REGULAR ENDOMORPHISM RINGS OF INJECTIVE MODULES

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ABSTRACT. The authors have investigated several properties of endomorphism rings of modules. In particular, a closedly simple module and a closedly semi-simple module were defined and then the regularities of its rings of endomorphisms were discussed.

1. Introduction

Assume that ring R is any associative ring with identity. The ring of all R -endomorphisms on a left R -module ${}_R M$ (a right R -module M_R) is denoted by $End_R(M)$ is studied. In the endomorphism ring $S = End_R(M)$ under the multiplication, i.e., the composition mapping, which is operating on the left side of ${}_R M$. Any left R -module ${}_R M$ is considered as a left R - and left S -module ${}_{R-S} M$. Precisely speaking, the composition of endomorphisms is written by the usual order of mappings as usual;

$$gf : M \xrightarrow[f]{} M \xrightarrow[g]{} M.$$

We have some materials to recall:

$$I_L = Hom_R(L, 0) = \{f \in End_R(M) \mid L \leq \ker f\}$$

is a left ideal of the endomorphism ring $End_R(M)$, for each submodule $L \leq M$ of M

For any subset J of $S = End_R(M)$, let $\ker J = \cap_{j \in J} \ker j$ be the intersection of all kernels of endomorphisms in J .

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We will call L a *closed* submodule if

$$L = \overline{L}, \text{ where } \overline{L} = \cap_{f \in S, L \leq \ker f} \ker f \text{ is the intersection of all kernels}$$

of endomorphisms whose kernels contain L .

Recall a module is *simple*, [3](*irreducible*, [7]) it has no non-trivial submodule. A module M is said to be *semi-simple* (*completely reducible*) if it is a sum of (possibly infinitely many) simple modules.

A submodule N of M is said to be *minimal* (*minimal closed*, resp.) if $0 \leq H \leq N$ for any (closed, resp.) submodule $H \leq M$, implies that $0 = H$ or $H = N$.

The Socle of a module M , denoted by $\text{Soc}(M)$, is the sum of all minimal submodules N_α of M , in fact,

$$\text{Soc}(M) = \sum N_\alpha, \text{ for all minimal submodules } N_\alpha \leq M.$$

Here we see the rhetorical question, which means, that a minimal submodule is simple in the submodule itself, and vice versa.

DEFINITION 1.1. For any left (left) R -module M ,

- (1) M is said to be *closedly simple* if there is no proper closed submodule of M i.e., only closed submodules are 0 and M itself [8].
- (2) Let $(M_\alpha)_{\alpha \in \Gamma}$ be an indexed set of left (right) R -modules M_α ($\alpha \in \Gamma$). Then a direct product $\prod_\Gamma M_\alpha$ and a weak direct product $\prod_\Gamma^w M_\alpha$ of R -modules $(M_\alpha)_{\alpha \in \Gamma}$ are defined, briefly, saying

$$\prod_\Gamma M_\alpha = \{(m_\alpha)_\alpha \mid m_\alpha \in M_\alpha (\alpha \in \Gamma)\}$$

under the coordinatewise addition and scalar multiplication;

$$(n_\alpha)_\alpha + (m_\alpha)_\alpha = (n_\alpha + m_\alpha)_\alpha \text{ and } r(n_\alpha)_\alpha = (rn_\alpha)_\alpha,$$

for every $(n_\alpha)_\alpha, (m_\alpha)_\alpha \in \prod_\Gamma M_\alpha$, and for every $r \in R$.

And $\prod_\Gamma^w M_\alpha = \{(m_\alpha) \in \prod_\Gamma M_\alpha \mid m_\beta = 0 \text{ for all but a finite number of indices } \beta \in \Gamma\}$.

With the α -th projection $\pi_\alpha : \prod_\Gamma M_\alpha \rightarrow M_\alpha$ defined by $\pi_\alpha((m_\alpha)_\Gamma) = m_\alpha$, ($\alpha \in \Gamma$)

we have that $(\prod_\Gamma M_\alpha, (\pi_\alpha)_\Gamma)$ is a direct product of $(M_\alpha)_{\alpha \in \Gamma}$ p.79 [3].

Moreover it is clearly known that any weak direct product of R -modules is a submodule of a direct product of R -modules.

The following definition is widely known.

DEFINITION 1.2. *A left R -module E is injective in case E is injective relative to every left R -module M . That is, E is injective in case whenever there is given the solid part of a diagram:*

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow{f} & M \\ & & \downarrow g & \nearrow h & \\ & & E & & \end{array}$$

in the category ${}_R\mathcal{M}$ of left R -modules with row exact, there is an

R -homomorphism h such that the whole diagram commutative; $hf = g$. In other words, the injective modules are the arrow-theoretic or categorical duals of the projective modules.

A left module ${}_R M$ is said to be *quasi-injective* in case it is M -injective (injective relative to M).

It is well-known that every injective module is quasi-injective.

Recall that 16.11 Corollary (p.186, [3]): for any indexed set $(E_\alpha)_{\alpha \in \Gamma}$ of left R -modules, we have $\prod_{\Gamma} E_\alpha$ is injective if and only if each E_α is injective.

The following proposition is 18.8. Corollary on p.206, [3].

PROPOSITION 1.3. *A ring R is semi-simple if and only if every left R -module is injective.*

Of course, another fact about semi-simple ring is found in [6]. If R is semi-simple, then every module is semi-simple. However the converse does not hold, in general. For instance, $\mathbb{Z}_p \oplus \mathbb{Z}_p$ (modulo p) is a semi-simple $\mathbb{Z}$ -module, but the integer ring $\mathbb{Z}$ is not semi-simple.

DEFINITION 1.4. *For a commutative ring R with identity, a left (right, resp.) R -module ${}_R M$ (M_R , resp.) is said to be a multiplication module if each submodule $K \leq M$, there is an ideal I of R such that $K = IM$ (MI , resp.).*

The following properties of multiplication modules are found from [10].

PROPOSITION 1.5. *For a commutative ring R with identity, R is semi-simple if and only if every multiplication module is semi-simple.*

LEMMA 1.6. *For any commutative ring R with identity and for any multiplication left (right, resp.) R -module ${}_R M$ (M_R resp.), if I is any minimal ideal of R , then we have IM (MI , resp.) is a minimal submodule of M or $IM = 0$ ($MI = 0$, resp.).*

PROPOSITION 1.7. *For any commutative ring R with identity, the following are equivalent:*

- (1) R is semi-simple;
- (2) R is a direct sum of simple rings;
- (3) Every multiplication module ${}_R M$ is semi-simple;
- (4) Every multiplication module ${}_R M$ is a direct sum of simple modules.

Recall that an associative ring R with identity is said to be *regular* if R is von-Neumann regular, that is, for any $a \in R$, there is $b \in R$ such that $a = aba$.

2. Results

To study the regular endomorphism rings of modules, we need to develop the properties of closed submodules and minimal closed submodules of left (right) R -modules.

PROPOSITION 2.1. *For any left (right) R -module M , we have the following:*

- (1) For submodule $K \leq M$, we have $K \leq \overline{K}$.
- (2) For submodules $K, L \leq M$, we have

$$\overline{K \cap L} \leq \overline{K} \cap \overline{L}$$

and

$$\overline{K + L} = \overline{\overline{K} + \overline{L}}.$$

- (3) If $N \leq M$ is minimal, then $\overline{N}$ is a minimal closed submodule of M .
- (4) The converse of the item (3) does not hold, in general.
- (5) For each pair of minimal submodules $N_1, N_2 \leq M$ we have

$$N_1 \cap N_2 = \begin{cases} 0, \\ N_i \ (i = 1, 2). \end{cases}$$

(6) For each pair of minimal closed submodules $N_1, N_2 \leq M$ we have

$$\overline{N_1 \cap N_2} = \begin{cases} 0, \\ \overline{N_i} = N_i \ (i = 1, 2). \end{cases}$$

Proof. (1): It is elementary.

(2): $\overline{K \cap L} \leq \overline{K} \cap \overline{L}$ is obvious.

Comparing left ideals $I_{K+L} = I_K \cap I_L$ with submodules K and L , we can see that their kernels are the same. So, $\ker I_{K+L} = \overline{K+L} = \ker(I_K \cap I_L) = \overline{K} + \overline{L}$.

(3): If K is any closed submodule of M such that $0 \leq K = \overline{K} \leq \overline{N}$. Then by the computation of sets; $\overline{N} \cap \overline{K} = \overline{N} \cap K = K \geq \overline{N \cap K}$. If $N \cap K = 0$, then since N is minimal we have $K = 0$. And if $N \cap K \neq 0$, then since N is minimal, we have $N \leq K$. Thus $\overline{N} \leq \overline{K} = K$. Then $\overline{N} = \overline{K} = K$. Hence $\overline{N}$ is a minimal closed submodule of M , for any minimal submodule N of M .

(4): Take the ring $\mathbb{Z}$ of all integers and the left $\mathbb{Z}$ -module ${}_Z\mathbb{Q}$ regarded as an additive group $\mathbb{Q}$, by the Generalized Schur's Lemma II with an injective closely simple module $\mathbb{Q}$ we have a division ring $End_Z(\mathbb{Q})$ (with keeping in mind, $End_{\mathbb{Q}}(\mathbb{Q})$ is a division ring by the Schur's Lemma), thence $\overline{k\mathbb{Z}} = \mathbb{Q}$ is a minimal closed submodule of $\mathbb{Q}$, however $k\mathbb{Z}$ (for any natural number $k \in \mathbb{N}$) is not minimal since $k\mathbb{Z} \leq \mathbb{Q}$, of course, $\mathbb{Z}$ is not Artinian [5].

(5) and (6) are proved by applying (3). □

From the property of (6) of Proposition 2.1 we can make a new definition of a module.

DEFINITION 2.2. A module M is said to be *closedly semi-simple* if it is a sum of closedly simple submodules of M .

NOTE 2.3. For any semi-simple module M ,

$$M = Soc(M) = \sum N_\alpha = \oplus N_\alpha \text{ for all minimal submodules } N_\alpha \text{ of } M.$$

And also for any closedly semi-simple module M ,

$$\begin{aligned} M &= \sum \overline{N_\alpha} = \oplus \overline{N_\alpha} \\ &= \left\{ \sum_{finite} n_\alpha \mid n_\alpha \in \overline{N_\alpha}, n_\beta = 0, \text{ for all but a finite number of indices } \beta \in \Gamma \right\} \\ &= \left\{ \oplus n_\alpha \mid n_\alpha \in \overline{N_\alpha}, n_\beta = 0, \text{ for all but a finite number of indices } \beta \in \Gamma \right\}, \end{aligned}$$

for all minimal closed submodules $\overline{N_\alpha}$ of M (See (5) and (6) of Proposition 2.1).

REMARK 2.4. *Every semi-simple module is always closedly semi-simple but the converse is not true, in general. The Abelian groups ${}_Z \oplus_1^\infty \mathbb{Q}$ (or $\oplus^n \mathbb{Q}$ ($n \in \mathbb{N}$)) is not semi-simple but closedly semi-simple $\mathbb{Z}$ -module.*

Usually we will ignore the trivial case of $M = 0 \oplus M = M + M$.

LEMMA 2.5. *For any left (right) R -module M , the following are equivalent :*

- (1) M is closedly simple;
- (2) Any non-zero endomorphism is a monomorphism.

Proof. For each $f \in \text{End}(M)$, we have a closed submodule $\ker f$ of M , thus $\ker f = 0$ if and only if f is a monomorphism. Hence the proof is completed. $\square$

PROPOSITION 2.6. *For any (quasi-)injective left (right) R -module M , if M is closedly simple, then $f : M \rightarrow M$ is left invertible, for any non-zero endomorphism f .*

Proof. From Lemma 1.4 because of the (quasi-)injectivity of M the following commutative diagram aids to complete the proof:

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{f \neq 0} & M \\ & & \downarrow 1_M & \swarrow \exists h & \\ & & M, & & \end{array}$$

where $1_M : M \rightarrow M$ is the identity mapping on M . Thus for any non-zero endomorphism $f : M \rightarrow M$ is a monomorphism, since M is injective there exists an endomorphism $h : M \rightarrow M$ such that $1_M = hf$, where $1_M : M \rightarrow M$ is the identity mapping on M . Thus f is left invertible. $\square$

LEMMA 2.7. *For any closedly simple module M , if $M = \sum_{\alpha \in \Gamma} N_\alpha$, for all closedly simple submodules N_α ($\alpha \in \Gamma$) of M , then $\oplus_\Gamma N_\alpha$ and $\prod_\Gamma^w N_\alpha$ are isomorphic, simply denoted by $M = \oplus_\Gamma N_\alpha \simeq \prod_\Gamma^w N_\alpha \leq (\prod_\Gamma N_\alpha)$.*

Proof. Since each closedly simple submodule N_α ($\alpha \in \Gamma$) is a minimal closed submodule of M , by (6) of Proposition 2.1, we have that $\sum_\Gamma N_\alpha = \oplus_\Gamma N_\alpha$. Define $\phi : \oplus_\Gamma N_\alpha \rightarrow \prod_\Gamma^w N_\alpha$ by $\oplus_\Gamma n_\alpha \mapsto (n_\alpha)_{\alpha \in \Gamma}$. Then it is easily proved ϕ is a well-defined, R -homomorphism, monomorphism, and epimorphism. Therefore ϕ is an R -isomorphism. $\square$

LEMMA 2.8. *If a left (right) R -module M is closedly semi-simple, then $\ker f$ is a direct summand of M , for any non-zero non-monomorphism f in $\text{End}(M)$.*

Proof. Let $f \in \text{End}_R(M)$ be not a monomorphism. Then clearly $\ker f \neq 0$. $\ker f \leq M$ by $f \neq 0$ and $\ker f \cap M = \ker f \cap (\sum N_\alpha)$, where $M = \sum N_\alpha$, for all minimal closed submodules N_α ($\alpha \in \Gamma$), for some index set Γ . Now then we partition the index set Γ into two disjoint sets A and B such that

$$A = \{\alpha \in \Gamma \mid \ker f \cap N_\alpha \neq 0\} \text{ and } B = \{\alpha \in \Gamma \mid \ker f \cap N_\alpha = 0\}.$$

Notice here $\ker f \cap N_\alpha (\neq 0)$ is a closed submodule contained in N_α , we have $N_\alpha \leq \ker f$, for each $\alpha \in A$.

From the following computations

$$M = \sum_{\Gamma} N_\alpha = \sum_A N_\alpha \oplus \sum_B N_\beta \leq \ker f \oplus \sum_B N_\beta$$

and $N_\alpha \leq \ker f$ for any $\alpha \in A$ by the minimal closed submodule N_α , it follows that $\sum_A N_\alpha = \ker f$, being a direct summand of M . $\square$

Applying 20.2 Proposition in [3] as functors direct sum $\oplus$ and direct product $\prod$ from the category ${}_R\mathcal{M}$ of left R -modules to the category ${}_Z\mathcal{M}$ of left Z -modules:

- (1) $\text{Hom}_R(\oplus_{\Gamma} U_\alpha, -) \simeq \prod_{\Gamma} \text{Hom}_R(U_\alpha, -)$;
- (2) $\text{Hom}_R(-, \prod_{\Gamma} U_\alpha) \simeq \prod_{\Gamma} \text{Hom}_R(-, U_\alpha)$.

We recall the following properties of direct product and weak direct product modules and the Hom functor are followed practically;

- (1) If $B \simeq C$ are isomorphic R -modules, then $\text{Hom}(B, A) \simeq \text{Hom}(C, A)$ and $\text{Hom}(A, B) \simeq \text{Hom}(A, C)$, for every R -module A .
- (2) For any indexed set $(M_\alpha)_{\alpha \in \Gamma}$ of left (right) R -modules M_α ($\alpha \in \Gamma$), then

$$\text{Hom}(\prod_{\Gamma}^w M_\alpha, \prod_{\Gamma}^w M_\alpha) \simeq \text{Hom}(\prod_{\Gamma} M_\alpha, \prod_{\Gamma} M_\alpha).$$

(Consider the mapping

$$\text{Hom}(\prod_{\Gamma} M_\alpha, \prod_{\Gamma} M_\alpha) \rightarrow \text{Hom}(\prod_{\Gamma}^w M_\alpha, \prod_{\Gamma}^w M_\alpha) : f \mapsto f|_{\prod_{\Gamma}^w M_\alpha},$$

where $f|_{\prod_{\Gamma}^w M_{\alpha}}$ is the restriction of f to the submodule $\prod_{\Gamma}^w M_{\alpha}$ of $\prod_{\Gamma} M_{\alpha}$, for each $f \in \text{Hom}(\prod_{\Gamma} M_{\alpha}, \prod_{\Gamma} M_{\alpha})$, which is required.)

- (3) For any closedly semi-simple module $M = \sum_{\Gamma} N_{\alpha}$, for all minimal closed submodules N_{α} of M , it is concluded immediately:

$$\text{Hom}(N_{\alpha}, \prod_{\Gamma}^w N_{\alpha}) \simeq \text{Hom}(N_{\alpha}, \prod_{\Gamma} N_{\alpha}) \simeq \prod_{\beta \in \Gamma} \text{Hom}(N_{\alpha}, N_{\beta}).$$

$$\begin{aligned} \text{End}(M) &= \text{End}(\oplus_{\Gamma} N_{\alpha}) = \text{Hom}(\oplus_{\Gamma} N_{\alpha}, \oplus_{\Gamma} N_{\beta}) \simeq \text{Hom}(\oplus_{\Gamma} N_{\alpha}, \prod_{\Gamma}^w N_{\beta}) \\ &\simeq \prod_{\Gamma} \text{Hom}(N_{\alpha}, \prod_{\Gamma}^w N_{\beta}) = \prod_{\Gamma} \text{Hom}(N_{\alpha}, \prod_{\Gamma} N_{\beta}) \\ &\simeq \prod_{\Gamma \times \Gamma} \text{Hom}(N_{\alpha}, N_{\beta}). \end{aligned}$$

THEOREM 2.9. *For any (quasi-)injective module M , if M is closedly semi-simple, then $\text{End}_R(M)$ is regular.*

Proof. Since on every (quasi-)injective module M every monomorphism f is left invertible. So f has a left inverse g in $\text{End}(M)$, and thus $f = fgf$, for some $g \in \text{End}(M)$. So the regularity of any monomorphism on M is obtained immediately.

The regularity of any non-zero non-monomorphism $f : M = \oplus N_{\alpha} \simeq \prod^w N_{\alpha} \rightarrow M = \oplus N_{\alpha}$ is only to be proved. By Lemma 2.8 the kernel $\ker f$ is a direct summand of M .

Considering the following commutative diagram with chasing element assign:

$$\begin{array}{ccccc} & & N_{\alpha} & & \\ & \swarrow j_{\alpha} & & & \\ M = \oplus N_{\alpha} & \longrightarrow & \prod^w N_{\alpha} & \longrightarrow & \prod N_{\alpha} \\ f \downarrow & & \phi f \phi^{-1} \downarrow & & \phi f \phi^{-1} \downarrow \\ M = \oplus N_{\alpha} & \xrightarrow{\phi} & \prod^w N_{\alpha} & \longrightarrow & \prod N_{\alpha} \\ & & & & \downarrow \pi_{\beta} \\ & & & & N_{\beta} \end{array}$$

$$\begin{array}{ccccc}
& & m_\alpha & & \\
& \swarrow j_\alpha & & & \\
\oplus m_\alpha \ (m_\beta = 0, \text{ if } \beta \neq \alpha) & \xrightarrow{\phi} & (\phi(\oplus m_\alpha))_\alpha & \longrightarrow & (\phi(\oplus m_\alpha))_\alpha \\
f \downarrow & & \phi f \phi^{-1} \downarrow & & \phi f \phi^{-1} \downarrow \\
f(\oplus m_\alpha) & \xrightarrow{\phi} & (\phi f(\oplus m_\alpha))_\alpha & \longrightarrow & (\phi f(\oplus m_\alpha))_\alpha \\
& & & & \downarrow \pi_\beta \\
& & & & \pi_\beta \phi f j_\alpha(m_\alpha) = f_{\alpha\beta}(m_\alpha),
\end{array}$$

where $j_\alpha : N_\alpha \rightarrow \oplus_\Gamma N_\alpha$ is the injection, $\pi_\beta : \prod_\Gamma N_\alpha \rightarrow N_\beta$ is the β -th projection, and $\phi : \oplus_\Gamma N_\alpha \rightarrow \prod_\Gamma^w N_\alpha$ is the R -isomorphism. Let $\pi_\beta \phi f j_\alpha = f_{\alpha\beta}$. Comparing the kernel of f with the kernel of $f_{\alpha\beta}$, we obtain that $\ker f = \oplus_{\alpha \in \Gamma} \ker f_{\alpha\beta}$ by [3]. Since $f \neq 0$ if and only if there are $\alpha, \beta \in \Gamma$ such that $f_{\alpha\beta} \neq 0$. Thus $\ker f_{\alpha\beta} = 0$, for some $(\alpha, \beta) \in \Gamma \times \Gamma$ (otherwise, it induces that $f = 0$ from $\ker f \geq N_\alpha$, for all α).

On the other hand, define a function $g_{\beta\alpha} = f_{\alpha\beta}^- : \text{Im } f_{\alpha\beta} \rightarrow M$ (pull-back of $f_{\alpha\beta}$) defined by the reversing assignment, i.e., $f_{\alpha\beta}(n_\alpha) \mapsto n_\alpha$ is an R -monomorphism from the R -submodule $\text{Im } f_{\alpha\beta} (\leq N_\beta)$ onto the preimage N_α (which is possible since $f_{\alpha\beta} : N_\alpha \rightarrow N_\beta$ is a monomorphism).

Precisely speaking, let $A' = \ker f$, $B' = \sum_{\Gamma \setminus \{\alpha\}} N_\gamma$, $C = \sum_{\Gamma \setminus \{\alpha, \beta\}} N_\delta$, and $D = \sum_{\Gamma \setminus \{\beta\}} N_\delta$. Then we have the following:

$$\begin{array}{c}
1_{A'} \oplus f_{\alpha\beta} \oplus g_{\beta\alpha} \oplus 1_C : A' \oplus N_\alpha \oplus \text{Im } f_{\alpha\beta} \oplus C \xrightarrow{\leq} A' \oplus N_\beta \oplus N_\alpha \oplus C \\
\parallel \\
A' \oplus N_\beta \oplus D \\
\parallel \\
A' \oplus B' \\
\parallel \\
\oplus_\Gamma N_\alpha = M
\end{array}$$

is the injective inclusion, since M is (quasi-)injective, there is an extension $h : M \rightarrow M$ such that the restriction $h|_{A' \oplus N_\alpha \oplus \text{Im } f_{\alpha\beta} \oplus D} = 1_{A'} \oplus g_{\beta\alpha} \oplus 1_D$ of h to the submodule $A' \oplus \text{Im } f_{\alpha\beta} \oplus N_\beta \oplus D$.

Take $h = (h_{\gamma\delta}) : M = \oplus N_\alpha \simeq \prod^w N_\alpha \rightarrow M = \oplus N_\alpha \simeq \prod^w N_\alpha$. Then $h_{\beta\alpha}f_{\alpha\beta} = 1_{N_\alpha}$, for every $(\alpha, \beta) \in \Gamma \times \Gamma$ follows from the composition $hf = 1_{B'}$ of h and f .

Consider that a direct sum $f : M = A' \oplus B' \rightarrow M = A' \oplus B'$ has a left inverse $h \in \text{End}(M)$. Then $f = fhf$ follows.

Therefore the endomorphism ring $\text{End}(M)$ is regular. The proof is completed. $\square$

COROLLARY 2.10. *For any (quasi-)injective R -module M , if M is semi-simple, then $\text{End}_R(M)$ is regular.*

Proof. Since every semi-simple module M is closedly semi-simple the proof is an easy consequence of the above Theorem 2.9. $\square$

Recall that a ring R is semi-simple if and only if every R -module is injective ([3]).

COROLLARY 2.11. *Let R be a semi-simple ring. Then $\text{End}(M)$ is regular, for every left (right) R -module M .*

Proof. By Corollary 2.6 p.274 [6], every R -module is semi-simple and thus every R -module is closedly semi-simple. On the other hand, R -modules is injective by Proposition 1.3. The hypotheses of Theorem 2.9 are satisfied. It follows that $\text{End}(M)$ is regular. $\square$

COROLLARY 2.12. *If a commutative ring R with identity is semi-simple, then $\text{End}_R(M)$ is regular, for any multiplication module M over R .*

Proof. Applying 18.8. Corollary on p. 206 in [3] to a commutative ring R , every left (right) R -module is injective. And by Proposition 1.7 every multiplication module M is semi-simple. Thus the hypotheses of Theorem 2.9 are satisfied. Hence $\text{End}(M)$ is regular, for every multiplication module M . $\square$

PROPOSITION 2.13. *If a module M is closedly semi-simple, then the following are equivalent:*

- (1) $\text{End}(M)$ is regular;
- (2) $Im f$ is a direct summand of M , for any non-zero $f \in \text{End}(M)$.

Proof. (1) $\implies$ (2): Automatically it is proved by Corollary 3.2, [2].

(2) $\implies$ (1): By Corollary 3.2 [2], the fact of a direct summand $\ker f$ of M follows from the closedly semi-simple module M , for every endomorphism $f : M \rightarrow M$ by Lemma 2.8. Together with the hypothesis of a direct summand $\operatorname{Im} f$ of M , immediately it follows that $\operatorname{End}(M)$ is regular. $\square$

COROLLARY 2.14. *If a module M is semi-simple, then the following are equivalent:*

- (1) $\operatorname{End}(M)$ is regular;
- (2) $\operatorname{Im} f$ of f is a direct summand of M , for any non-zero $f \in \operatorname{End}(M)$.

Proof. Since each semi-simple module is a closedly semi-simple module the proof is completed by the Proposition 2.13. $\square$

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