APPROXIMATE GENERALIZED EXPONENTIAL FUNCTIONS

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ABSTRACT. In this paper we prove the superstability of a generalized exponential functional equation

$$f(x+y) = a^{2xy-1}g(x)f(y).$$

It is a generalization of the superstability theorem for the exponential functional equation proved by Baker. Also we investigate the stability of this functional equation in the following form :

$$\frac{1}{1+\delta} \leq \frac{f(x+y)}{a^{2xy-1}g(x)f(y)} \leq 1+\delta.$$

1. Introduction

In 1940, S. M. Ulam gave a wide ranging talk in the Mathematical Club of the University of Wisconsin in which he discussed a number of important unsolved problems (ref. [22]). One of those was the question concerning the stability of homomorphisms :

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In the next year, Hyers [5] answered the Ulam's question for the case of the additive mapping on the Banach spaces G_1, G_2 . Thereafter, the result of

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Hyers has been generalized by Rassias [15]. Since then, the stability problems of various functional equations have been investigated by many authors (see [1,3, 4, 6-13, 16-21]).

In particular, Baker et al. in [4] introduced the stability of the exponential functional equation in the following form : if f satisfies the inequality $|f(x+y) - f(x)f(y)| \leq \varepsilon$, then either f is bounded or $f(x+y) = f(x)f(y)$. This is frequently referred to as *Superstability*.

In this paper, we will investigate the solution and the superstability of the exponential type functional equation

$$(1) \quad f(x+y) = a^{2xy-1}g(x)f(y),$$

which is a generalization of the superstability of the exponential functional equation given by Baker et al.[2]. Also we investigate the stability of this functional equation in the following form :

$$\frac{1}{1+\delta} \leq \frac{f(x+y)}{a^{2xy-1}g(x)f(y)} \leq 1+\delta.$$

Note that $f(x) = a^{x^2+1}$ and $f(x) = a^{x^2+x+1}$ are solutions of the equation (1).

2. Superstability of the generalized exponential functional equation

Baker et al.[2] proved the superstability of Cauchy's exponential equation

$$f(x+y) = f(x)f(y).$$

That is, if the Cauchy difference $f(x+y) - f(x)f(y)$ of a real-valued function f defined on a real vector space is bounded for all x, y , then f is either bounded or exponential. Their result was generalized by Baker [1] : let S be a semi-group and let f be a complex-valued function defined on S such that

$$|f(xy) - f(x)f(y)| < \delta$$

for all $x, y \in S$, then f is either bounded or multiplicative. The following our result is a generalization of Baker's theorem.

Theorem 1. *Let $\delta > 0$ and $a \geq 1$ be given. Let $f, g : (0, \infty) \rightarrow (0, \infty)$ be functions with $g(m) \geq \max\{2, 4\delta/f(m)\}$ for some integer $m \geq 1$ such that*

$$(2) \quad |f(x+y) - a^{2xy-1}g(x)f(y)| < \delta$$

for all $x, y \in (0, \infty)$. Then

$$g(x+y) = a^{2xy-1}g(x)g(y)$$

for all $x, y \in (0, \infty)$.

Proof. If we replace x and y by m in (2), simultaneously, we get

$$|f(2m) - a^{2m^2-1}g(m)f(m)| < \delta.$$

By induction, we get that for all $m \geq 2$

$$(3) \quad \begin{aligned} & |f(nm) - a^{2(m^2+2m^2\cdots+(n-1)m^2)-(n-1)}g(m)^{n-1}f(m)| \\ & \leq \delta + a^{2(n-1)m^2-1}g(m)\delta + a^{2((n-1)m^2+(n-2)m^2)-2}g(m)^2\delta \\ & + \cdots + a^{2((n-1)m^2+(n-2)m^2\cdots+2m^2)-(n-2)}g(m)^{n-2}\delta. \end{aligned}$$

In fact, if the inequality (3) holds, we have

$$\begin{aligned} & |f((n+1)m) - a^{2(m^2+2m^2\cdots+nm^2)-n}g(m)^n f(m)| \\ & \leq |f((n+1)m) - a^{2nm^2-1}g(m)f(nm)| \\ & + |f(nm) - a^{2(m^2+2m^2\cdots+(n-1)m^2)-(n-1)}g(m)^{n-1}f(m)|a^{2nm^2-1}g(m) \\ & \leq \delta + a^{2nm^2-1}g(m)\delta + a^{2nm^2}a^{(n-1)m^2-2}g(m)^2\delta \\ & + \cdots + a^{2nm^2}a^{(n-1)m^2}\cdots a^{2m^2-n}g(m)^{n-1}\delta \end{aligned}$$

for all $n \geq 2$. Note that $2(m^2 + 2m^2 + \cdots + (n-1)m^2) - (n-1) \geq 0$ for any $n \geq 1$. By (3), we get

$$\begin{aligned} & \left| \frac{f(nm)}{a^{2(m^2+2m^2\cdots+(n-1)m^2)-(n-1)}g(m)^{n-1}f(m)} - 1 \right| \\ & \leq \left(\frac{1}{g(m)^{n-1}f(m)} + \frac{1}{g(m)^{n-2}f(m)} + \cdots + \frac{1}{g(m)f(m)} \right) \delta \\ & < \frac{1}{g(m)f(m)} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \right) \delta = \frac{2\delta}{g(m)f(m)} \leq \frac{1}{2} \end{aligned}$$

for all positive integer n . Since $g(m)^n \rightarrow \infty$ as $n \rightarrow \infty$ and

$$a^{2(m^2+2m^2\cdots+(n-1)m^2)-(n-1)} \geq 1,$$

$$f(nm) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Then for any $x, y \in (0, \infty)$ we have

$$\begin{aligned} & f(nm)|g(x+y) - a^{2xy-1}g(x)g(y)| \\ & \leq |a^{2nm(x+y)-1}g(x+y)f(nm) - f(nm+x+y)| \frac{1}{a^{2nm(x+y)-1}} \\ & \quad + |f(nm+x+y) - a^{2x(y+nm)-1}g(x)f(y+nm)| \frac{1}{a^{2nm(x+y)-1}} \\ & \quad + |f(y+nm) - a^{2y nm-1}g(y)f(nm)|g(x) \cdot \frac{a^{2xy}}{a^{2y nm}} \end{aligned}$$

and so

$$\begin{aligned} & |g(x+y) - a^{2xy-1}g(x)g(y)| \\ & \leq \frac{2\delta}{f(nm)a^{2nm(x+y)-1}} + \frac{a^{2xy}\delta g(x)}{f(nm)a^{2y nm}} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Thus it follows that

$$g(x+y) = a^{2xy-1}g(x)g(y).$$

for any $x, y \in (0, \infty)$. □

Corollary 2. *Let $\delta > 0$ and $a \geq 1$ be given. Let $f : (0, \infty) \rightarrow (0, \infty)$ be a function with $f(m) \geq \max\{2, 2\sqrt{\delta}\}$ for some integer $m \geq 1$ such that*

$$|f(x+y) - a^{2xy-1}f(x)f(y)| < \delta$$

for all $x, y \in (0, \infty)$. Then

$$f(x+y) = a^{2xy-1}f(x)f(y)$$

for all $x, y \in (0, \infty)$.

Corollary 3. *Let $\delta > 0$ and $a \geq 1$ be given. Let $f : (0, \infty) \rightarrow (0, \infty)$ be a function such that*

$$|f(x+y) - a^{2xy-1}f(x)f(y)| < \delta$$

for all $x, y \in (0, \infty)$. Then either f is bounded or

$$f(x+y) = a^{2xy-1}f(x)f(y)$$

for all $x, y \in (0, \infty)$.

Suppose that $H_p : (0, \infty) \times (0, \infty) \rightarrow (0, \infty)$ be a monotonically increasing (in both variables) homogeneous mapping, for which $H_p(tx, ty) = t^p H_p(x, y)$ holds for some $p > 1$, and for all $t, x, y \in (0, \infty)$. For examples, let $H_p(x, y) = ax^p + by^p$ for $a, b, x, y \in (0, \infty)$. Then H_p is a monotonically increasing (in both variables) homogeneous mapping.

Theorem 4. Let $a \geq 1$ be given. If $f, g : (0, \infty) \rightarrow (0, \infty)$ satisfy the functional inequality

$$|f(x+y) - a^{2xy-1}g(x)f(y)| \leq H_p(x, y),$$

then either $g(x) = o(x^p)$ as $x \rightarrow \infty$ or $g(x+y) = a^{2xy-1}g(x)g(y)$ for every $x, y \in (0, \infty)$.

Proof. By the same method as the proof of Theorem 1, we have

$$\begin{aligned} & |f(nx) - a^{2(x^2+2x^2+\dots+(n-1)x^2)-(n-1)}g(x)^{n-1}f(x)| \\ & \leq H_p((n-1)x, x) + H_p((n-2)x, x)g(x)a^{2(n-1)x^2-1} \\ & + H_p((n-3)x, x)g(x)^2a^{2((n-1)x^2+(n-2)x^2)-2} \\ & + \dots + H_p(x, x)g(x)^{n-2}a^{2(2x^2+\dots+(n-2)x^2)-(n-2)} \end{aligned}$$

for all positive integer $n \geq 2$ and $x > 0$. Since $a \geq 1$, we get

$$\begin{aligned} & \left| \frac{f(nx)}{g(x)^{n-1}f(x)a^{2(x^2+2x^2+\dots+(n-1)x^2)-(n-1)}} - 1 \right| \leq \sum_{i=2}^{n-1} \frac{H_p(ix, x)}{g(x)^{i-1}f(x)} \\ & \leq \sum_{i=1}^{\infty} \frac{i^p H_p(x, x)}{g(x)^i f(x)} \leq \frac{H_p(x, x)}{g(x)f(x)} \sum_{i=1}^{\infty} \frac{i^p}{g(x)^{i-1}} \end{aligned}$$

for any positive integer $n \geq 2$ and $x > 0$. Assume that $g(x) \neq o(x^p)$ as $x \rightarrow \infty$, that is, there exist some $\alpha > 0$ and a sequence $\{x_k\}$ in $(0, \infty)$ such

that $x_k \rightarrow \infty$ as $k \rightarrow \infty$ and $g(x_k) \geq \alpha x_k^p > 1$ for sufficiently large k . Then $\frac{i^p}{g(x_k)^{i-1}} \leq \frac{i^p}{(\alpha x_k^p)^{i-1}}$ and $\sum_{i=1}^{\infty} \frac{i^p}{(\alpha x_k^p)^{i-1}}$ converges. We can then let the series $\sum_{i=1}^{\infty} \frac{i^p}{g(x_k)^{i-1}}$ converge to a value less than $\frac{g(x)f(x)}{2H_p(x,x)}$ by taking sufficiently large k . Thus for some sufficiently large k and any $n \geq 2$, we have

$$\left| \frac{f(nx_k)}{g(x_k)^{n-1} f(x) a^{-(n-1)} \prod_{i=1}^{n-1} a^{ix_k^2}} - 1 \right| < \frac{1}{2}.$$

By the convergence of $\sum_{i=1}^{\infty} \frac{i^p}{g(x_k)^{i-1}}$, we have $\frac{n^p H_p(x_k, x_k)}{g(x_k)^{n-1}} \rightarrow 0$ as $n \rightarrow \infty$. Note that $a^{-(n-1)} \prod_{i=1}^{n-1} a^{ix_k^2} \geq 1$, we have

$$\left| \frac{\frac{f(nx_k)}{n^p H_p(x_k, x_k)}}{\frac{g(x_k)^{n-1} f(x) a^{-(n-1)} \prod_{i=1}^{n-1} a^{ix_k^2}}{n^p H_p(x_k, x_k)}} - 1 \right| < \frac{1}{2}$$

and so $\frac{n^p H_p(x_k, x_k)}{f(nx_k)} \rightarrow 0$ as $n \rightarrow \infty$.

Now let $x, y > 0$ be given. If k is sufficiently large, we have

$$\begin{aligned} & f(nx_k) |g(x+y) - a^{2xy-1} g(x)g(y)| \\ & \leq |a^{2nx_k(x+y)-1} f(nx_k)g(x+y) - f(nx_k+x+y)| \frac{1}{a^{2nx_k(x+y)-1}} \\ & + |f(nx_k+x+y) - a^{2x(y+nx_k)-1} g(x)f(y+nx_k)| \frac{1}{a^{2nx_k(x+y)-1}} \\ & + |f(y+nx_k) - a^{2ynx_k-1} g(y)f(nx_k)| \frac{g(x)a^{2xy-1}}{a^{2ynx_k-1}} \\ & \leq C_1 H_p(x+y, nx_k) + C_2 H_p(x, y+nx_k) + C_3 H_p(y, nx_k) \\ & \leq (C_1 + C_2 + C_3) H_p(nx_k, nx_k) = (C_1 + C_2 + C_3) n^p H_p(x_k, x_k) \end{aligned}$$

for some $C_1, C_2, C_3 > 0$ and sufficiently large n . Thus

$$|g(x+y) - a^{2xy-1} g(x)g(y)| \leq \frac{(C_1 + C_2 + C_3) n^p H_p(x_k, x_k)}{f(nx_k)} \rightarrow 0$$

as $n \rightarrow \infty$. □

Corollary 5. *Let $a \geq 1$ be given. If $f : (0, \infty) \rightarrow (0, \infty)$ satisfies the functional inequality*

$$|f(x+y) - a^{2xy-1} f(x)f(y)| \leq H_p(x, y),$$

then either $f(x) = o(x^p)$ as $x \rightarrow \infty$ or $f(x+y) = a^{2xy-1} f(x)f(y)$ for every $x, y \in (0, \infty)$.

3. Stability of the equation (1)

R. Ger [4] introduced a stability for the exponential equation in the following type :

$$\left| \frac{f(x+y)}{f(x)f(y)} - 1 \right| \leq \delta.$$

Now we prove the stability of the equation (1) in a similar type of Ger.

Theorem 6. *Let $0 < \delta < 1$ and $a \geq 1$ be given. If functions $f, g : (0, \infty) \rightarrow (0, \infty)$ satisfies the inequality*

$$(4) \quad \frac{1}{1+\delta} \leq \frac{f(x+y)}{a^{2xy-1}g(x)f(y)} \leq 1+\delta$$

for all $x, y \in (0, \infty)$ and $e^{-M} \leq \frac{f(x)}{g(x)} \leq e^M$ for some M , then there exists a function $F : (0, \infty) \rightarrow (0, \infty)$ such that

$$F(x+y) = a^{2xy-1}F(x)F(y)$$

for all $x, y \in (0, \infty)$,

$$\frac{1}{1+\delta} \leq \frac{F(x)}{f(x)} \leq (1+\delta)e^M,$$

and

$$\frac{1}{(1+\delta)e^M} \leq \frac{F(x)}{g(x)} \leq (1+\delta)e^{2M}$$

for all $x \in (0, \infty)$.

Proof. If we define functions $G, H : (0, \infty) \rightarrow (0, \infty)$ by

$$G(x) = \ln g(x), H(x) = \ln f(x)$$

for all $x > 0$, then the equality (4) may be transformed into

$$\ln \frac{1}{1+\delta} \leq H(x+y) - \ln a^{2xy-1} - G(x) - H(y) \leq \ln(1+\delta) := \delta_1$$

and so

$$|H(x+y) - \ln a^{2xy-1} - G(x) - H(y)| \leq \delta_1$$

for all $x, y > 0$. By assumption, we have

$$|H(x+y) - \ln a^{2xy-1} - H(x) - H(y)| \leq \delta_1 + M := \theta$$

for all $x, y > 0$. Replacing y by x and dividing by 2, we get

$$(5) \quad \left| \frac{H(2x)}{2} - \ln a^{\frac{2x^2-1}{2}} - H(x) \right| \leq \frac{\theta}{2}$$

for all $x > 0$. We use induction on n to prove

$$(6) \quad \left| \frac{H(2^n x)}{2^n} - \ln a^{\frac{2x^2-1}{2} + \frac{2^3 x^2-1}{2^2} + \dots + \frac{2^{(2n-1)} x^2-1}{2^{n-1}}} - H(x) \right| \leq \theta \sum_{i=1}^n \frac{1}{2^i}$$

for all $x > 0$. On account of (5), the inequality holds for $n = 1$. Suppose that inequality (6) holds true for some $n > 1$. Then (5) and (6) imply

$$\begin{aligned} & \left| \frac{H(2^{n+1}x)}{2^{n+1}} - \ln a^{\frac{2x^2-1}{2} + \frac{2^3 x^2-1}{2^2} + \dots + \frac{2^{(2n+1)} x^2-1}{2^{n+1}}} - H(x) \right| \\ & \leq \left| \frac{H(2^n 2x)}{2^{n+1}} - \frac{1}{2} \ln a^{\frac{2(2x)^2-1}{2} + \frac{2^3 (2x)^2-1}{2^2} + \dots + \frac{2^{(2n-1)} (2x)^2-1}{2^{n-1}}} - \frac{H(2x)}{2} \right| \\ & \quad + \left| \frac{H(2x)}{2} - \ln a^{\frac{2x^2-1}{2}} - H(x) \right| \\ & \leq \theta \sum_{i=1}^{n+1} \frac{1}{2^i}, \end{aligned}$$

which ends the proof of (6). For any $x > 0$ and for every positive integer n we define

$$P_n(x) = \frac{H(2^n x)}{2^n} - \ln \prod_{i=1}^{n-1} a^{\frac{2^{(2^i-1)} x^2-1}{2^i}}.$$

Let $m, n > 0$ be integers with $n > m$. Then it follows from (6)

$$\begin{aligned} & |P_n(x) - P_m(x)| \\ & = \frac{1}{2^m} \left| \frac{H(2^{n-m}(2^m x))}{2^{n-m}} - \ln \left(\prod_{i=1}^{n-m} a^{2^{2^i-1}(2^m x)^2-1} \right) - H(2^m x) \right| \\ & \leq \theta \sum_{i=m+1}^n \frac{1}{2^i} \rightarrow 0 \end{aligned}$$

as $m \rightarrow \infty$. Therefore, the sequence $\{P_n(x)\}$ is a Cauchy sequence, and we may define a function $L : (-\infty, \infty) \rightarrow (-\infty, \infty)$ by

$$L(x) := \lim_{n \rightarrow \infty} P_n(x)$$

and

$$F(x) := e^{L(x)} = \lim_{n \rightarrow \infty} \frac{f(2^n x)^{\frac{1}{2^n}}}{\prod_{i=1}^n a^{\frac{2^{(2i-1)}x^2-1}{2^i}}}$$

for all $x > 0$. Thus

$$\begin{aligned} & \frac{F(x+y)}{a^{2xy-1}F(x)F(y)} \\ &= \lim_{n \rightarrow \infty} \frac{f(2^n x + 2^n y)^{\frac{1}{2^n}} \prod_{i=1}^n a^{\frac{2^{(2i-1)}x^2-1}{2^i}} \prod_{i=1}^n a^{\frac{2^{(2i-1)}y^2-1}{2^i}}}{a^{2xy-1} f(2^n x)^{\frac{1}{2^n}} f(2^n y)^{\frac{1}{2^n}} \prod_{i=1}^n a^{\frac{2^{(2i-1)}(x+y)^2-1}{2^i}}} \\ &= \lim_{n \rightarrow \infty} \left[\frac{f(2^n x + 2^n y)}{f(2^n x)f(2^n y)} \right]^{\frac{1}{2^n}} \cdot \frac{a^{-(\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n})}}{a^{2xy-1} a^{2xy(1+2+\dots+2^{n-1})}} \\ &= \lim_{n \rightarrow \infty} \left[\frac{f(2^n x + 2^n y)}{a^{2(2^n x)(2^n y)-1} f(2^n x)f(2^n y)} \right]^{\frac{1}{2^n}} \end{aligned}$$

and

$$\begin{aligned} \left(\frac{1}{1+\delta} \right)^{\frac{1}{2^n}} &\leq \left[\frac{f(2^n x + 2^n y)}{a^{2(2^n x)(2^n y)-1} g(2^n x)f(2^n y)} \right]^{\frac{1}{2^n}} \\ &\leq \left[\frac{f(2^n x + 2^n y)}{a^{2(2^n x)(2^n y)-1} f(2^n x)f(2^n y)} \right]^{\frac{1}{2^n}} (e^M)^{\frac{1}{2^n}} \\ &\leq \left[\frac{f(2^n x + 2^n y)}{a^{2(2^n x)(2^n y)-1} g(2^n x)f(2^n y)} \right]^{\frac{1}{2^n}} (e^{2M})^{\frac{1}{2^n}} \\ &\leq (1+\delta)^{\frac{1}{2^n}} (e^{2M})^{\frac{1}{2^n}} \end{aligned}$$

for all $x, y > 0$ and for every positive integer n . Therefore we have

$$F(x+y) = a^{2xy-1}F(x)F(y)$$

for all $x, y > 0$. We can easily see from (4) that

$$\begin{aligned} \left(\frac{1}{1+\delta} \right)^{\frac{1}{2}} &\leq \frac{f(2x)^{\frac{1}{2}}}{a^{\frac{2x^2-1}{2}} g(x)^{\frac{1}{2}} f(x)^{\frac{1}{2}}} \leq \frac{f(2x)^{\frac{1}{2}}}{a^{\frac{2x^2-1}{2}} f(x)} \cdot e^{\frac{M}{2}} \\ &\leq \frac{f(2x)^{\frac{1}{2}}}{a^{\frac{2x^2-1}{2}} f(x)^{\frac{1}{2}} g(x)^{\frac{1}{2}}} \cdot e^{\frac{2M}{2}} \leq (1+\delta)^{\frac{1}{2}} \cdot e^{\frac{2M}{2}} \end{aligned}$$

for all $x > 0$. Note that for all $x > 0$ and for every positive integer n

$$\begin{aligned} & \frac{f(2^n x)^{\frac{1}{2^n}}}{\prod_{i=1}^{i=n} a^{\frac{2^{(2i-1)}x^2-1}{2^i}} f(x)} \\ &= \frac{f(2^n x)^{\frac{1}{2^n}}}{(a^{2(2^{n-1}x)^2-1})^{\frac{1}{2^n}} f(2^{n-1}x)^{\frac{1}{2^{n-1}}}} \cdot \frac{f(2^{n-1}x)^{\frac{1}{2^{n-1}}}}{(a^{2(2^{n-2}x)^2-1})^{\frac{1}{2^{n-1}}} f(2^{n-2}x)^{\frac{1}{2^{n-2}}}} \\ & \dots \frac{f(2x)^{\frac{1}{2}}}{(a^{2x^2-1})^{\frac{1}{2}} f(x)}. \end{aligned}$$

Thus we have

$$\begin{aligned} \left(\frac{1}{1+\delta}\right)^{\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^n}} &\leq \frac{f(2^n x)^{\frac{1}{2^n}}}{a^{\frac{x^2}{2}} a^{x^2} \dots a^{2^{n-2}x^2} f(x)} \\ &\leq ((1+\delta)e^M)^{\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^n}} \end{aligned}$$

and so

$$\frac{1}{1+\delta} \leq \frac{F(x)}{f(x)} \leq (1+\delta)e^M,$$

and thus

$$\frac{1}{(1+\delta)e^M} \leq \frac{F(x)}{g(x)} \leq (1+\delta)e^{2M}$$

for all $x > 0$. □

Corollary 7. *Let $0 < \delta < 1$ and $a \geq 1$ be given. If a function $f : (0, \infty) \rightarrow (0, \infty)$ satisfies the inequality*

$$\frac{1}{1+\delta} \leq \frac{f(x+y)}{a^{2xy-1}f(x)f(y)} \leq 1+\delta$$

for all $x, y \in (0, \infty)$, then there exists a function $F : (0, \infty) \rightarrow (0, \infty)$ such that

$$F(x+y) = a^{2xy-1}F(x)F(y)$$

for all $x, y \in (0, \infty)$ and

$$\frac{1}{1+\delta} \leq \frac{F(x)}{f(x)} \leq (1+\delta)$$

for all $x \in (0, \infty)$.

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