeometric functions as the Gauss and Appell functions. In this sequel to the works of Hasanov and Srivastava [13, 14], it can be shown that some rather elementary techniques based upon certain inverse pairs of symbolic operators

$$\begin{aligned}\tilde{\nabla}_{x_1; x_2 \dots x_r}(h) &:= \frac{\Gamma(h)\Gamma(\delta_1+\delta_2+\dots+\delta_r+h)}{\Gamma(\delta_1+h)\Gamma(\delta_2+\dots+\delta_r+h)} \\ &= \sum_{m_2, \dots, m_r=0}^{\infty} \frac{(-\delta_1)_{m_2+\dots+m_r}(-\delta_2)_{m_2} \dots (-\delta_r)_{m_r}}{(h)_{m_2+\dots+m_r} m_2! \dots m_r!},\end{aligned}\tag{1.4}$$

$$\begin{aligned}(1.5) \quad \tilde{\Delta}_{x_1; x_2 \dots x_r}(h) &:= \frac{\Gamma(\delta_1+h)\Gamma(\delta_2+\dots+\delta_r+h)}{\Gamma(h)\Gamma(\delta_1+\delta_2+\dots+\delta_r+h)} \\ &= \sum_{m_2, \dots, m_r=0}^{\infty} \frac{(-\delta_1)_{m_2+\dots+m_r}(-\delta_2)_{m_2} \dots (-\delta_r)_{m_r}}{(1-h-\delta_1-\dots-\delta_r)_{m_2+\dots+m_r} m_2! \dots m_r!} \\ &= \sum_{m_2, \dots, m_r=0}^{\infty} \frac{(-1)^{m_2+\dots+m_r} (h)_{2(m_2+\dots+m_r)}}{(h+m_2+\dots+m_r-1)_{m_2+\dots+m_r} (\delta_1+h)_{m_2+\dots+m_r}} \\ &\quad \cdot \frac{(-\delta_1)_{m_2+\dots+m_r}(-\delta_2)_{m_2} \dots (-\delta_r)_{m_r}}{(\delta_2+\dots+\delta_r+h)_{m_2+\dots+m_r} m_2! \dots m_r!} \\ &\quad \left(\delta_j := x_j \frac{\partial}{\partial x_j}; j = 1, 2, \dots, r \right),\end{aligned}$$

would lead us easily to several decomposition formulas associated with Lauricella's hypergeometric function $F_A^{(r)}$, $F_B^{(r)}$, $F_C^{(r)}$ and $F_D^{(r)}$ in r variables and with other multiple hypergeometric functions. It is noted in

passing that, very recently, Choi [6] used the Lauricella series $F_D^{(r)}$ to give a generating function for Gottlieb polynomials in several variables.

We introduce the following multivariable symbolic operators (see [7]):

$$\begin{aligned}
 (1.6) \quad H_{x_1, \dots, x_l}(\alpha, \beta) &:= \frac{\Gamma(\beta) \Gamma(\alpha + \delta_1 + \dots + \delta_l)}{\Gamma(\alpha) \Gamma(\beta + \delta_1 + \dots + \delta_l)} \\
 &= \sum_{k_1, \dots, k_l=0}^{\infty} \frac{(\beta - \alpha)_{k_1 + \dots + k_l} (-\delta_1)_{k_1} \dots (-\delta_l)_{k_l}}{(\beta)_{k_1 + \dots + k_l} k_1! \dots k_l!}
 \end{aligned}$$

and

$$\begin{aligned}
 (1.7) \quad \bar{H}_{x_1, \dots, x_l}(\alpha, \beta) &:= \frac{\Gamma(\alpha) \Gamma(\beta + \delta_1 + \dots + \delta_l)}{\Gamma(\beta) \Gamma(\alpha + \delta_1 + \dots + \delta_l)} \\
 &= \sum_{k_1, \dots, k_l=0}^{\infty} \frac{(\beta - \alpha)_{k_1 + \dots + k_l} (-\delta_1)_{k_1} \dots (-\delta_l)_{k_l}}{(1 - \alpha - \delta_1 - \dots - \delta_l)_{k_1 + \dots + k_l} k_1! \dots k_l!}
 \end{aligned}$$

$$\left(\delta_j := x_j \frac{\partial}{\partial x_j}, j = 1, \dots, l; l \in \mathbb{N} := \{1, 2, 3, \dots\} \right).$$

In order to obtain the above-mentioned operators we have applied the following known multiple hypergeometric summation formulas (see [1, p. 117]):

$$\begin{aligned}
 F_D^{(r)}(\alpha; \beta_1, \dots, \beta_r; \gamma; 1, \dots, 1) &= \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta_1 - \dots - \beta_r)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta_1 - \dots - \beta_r)} \\
 (\Re(\gamma - \alpha - \beta_1 - \dots - \beta_r) > 0, \gamma \notin \mathbb{Z}_0^- := \{0, -1, -2, -3, \dots\})
 \end{aligned}$$

for the Lauricella function $F_D^{(r)}$ in r variables. It is noted that, using multidimensional symbolic operators presented in [13, 14], Kim *et al.* [15] proved several decomposition formulas and integral representations for the Saran's function (1.1). Here, we aim at presenting further decomposition formulas for Saran function F_E , which are used to give some integral representations of the function F_E . We also present a system of partial differential equations for the Saran function F_E .

2. Families of decomposition formulas for Saran function F_E

It is easy to derive the following relations by using suitable cases of the symbolic operators defined by (1.6) and (1.7):

$$(2.1) \quad \begin{aligned} &F_E(\alpha; \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) \\ &= H_x(\beta_1, \gamma_1)(1-x)^{-\alpha} F_4\left(\alpha, \beta_2; \gamma_2, \gamma_3; \frac{y}{1-x}, \frac{z}{1-x}\right); \end{aligned}$$

$$(2.2) \quad \begin{aligned} &(1-x)^{-\alpha} F_4\left(\alpha, \beta_2; \gamma_2, \gamma_3; \frac{y}{1-x}, \frac{z}{1-x}\right) \\ &= \bar{H}_x(\beta_1, \gamma_1) F_E(\alpha; \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z); \end{aligned}$$

$$F_E(\alpha; \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) = H_{x,y,z}(\alpha, \tau) F_E(\tau; \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z); \quad (2.3)$$

$$F_E(\alpha; \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) = H_x(\beta_1, \tau) F_E(\alpha; \tau, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z); \quad (2.4)$$

$$F_E(\alpha; \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) = H_{y,z}(\beta_2, \tau) F_E(\alpha; \beta_1, \tau; \gamma_1, \gamma_2, \gamma_3; x, y, z); \quad (2.5)$$

$$F_E(\alpha; \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) = H_x(\tau, \gamma_1) F_E(\alpha; \beta_1, \beta_2; \tau, \gamma_2, \gamma_3; x, y, z). \quad (2.6)$$

In view of the known Mellin-Barnes contour integral representations for the Saran function F_E , it is not difficult to give alternative proofs of the operator identities (2.1) to (2.6) above by using the Mellin and the inverse Mellin transformations (see, e.g., [17]). The details involved in these alternative derivations of the operator identities (2.1) to (2.6) are omitted here. By virtue of the derivative formulas for the Saran function, and also of some standard properties of hypergeometric functions, we find each of the following decomposition formulas for the Saran function F_E in triple variables:

$$(2.7) \quad \begin{aligned} &F_E(\alpha; \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) \\ &= (1-x)^{-\alpha} \sum_{i=0}^{\infty} \frac{(\alpha)_i (\gamma_1 - \beta_1)_i}{(\gamma_1)_i i!} \left(\frac{x}{x-1}\right)^i F_4\left(\alpha + i, \beta_2; \gamma_2, \gamma_3; \frac{y}{1-x}, \frac{z}{1-x}\right); \end{aligned}$$

$$\begin{aligned}
(2.8) \quad & (1-x)^{-\alpha} F_4 \left(\alpha, \beta_2; \gamma_2, \gamma_3; \frac{y}{1-x}, \frac{z}{1-x} \right) \\
&= \sum_{i=0}^{\infty} \frac{(\alpha)_i (\gamma_1 - \beta_1)_i}{(\gamma_1)_i i!} x^i F_E (\alpha + i; \beta_1, \beta_2; \gamma_1 + i, \gamma_2, \gamma_3; x, y, z); \\
F_E (\alpha; \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) &= \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k} (\tau - \alpha)_{i+j+k} (\beta_1)_i (\beta_2)_{j+k}}{(\gamma_1)_i (\gamma_2)_j (\gamma_3)_k i! j! k!} x^i y^j z^k \\
&\quad \times F_E (\tau + i + j + k; \beta_1 + i, \beta_2 + j + k; \gamma_1 + i, \gamma_2 + j, \gamma_3 + k; x, y, z); \quad (2.9)
\end{aligned}$$

$$\begin{aligned}
(2.10) \quad & F_E (\alpha; \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i (\alpha)_i (\tau - \beta_1)_i}{(\gamma_1)_i i!} x^i F_E (\alpha + i; \tau + i, \beta_2; \gamma_1 + i, \gamma_2, \gamma_3; x, y, z); \\
F_E (\alpha; \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) &= \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k} (\alpha)_{j+k} (\tau - \beta_2)_{j+k}}{(\gamma_2)_j (\gamma_3)_k j! k!} y^j z^k \\
&\quad \times F_E (\alpha + j + k; \beta_1, \tau + j + k; \gamma_1, \gamma_2 + j, \gamma_3 + k; x, y, z); \quad (2.11)
\end{aligned}$$

$$\begin{aligned}
F_E (\alpha; \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) &= \sum_{i=0}^{\infty} \frac{(-1)^i (\alpha)_i (\beta_1)_i (\gamma_1 - \tau)_i}{(\gamma_1)_i (\tau)_i i!} x^i \\
&\quad \times F_E (\alpha + i; \beta_1 + i, \beta_2; \tau + i, \gamma_2, \gamma_3; x, y, z); \quad (2.12)
\end{aligned}$$

$$\begin{aligned}
& F_E (\alpha; \beta_1, \gamma_2; \gamma_1, \gamma_2, \gamma_2; x, y, z) \\
&= (1-y-z)^{-\alpha} \sum_{i=0}^{\infty} \frac{(\alpha)_{2i}}{(\gamma_2)_i i!} \left(\frac{yz}{(1-y-z)^2} \right)^i F \left(\alpha + 2i, \beta_1; \gamma_1; \frac{x}{1-y-z} \right). \quad (2.13)
\end{aligned}$$

Our operational derivations of the decomposition formulas (2.7) to (2.13) would indeed run parallel to those presented in the earlier works, which we have already cited in the preceding sections. In addition to the various operator expressions, we also make use of the following operator identities [22, p. 93]:

$$(\delta + \alpha)_n \{f(\xi)\} = \xi^{1-\alpha} \frac{d^n}{d\xi^n} \{\xi^{\alpha+n-1} f(\xi)\} \quad (2.14)$$

$$\left(\delta := \xi \frac{d}{d\xi}; \alpha \in \mathbb{C}; n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \right)$$

and

$$(-\delta)_n \{f(\xi)\} = (-1)^n \xi^n \frac{d^n}{d\xi^n} \{f(\xi)\}, \quad \left(\delta := \xi \frac{d}{d\xi}; n \in \mathbb{N}_0 \right) \quad (2.15)$$

for every analytic function $f(\xi)$.

3. Integral representations via decomposition formulas

Here, we observe that several integral representations of the Eulerian type can be deduced from the corresponding decomposition formulas in Section 2. For example, some integral representations for the Saran function F_E are given:

$$\begin{aligned} F_E(\alpha; \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) &= \frac{\Gamma(\gamma_1)}{\Gamma(\beta_1) \Gamma(\gamma_1 - \beta_1)} \\ &\times \int_0^1 \xi^{\beta_1-1} (1-\xi)^{\gamma_1-\beta_1-1} (1-x\xi)^{-\alpha} F_4\left(\alpha, \beta_2; \gamma_2, \gamma_3; \frac{y}{1-x\xi}, \frac{z}{1-x\xi}\right) d\xi \end{aligned} \quad (3.1)$$

$$(\Re(\gamma_1) > \Re(\beta_1) > 0);$$

(3.2)

$$\begin{aligned} F_E(\alpha; \beta_1, \gamma_2; \gamma_1, \gamma_2, \gamma_2; x, y, z) &= \frac{\Gamma(\gamma_1)}{\Gamma(\beta_1) \Gamma(\gamma_1 - \beta_1)} \\ &\times \int_0^1 t^{\beta_1-1} (1-t)^{\gamma_1-\beta_1-1} (1-y-z-xt)^{-\alpha} \\ &\cdot F\left(\frac{\alpha}{2}, \frac{1+\alpha}{2}; \gamma_2; \frac{4yz}{(1-y-z-xt)^2}\right) dt \quad (\Re(\gamma_1) > \Re(\beta_1) > 0); \end{aligned}$$

$$\begin{aligned}
 (3.3) \quad & F_E(\alpha; \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) \\
 &= \frac{\Gamma(\alpha + \beta_1 + \beta_2)}{\Gamma(\alpha) \Gamma(\beta_1) \Gamma(\beta_2)} \int_0^1 \int_0^1 \xi^{\alpha-1} \eta^{\alpha+\beta_1-1} (1-\xi)^{\beta_1-1} (1-\eta)^{\beta_2-1} \\
 &\quad \times F_4\left(\frac{\alpha + \beta_1 + \beta_2}{2}, \frac{\alpha + \beta_1 + \beta_2 + 1}{2}; \right. \\
 &\quad \left. \gamma_1, \gamma_2, \gamma_3; 4x\xi\eta^2(1-\xi), 4y\xi\eta(1-\eta), 4z\xi\eta(1-\eta)\right) d\xi d\eta.
 \end{aligned}$$

4. System of partial differential equations for F_E

According to the theory of multiple hypergeometric functions [1], the system of partial differential equations for the Saran hypergeometric function F_E is readily seen to be given as follows:

$$(4.1) \quad \begin{cases} x(1-x)u_{xx} - xyu_{xy} - xzu_{xz} + [\gamma_1 - (\alpha + \beta_1 + 1)x]u_x \\ \quad - \beta_1 y u_y - \beta_1 z u_z - \alpha \beta_1 u = 0, \\ y(1-y)u_{yy} - xyu_{xy} - xzu_{xz} - 2yzu_{yz} - z^2 u_{zz} \\ \quad - \beta_2 x u_x + [\gamma_2 - (\alpha + \beta_2 + 1)y]u_y - (\alpha + \beta_2 + 1)z u_z - \alpha \beta_2 u = 0, \\ z(1-z)u_{zz} - xyu_{xy} - xzu_{xz} - 2yzu_{yz} - y^2 u_{yy} \\ \quad - \beta_2 x u_x - (\alpha + \beta_2 + 1) y u_y + [\gamma_3 - (\alpha + \beta_2 + 1)z]u_z - \alpha \beta_2 u = 0, \end{cases}$$

where $u = F_E(\alpha; \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z)$. It is noted that three equations of the system (4.1) are simultaneous, because the hypergeometric function F_E satisfies the system. Now in order to find the linearly independent solutions of the system (4.1) we consider u as in the form $u = x^\tau y^\nu z^\lambda w$, where w is an unknown function, and τ, ν and λ are constants which are to be determined. So, substituting $u = x^\tau y^\nu z^\lambda w$

into the system (4.1), we obtain

$$(4.2) \quad \left\{ \begin{array}{l} x(1-x)w_{xx} - xyw_{xy} - xzw_{xz} - (\tau + \beta_1)yw_y - (\tau + \beta_1)zw_z \\ \quad + \{2\tau + \gamma_1 - [(\tau + \nu + \lambda + \alpha) + (\tau + \beta_1) + 1]x\}w_x \\ \quad - \{-\tau(\tau + \gamma_1 - 1)x^{-1} + (\tau + \beta_1)(\tau + \nu + \lambda + \alpha)\}w = 0, \\ y(1-y)w_{yy} - xyw_{xy} - xzw_{xz} - 2yzw_{yz} - z^2w_{zz} - (\nu + \lambda + \beta_2)xw_x \\ \quad + \{2\nu + \gamma_2 - [(\tau + \nu + \lambda + \alpha) + (\nu + \lambda + \beta_2) + 1]y\}w_y \\ \quad - [(\tau + \nu + \lambda + \alpha) + (\nu + \lambda + \beta_2) + 1]zw_z \\ \quad - \{\nu(\nu + \gamma_2 - 1)y^{-1} + (\tau + \nu + \lambda + \alpha)(\nu + \lambda + \beta_2)\}w = 0, \\ z(1-z)w_{zz} - xyw_{xy} - xzw_{xz} - 2yzw_{yz} - y^2w_{yy} - (\nu + \lambda + \beta_2)xw_x \\ \quad - [(\tau + \nu + \lambda + \alpha) + (\nu + \lambda + \beta_2) + 1]yw_y \\ \quad + \{2\lambda + \gamma_3 - [(\tau + \nu + \lambda + \alpha) + (\nu + \lambda + \beta_2) + 1]z\}w_z \\ \quad - \{-\lambda(\lambda - 1 + \gamma_3)z^{-1} + (\tau + \nu + \lambda + \alpha)(\nu + \lambda + \beta_2)\}w = 0. \end{array} \right.$$

Therefore, it is observed that the following conditions should be satisfied:

$$(4.3) \quad \left\{ \begin{array}{l} \tau(\tau + \gamma_1 - 1) = 0, \\ \nu(\nu + \gamma_2 - 1) = 0, \\ \lambda(\lambda + \gamma_3 - 1) = 0. \end{array} \right.$$

It is found that the system (4.3) satisfies the following solutions:

$$(4.4) \quad \begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \tau : & 0 & 1 - \gamma_1 & 0 & 0 & 1 - \gamma_1 & 1 - \gamma_1 & 0 & 1 - \gamma_1 \\ \nu : & 0 & 0 & 1 - \gamma_2 & 0 & 1 - \gamma_2 & 0 & 1 - \gamma_2 & 1 - \gamma_2 \\ \lambda : & 0 & 0 & 0 & 1 - \gamma_3 & 0 & 1 - \gamma_3 & 1 - \gamma_3 & 1 - \gamma_3 \end{array}$$

Finally, substituting all eight solutions (4.4), we find the following linearly independent solutions of the system (4.1):

$$u_1 = k_1 F_E(\alpha; \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z), \quad (4.4)$$

$$u_2 = k_2 x^{1-\gamma_1} F_E(1 - \gamma_1 + \alpha; 1 - \gamma_1 + \beta_1, \beta_2; 2 - \gamma_1, \gamma_2, \gamma_3; x, y, z), \quad (4.5)$$

$$u_3 = k_3 y^{1-\gamma_2} F_E(1 - \gamma_2 + \alpha; \beta_1, 1 - \gamma_2 + \beta_2; \gamma_1, 2 - \gamma_2, \gamma_3; x, y, z), \quad (4.6)$$

$$u_4 = k_4 z^{1-\gamma_3} F_E(1 - \gamma_3 + \alpha; \beta_1, 1 - \gamma_3 + \beta_2; \gamma_1, \gamma_2, 2 - \gamma_3; x, y, z), \quad (4.7)$$

$$(4.8) \quad u_5 = k_5 x^{1-\gamma_1} y^{1-\gamma_2} \cdot F_E(2-\gamma_1-\gamma_2+\alpha; 1-\gamma_1+\beta_1, 1-\gamma_2+\beta_2; 2-\gamma_1, 2-\gamma_2, \gamma_3; x, y, z),$$

$$(4.9) \quad u_6 = k_6 x^{1-\gamma_1} z^{1-\gamma_3} \cdot F_E(2-\gamma_1-\gamma_3+\alpha; 1-\gamma_1+\beta_1, 1-\gamma_3+\beta_2; 2-\gamma_1, \gamma_2, 2-\gamma_3; x, y, z),$$

$$(4.10) \quad u_7 = k_7 y^{1-\gamma_2} z^{1-\gamma_3} \cdot F_E(2-\gamma_2-\gamma_3+\alpha; \beta_1, 2-\gamma_2-\gamma_3+\beta_2; \gamma_1, 2-\gamma_2, 2-\gamma_3; x, y, z),$$

$$(4.11) \quad u_8 = k_8 x^{1-\gamma_1} y^{1-\gamma_2} z^{1-\gamma_3} F_E(3-\gamma_1-\gamma_2-\gamma_3+\alpha; 1-\gamma_1+\beta_1, 2-\gamma_2-\gamma_3+\beta_2; 2-\gamma_1, 2-\gamma_2, 2-\gamma_3; x, y, z).$$

where k_i ($i = 1, 2, \dots, 8$) are constants.

Remark. By choosing suitably specialized parameters in the results in Section 2, we can get the following relations for hypergeometric function F_E :

$$(4.12) \quad F_E(\alpha; \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) = (1-x)^{-\alpha} F_E\left(\alpha; \gamma_1-\beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; \frac{x}{x-1}, \frac{y}{1-x}, \frac{z}{1-x}\right);$$

$$(4.13) \quad F_E(\alpha; \gamma_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) = (1-x)^{-\alpha} F_4\left(\alpha, \beta_2; \gamma_2, \gamma_3; \frac{y}{1-x}, \frac{z}{1-x}\right);$$

$$(4.14) \quad F_E(\alpha; \beta_1, \gamma_2; \gamma_1, \gamma_2, \gamma_2; x, y, z) = (1-x-y-z)^{-\alpha} \times H_4\left(\alpha, \gamma_1-\beta_1; \gamma_2, \gamma_1; \frac{x}{x+y+z-1}, \frac{yz}{(1-x-y-z)^2}\right),$$

where H_4 denotes one of Horn's double series defined by

$$(4.15) \quad H_4(\alpha, \beta; \gamma, \delta; x, y) := \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_n}{(\gamma)_m(\delta)_n m! n!} x^m y^n.$$

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