INTUITIONISTIC FUZZY QUICK IDEALS IN d -ALGEBRAS

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Abstract. Quick ideals and the fuzzification of quick ideals in d -algebras are considered, and some related properties are investigated. The intuitionistic fuzzification of quick ideals of a d -algebra is established, and related results are studied. The notion of equivalence relations on the family of all intuitionistic fuzzy quick ideals of a d -algebra is introduced, and then some properties are discussed.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK -algebras and BCI -algebras ([3,4]). It is known that the class of BCK -algebras is a proper subclass of the class of BCI -algebras. J. Neggers and H. S. Kim ([9]) introduced the notion of d -algebras which is another useful generalization of BCK -algebras, and investigated several relations between d -algebras and BCK -algebras. In the same paper they also investigated other relations between d -algebras and oriented digraphs. J. Neggers, Y. B. Jun and H. S. Kim ([10]) discussed ideal theory in d -algebras, and introduced the notions of d -subalgebra, d -ideal, $d^\#$ -ideal and d^* -ideal, and investigated some relations among them. Y. C. Lee and H. S. Kim ([7]) introduced the notion of d -transitive d^* -algebra which is another interesting generalization of BCK -algebras. Y. B. Jun, H. S. Kim and D. S. Yoo ([5]) established the notion of intuitionistic fuzzy d -algebra X and obtained the fundamental results needed to develop a further theory of these objects.

In this paper, we introduce the notion of quick ideals and the fuzzification of quick ideals in d -algebras, and investigate some related properties in d -algebras. We also discuss the product of fuzzy quick ideals and

Received July 13, 2009. Accepted September 9, 2009.

2000 *Mathematics Subject Classification.* 06F35, 03G25.

Key words and phrases. (fuzzy) quick ideals, (fuzzy) d -subalgebra, (fuzzy) d -ideals.

projections of fuzzy quick ideals in d -algebras. We establish the notion of intuitionistic fuzzy BCK -ideal/ d -ideal/quick ideal of a d -algebra and the notion of equivalence relations on the family of all intuitionistic fuzzy quick ideals of a d -algebra. We also discuss some related properties.

2. Preliminaries

A d -algebra ([9]) is a non-empty set X with a constant 0 and a binary operation “ $*$ ” satisfying axioms :

- (I) $x * x = 0$,
- (II) $0 * x = 0$,
- (III) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y \in X$.

A BCK -algebra is a d -algebra $(X; *, 0)$ satisfying additional axioms:

- (IV) $((x * y) * (x * z)) * (z * y) = 0$,
- (V) $(x * (x * y)) * y = 0$, for all $x, y, z \in X$.

For brevity we also call X a d -algebra. In X we can define a binary relation “ $\leq$ ” by $x \leq y$ if and only if $x * y = 0$. A mapping $\alpha : X \rightarrow Y$ of d -algebras is called a d -homomorphism if $\alpha(x * y) = \alpha(x) * \alpha(y)$ for all $x, y \in X$.

Definition 2.1. ([10]) Let X be a d -algebra and let $\emptyset \neq I \subseteq X$. I is called a d -subalgebra of X if $x * y \in I$ whenever $x \in I$ and $y \in I$. I is called a BCK -ideal of X if it satisfies:

- (D₀) $0 \in I$,
- (D₁) $x * y \in I$ and $y \in I$ imply $x \in I$.

I is called a d -ideal of X if it satisfies (D₁) and

- (D₂) $x \in I$ and $y \in X$ imply $x * y \in I$, i.e., $I * X \subseteq I$.

Definition 2.2. ([2]) An intuitionistic fuzzy set (IFS for short) D in a d -algebra X is an object having the form

$$D = \{ \langle x, \mu_D(x), \gamma_D(x) \rangle \mid x \in X \}$$

where the functions $\mu_D : X \rightarrow [0, 1]$ and $\gamma_D : X \rightarrow [0, 1]$ denote the degree of membership (namely $\mu_D(x)$) and the degree of nonmembership (namely $\gamma_D(x)$) of each element $x \in X$ to the set D , respectively, and where $0 \leq \mu_D(x) + \gamma_D(x) \leq 1$ for each $x \in X$.

For the sake of simplicity, we shall use the notation $D = \langle x, \mu_D, \gamma_D \rangle$ instead of $D = \{ \langle x, \mu_D(x), \gamma_D(x) \rangle \mid x \in X \}$. Let f be a mapping from a set X to a set Y . If

$$B = \{ \langle y, \mu_B(y), \gamma_B(y) \rangle \mid y \in Y \}$$

is an IFS in Y , then the *preimage* of B under f , denoted by $f^{-1}(B)$, is the IFS in X defined by

$$f^{-1}(B) = \{ \langle x, f^{-1}(\mu_B)(x), f^{-1}(\gamma_B)(x) \rangle \mid x \in X \},$$

where $f^{-1}(\mu_B)(x) = \mu_B(f(x))$ and $f^{-1}(\gamma_B)(x) = \gamma_B(f(x))$ for all $x \in X$.

3. Quick ideals in d -algebras

Definition 3.1. Let X be a d -algebra and let $0 \in I \subseteq X$. I is called a *quick ideal* of X if for any $x, y \in X$ with $x * y \neq 0$, $x * y \in I$ implies $x, y \in I$.

The notion of a quick ideal is different from the notion of an ideal in d -algebras.

Example 3.2. Let $X := \{0, a, b, c\}$ be a d -algebra ([6]) which is not a BCK -algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	c	0	b
c	c	c	c	0

Then $I := \{0, a\}$ is a quick ideal of X and a BCK -ideal of X .

Proposition 3.3. Let X be a d -algebra and let I be a quick ideal of X . Assume that $x * y = 0, y \in I$ implies $x \in I$. Then I is a BCK -ideal of X .

Proof. Assume that $x * y = 0, y \in I$ implies $x \in I$. Then I is a BCK -ideal of X since $x * y \neq 0, x * y \in I$, also implies $x \in I, y \in I$. $\square$

Example 3.4. (1) Let $X := \{0, a, b, c\}$ be a d -algebra ([10]) which is not a BCK -algebra with the Cayley table as follows:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	b	0	0
c	c	c	c	0

Then $A := \{0, a\}$ is a quick ideal of X , a BCK -ideal of X , but not a d -ideal of X , since $a * c = b \notin A$. Also A is a d -subalgebra of X . $B := \{0, b\}$ is a d -subalgebra of X , but not a quick ideal of X , since $a * c = b \in B$,

$a, c \notin B$. Also B is not a BCK -ideal of X , since $a * b = 0, b \in B$ and $a \notin B$.

(2) Let $X := \{0, a, b, c\}$ be a d -algebra ([9]) which is not a BCK -algebra with the Cayley table as follows:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	c
b	b	b	0	0
c	c	0	b	0

Then $C := \{0, a, c\}$ is a quick ideal of X , a d -subalgebra of X , but not a BCK -ideal of X since $b * c = 0, c \in C$ and $b \notin C$.

(3) Let $X := \{0, a, b, c\}$ be a d -algebra ([9]) which is not a BCK -algebra with the Cayley table as follows:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	b	0	0
c	c	0	b	0

Then $D := \{0, a, c\}$ is a quick ideal of X , but not a d -subalgebra of X since $a * c = b \notin D$. D is not a BCK -ideal of X , since $b * c = 0, c \in D$ and $b \notin D$.

(4) Let $X := \{0, a, b, c\}$ be a d -algebra which is not a BCK -algebra with the Cayley table as follows:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	b	0	b
c	c	c	c	0

Then $E := \{0, a, c\}$ is a BCK -ideal of X , not a quick ideal of X , since $c * b = c \in E$ and $b \notin E$. Also E is not a d -subalgebra of X since $a * c = b \notin E$.

4. Fuzzy quick ideals in d -algebras

Definition 4.1. ([6]) Let μ be a fuzzy set in a d -algebra X . Then μ is called a *fuzzy d -subalgebra* of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$, $\forall x, y \in X$. μ is called a *fuzzy BCK -ideal* of X if

$$(F_0) \quad \mu(0) \geq \mu(x),$$

(F_1) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$, for all $x, y \in X$.

μ is called a *fuzzy d -ideal* of X if it satisfies (F_1) and

(F_2) $\mu(x * y) \geq \mu(x)$ for all $x, y \in X$.

Definition 4.2. Let X be a d -algebra. A map $\mu : X \rightarrow [0, 1]$ is called a *fuzzy quick ideal* of X if it satisfies (F_0) and

(F_3) for any $x, y \in X$ with $x * y \neq 0$, $\min\{\mu(x), \mu(y)\} \geq \mu(x * y)$.

Theorem 4.3. Let μ be a fuzzy quick ideal of a d -algebra X . Assume μ is order reversing. Then μ is a fuzzy BCK -ideal of X .

Proof. Let $x, y \in X$. Since μ is order reversing, if $x * y = 0$, then $\mu(x) \geq \mu(y)$. Hence $\mu(x) \geq \mu(y) \geq \min\{\mu(x * y) = \mu(0), \mu(y)\}$. If $x * y \neq 0$, then we have

$$\mu(x) \geq \min\{\mu(x), \mu(y)\} \geq \mu(x * y) \geq \min\{\mu(x * y), \mu(y)\}.$$

Therefore μ is a fuzzy BCK -ideal of X . $\square$

The converse of Theorem 4.3 need not be true.

Example 4.4. (1) Let $X := \{0, a, b, c\}$ be a d -algebra as in Example 3.4-(2). Then $C := \{0, a, c\}$ is a quick ideal of X , a d -subalgebra of X , but not a BCK -ideal of X . Define $\mu : X \rightarrow [0, 1]$ be a fuzzy set defined by $\mu(0) = \mu(a) = \mu(c) = t_1$ and $\mu(b) = t_2$ for any $t_1, t_2 \in [0, 1]$ with $t_1 > t_2$. Then μ is a fuzzy quick ideal of X but not a fuzzy BCK -ideal of X since $\mu(b) = t_2 \leq \min\{\mu(b * c) = \mu(0), \mu(c)\} = t_1$. μ is not order reversing since $b * c = 0$, i.e., $b \leq c$ implies $\mu(b) = t_2 \leq \mu(c) = t_1$. Also μ is a fuzzy d -subalgebra of X .

(2) Let $X := \{0, a, b, c\}$ be a d -algebra as in Example 3.4-(1). Then $F := \{0, a, b\}$ is a BCK -ideal of X , not a quick ideal of X since $a * c = b, a \in F$ and $c \notin F$. Define $\rho : X \rightarrow [0, 1]$ be a fuzzy set defined by $\rho(0) = \rho(a) = \rho(b) = t_1$ and $\rho(c) = t_2$. Then ρ is a fuzzy BCK -ideal of X , order reversing, but not a fuzzy quick ideal of X , since $\min\{\rho(a), \rho(c)\} = \rho(c) = t_2 < \rho(a * c) = \rho(b) = t_1$ for some $a * c = c \neq 0$.

Definition 4.5. Let μ be a fuzzy set in a set X . For $t \in [0, 1]$, the set

$$\mu_t := \{x \in X \mid \mu(x) \geq t\}$$

is called a *level subset* of μ .

Theorem 4.6. Let μ be a fuzzy set of a d -algebra X . Then μ is a fuzzy quick ideal of X if and only if for any $t \in [0, 1]$ with $\mu_t \neq \emptyset$, μ_t is a quick ideal of X .

Proof. For any $t \in [0, 1]$ with $\mu_t \neq \emptyset$, let $x, y \in X$ with $x * y \neq 0$, $x * y \in \mu_t$. Then $\mu(x * y) \geq t$. Since μ is a fuzzy quick ideal of X , we have $\min\{\mu(x), \mu(y)\} \geq \mu(x * y) \geq t$. Hence $x, y \in \mu_t$. From (F_0) , we have $0 \in \mu_t$. Therefore μ_t is a quick ideal of X .

Conversely, assume that there exist $x, y \in X$ such that $x * y \neq 0$, $\mu(x * y) > \min\{\mu(x), \mu(y)\}$. Let $t_1 := \mu(x * y)$, $t_2 := \mu(x)$, $t_3 := \mu(y)$. Then $x * y \in \mu_{t_1}$, but at least one of x or y does not belong to μ_{t_1} . Since μ_{t_1} is a quick ideal of X , we have $x, y \in \mu_{t_1}$, which is a contradiction. $\square$

Theorem 4.7. Let I be a quick ideal of a d -algebra X and let μ be a fuzzy set in X defined by

$$\mu(x) = \begin{cases} t_0, & \text{if } x \in I \\ t_1, & \text{if } x \notin I \end{cases}$$

for all $x \in X$, where $t_0, t_1 \in [0, 1]$, $t_0 > t_1$. Then μ is a fuzzy quick ideal of X and $\mu_{t_0} = I$.

Proof. Let $x, y \in X$ with $x * y \neq 0$. If any one of x and y does not belong to I , then $x * y \notin I$. Hence $\min\{\mu(x), \mu(y)\} = t_1 \geq \mu(x * y) = t_1$. If $x, y \in I$, then $\mu(x) = \mu(y) = t_0$ and $\min\{\mu(x), \mu(y)\} = t_0 \geq \mu(x * y)$. If $x, y \notin I$, then $x * y \notin I$. Hence $\min\{\mu(x), \mu(y)\} = t_1 = \mu(x * y)$. Therefore μ is a quick ideal of X . It is clear that $\mu_{t_0} = I$. $\square$

Corollary 4.8. Any quick ideal of a d -algebra X can be realized as a level subset of some fuzzy quick ideal of X .

The following theorem shows that the concept of a fuzzy quick ideal of X is a generalization of a quick ideal. Using Corollary 4.8, the proof is straightforward.

Theorem 4.9. Let I be a non-empty subset of a d -algebra of X and let μ be a fuzzy set in X such that μ is into $\{0, 1\}$, so that μ is the characteristic function of I . Then μ is a fuzzy quick ideal of X if and only if I is a quick ideal of X .

Lemma 4.10. Let Λ be a totally ordered set and let $\{I_t | t \in \Lambda\}$ be a family of quick ideals of X such that for all $s, t \in \Lambda$, $s > t$ if and only if $I_s \subset I_t$. Then $\cap_{t \in \Lambda} I_t$ and $\cup_{t \in \Lambda} I_t$ are quick ideals of X .

Let Λ be a non-empty subset of $[0, 1]$.

Theorem 4.11. Let $\{I_t : t \in \Lambda\}$ be a collection of quick ideals of X such that $X = \cup_{t \in \Lambda} I_t$ for all $s, t \in \Lambda$, $s > t$ if and only if $I_s \subset I_t$. Then a map μ defined by $\mu(x) := \sup\{t | x \in I_t\}$ for all $x \in X$ is a fuzzy quick ideal of X .

Proof. Following Theorem 4.6, it is sufficient to show that μ_s is a quick ideal of X for every $s \in [0, 1]$. To do this, we divide into the following two cases:

- (i) $s = \sup\{t \in \Lambda \mid t < s\}$ and
- (ii) $s \neq \sup\{t \in \Lambda \mid t < s\}$.

Case (i) implies that $x \in \mu_s \Leftrightarrow x \in I_t$ for all $t < s \Leftrightarrow x \in \bigcap_{t < s} I_t$, so that $\mu_s = \bigcap_{t < s} I_t$, which is a quick ideal of X by Lemma 4.10. For the case (ii), we claim that $\mu_s = \bigcup_{t \geq s} I_t$. If $x \in \bigcup_{t \geq s} I_t$, then $x \in I_t$ for some $t \geq s$. It follows that $\mu(x) \geq t \geq s$, so that $x \in \mu_s$. This proves that $\bigcup_{t \geq s} I_t \subseteq \mu_s$.

Now assume that $x \notin \bigcup_{t \geq s} I_t$. Then $x \notin I_t$ for all $t \geq s$. Since $s \neq \sup\{t \in \Lambda \mid t < s\}$, there exists $\epsilon > 0$ such that $(s - \epsilon, s) \cap \Lambda = \emptyset$. Hence $x \notin I_t$ for all $t > s - \epsilon$, which means that if $x \in I_t$ then $t \leq s - \epsilon$. Thus $\mu(x) \leq s - \epsilon < s$, and so $x \notin \mu_s$. Therefore $\mu_s \subseteq \bigcup_{t \geq s} I_t$. Using Lemma 4.10, $\mu_s = \bigcup_{t \geq s} I_t$ is a quick ideal of X . Consequently, we conclude that μ is a quick ideal of X . $\square$

Definition 4.12. Let μ and ν be fuzzy quick ideals of X_1 and X_2 , respectively. The product $\mu \times \nu$ of μ and ν is a fuzzy set of $X_1 \times X_2$ which is defined by

$$(\mu \times \nu) := \min\{\mu(x), \nu(y)\}, \quad \forall (x, y) \in X_1 \times X_2.$$

Theorem 4.13. If μ_i are fuzzy quick ideals of X_i ($i = 1, 2$), then $\mu_1 \times \mu_2$ is a fuzzy quick ideal of $X_1 \times X_2$.

Proof. For any $(x, y) \in X_1 \times X_2$, we have $(\mu_1 \times \mu_2)(0, 0) = \mu\{\mu_1(0), \mu_2(0)\} \geq \min\{\mu_1(x), \mu_2(y)\} = (\mu_1 \times \mu_2)(x, y)$. The condition (F_0) is true. Let $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$ with $(x_1, x_2) * (y_1, y_2) \neq (0, 0)$. Then

$$\begin{aligned} & \min\{(\mu_1 \times \mu_2)(x_1, x_2), (\mu_1 \times \mu_2)(y_1, y_2)\} \\ &= \min\{\min\{\mu_1(x_1), \mu_2(x_2)\}, \min\{\mu_1(y_1), \mu_2(y_2)\}\} \\ &= \min\{\min\{\mu_1(x_1), \mu_1(y_1)\}, \min\{\mu_2(x_2), \mu_2(y_2)\}\} \\ &\geq \min\{\mu_1(x_1 * y_1), \mu_2(x_2 * y_2)\} \\ &= (\mu_1 \times \mu_2)(x_1 * y_1, x_2 * y_2) \\ &= (\mu_1 \times \mu_2)((x_1, x_2) * (y_1, y_2)) \end{aligned}$$

and hence the condition (F_3) holds. Thus $\mu_1 \times \mu_2$ is a fuzzy quick ideal of $X_1 \times X_2$. $\square$

For an endomorphism f of a d -algebra X and a fuzzy set μ in X , we define a new fuzzy set μ^f by $\mu^f(x) = \mu(f(x))$ for all $x \in X$. Note that

for any endomorphism f of a d -algebra X , we have $f(0) = 0$ for any $x \in X$, since $f(0 * 0) = f(0) * f(0) = 0$.

Proposition 4.14. *If μ is a fuzzy quick ideal of a d -algebra X , then so is μ^f .*

Proof. For any $x \in X$, we have $\mu^f(0) = \mu(f(0)) \geq \mu^f(x) = \mu(f(x))$. The condition (F_0) is true. For any $x, y \in X$ with $x * y \neq 0$, we have

$$\begin{aligned} \min\{\mu^f(x), \mu^f(y)\} &= \min\{\mu(f(x)), \mu(f(y))\} \\ &\geq \mu(f(x) * f(y)) \\ &= \mu(f(x * y)) \\ &= \mu^f(x * y). \end{aligned}$$

The condition (F_3) holds. This completes the proof. $\square$

Example 4.15. Let $X := \{0, a, b, c\}$ be a d -algebra as in Example 3.4-(1). Then $A = \{0, a\}$ is a quick ideal of X . Let $\mu : X \rightarrow [0, 1]$ be a fuzzy set defined by $\mu(0) = \mu(a) = t_1$ and $\mu(b) = \mu(c) = t_2$ for $t_1, t_2 \in [0, 1]$ with $t_1 > t_2$. Then μ is a fuzzy quick ideal of X . There exist 5 endomorphisms of a d -algebra X as follows:

$$\begin{aligned} f_1 : 0 &\rightarrow 0, a \rightarrow 0, b \rightarrow 0, c \rightarrow 0 \\ f_2 : 0 &\rightarrow 0, a \rightarrow 0, b \rightarrow 0, c \rightarrow a \\ f_3 : 0 &\rightarrow 0, a \rightarrow 0, b \rightarrow 0, c \rightarrow b \\ f_4 : 0 &\rightarrow 0, a \rightarrow 0, b \rightarrow 0, c \rightarrow c \\ f_5 : 0 &\rightarrow 0, a \rightarrow a, b \rightarrow b, c \rightarrow c. \end{aligned}$$

By Proposition 4.14, the maps μ^{f_i} ($i = 1, 2, 3, 4, 5$) are fuzzy quick ideals of X .

Definition 4.16. A fuzzy quick ideal I of a d -algebra of X is said to be *characteristic* if $f(I) = I$ for all $f \in \text{Aut}(X)$, where $\text{Aut}(X)$ is the set of all automorphisms of X .

Definition 4.17. A fuzzy quick ideal of a d -algebra X is said to be *fuzzy characteristic* if $\mu^f(x) = \mu(x)$ for all $x \in X$ and $f \in \text{Aut}(X)$.

Example 4.18. In Example 4.15, f_5 is an automorphism of X . It is easy to check that $f_5(\mu(x)) = \mu(x)$ for all $x \in X$. Hence μ is characteristic. Also $\mu^{f_5}(x) = \mu(f_5(x)) = \mu(x)$ for all $x \in X$. Therefore μ is fuzzy characteristic.

5. Intuitionistic fuzzy quick ideals in d -algebras

Definition 5.1. ([5]) Let X be a d -algebra. An IFS $D = \langle x, \mu_D, \gamma_D \rangle$ in X is called an *intuitionistic fuzzy d -algebra* if satisfies:

$$\mu_D(x * y) \geq \min\{\mu_D(x), \mu_D(y)\} \text{ and } \gamma_D(x * y) \leq \max\{\gamma_D(x), \gamma_D(y)\}$$

for all $x, y \in X$.

Definition 5.2. Let X be a d -algebra. An IFS $D = \langle x, \mu_D, \gamma_D \rangle$ in X is called an *intuitionistic fuzzy BCK-ideal* of X if satisfies:

$$\begin{aligned} (a_0) \quad & (\forall x \in X) (\mu_D(0) \geq \mu_D(x), \gamma_D(0) \leq \gamma_D(x)), \\ (a_1) \quad & (\forall x, y \in X) (\mu_D(x) \geq \min\{\mu_D(x*y), \mu_D(y)\} \text{ and } \gamma_D(x) \leq \max\{\gamma_D(x*y), \gamma_D(y)\}). \end{aligned}$$

An IFS $D = \langle x, \mu_D, \gamma_D \rangle$ in X is called an *intuitionistic fuzzy d -ideal* of X if satisfies (a_1) and

$$(a_2) \quad (\forall x, y \in X) (\mu_D(x * y) \geq \mu_D(x) \text{ and } \gamma_D(x * y) \leq \gamma_D(x)).$$

An IFS $D = \langle x, \mu_D, \gamma_D \rangle$ in X is called an *intuitionistic fuzzy quick ideal* of X if satisfies (a_0) and

$$(a_3) \quad (\forall x, y \in X \text{ with } x * y \neq 0) (\min\{\mu_D(x), \mu_D(y)\} \geq \mu_D(x * y) \text{ and } \gamma_D(x * y) \geq \max\{\gamma_D(x), \gamma_D(y)\}).$$

Example 5.3. Let $X := \{0, 1, 2, 3, 4\}$ be a d -algebra which is not a BCK-algebra with the following Cayley table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	0	2
3	3	3	3	0	0
4	4	3	3	1	0

Let $D = \langle x, \mu_D(x), \gamma_D(x) \rangle$ be an IFS in X defined by $\mu_D(0) = \mu_D(1) = 0.5$, $\mu_D(2) = \mu_D(3) = \mu_D(4) = 0.03$, $\gamma_D(0) = \gamma_D(1) = 0.05$, $\gamma_D(2) = \gamma_D(3) = \gamma_D(4) = 0.3$. Then $D = \langle x, \mu_D(x), \gamma_D(x) \rangle$ is an intuitionistic fuzzy BCK-ideal of X and an intuitionistic fuzzy d -algebra of X .

Example 5.4. Let $X := \{0, 1, 2, 3, 4\}$ be a d -algebra which is not a BCK -algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	0	2
3	3	3	3	0	3
4	4	3	3	3	0

Let $D = \langle x, \mu_D(x), \gamma_D(x) \rangle$ be an IFS in X defined by $\mu_D(3) = 0.03, \mu_D(x) = 0.5, \gamma_D(3) = 0.3$, and $\gamma_D(x) = 0.05$ for $x \neq 3$. Then $D = \langle x, \mu_D(x), \gamma_D(x) \rangle$ is an intuitionistic fuzzy BCK -ideal of X which is not an intuitionistic fuzzy d -ideal of X since $\mu_D(4*3) = \mu_D(3) = 0.03 \leq \mu_D(4) = 0.5$ and $\gamma_D(4*3) = \gamma_D(3) = 0.3 \geq \gamma_D(4) = 0.05$.

Example 5.5. Let $X := \{0, a, b, c\}$ be a d -algebra as in Example 3.4-(2). Let $D = \langle x, \mu_D(x), \gamma_D(x) \rangle$ be an IFS in X defined by $\mu_D(b) = 0.3, \mu_D(x) = 0.8, \gamma_D(b) = 0.08$, and $\gamma_D(x) = 0.03$ for $x \neq b$. Then $D = \langle x, \mu_D(x), \gamma_D(x) \rangle$ is an intuitionistic fuzzy quick ideal of X , but not an intuitionistic fuzzy BCK -ideal of X since $\mu_D(b) = 0.3 \leq \min\{\mu_D(b*c) = \mu_D(0), \mu_D(c)\} = 0.8$ and $\gamma_D(b) = 0.08 \geq \max\{\gamma_D(b*c) = \gamma_D(0), \gamma_D(c)\} = 0.03$. Also D is an intuitionistic fuzzy d -algebra.

Example 5.6. Let $X := \{0, a, b, c\}$ be a d -algebra as in Example 3.4-(1). Let $D = \langle x, \mu_D(x), \gamma_D(x) \rangle$ be an IFS in X defined by $\mu_D(c) = 0.2, \mu_D(x) = 0.7, \gamma_D(c) = 0.08$, and $\gamma_D(x) = 0.01$ for $x \neq c$. Then $D = \langle x, \mu_D(x), \gamma_D(x) \rangle$ is an intuitionistic fuzzy BCK -ideal of X , but not an intuitionistic fuzzy quick ideal of X since $a*c = b \neq 0$, $\min\{\mu_D(a), \mu_D(c)\} = 0.2 < \mu_D(a*c) = \mu_D(b) = 0.7$ and $\gamma_D(a*c) = \gamma_D(b) = 0.01 < \max\{\gamma_D(a), \gamma_D(c)\} = 0.08$.

Example 5.7. Let $X := \{0, a, b, c\}$ be a d -algebra as in Example 3.4-(3). Let $D = \langle x, \mu_D(x), \gamma_D(x) \rangle$ be an IFS in X defined by $\mu_D(b) = 0.3, \mu_D(x) = 0.8, \gamma_D(b) = 0.08$, and $\gamma_D(x) = 0.03$ for $x \neq b$. Then $D = \langle x, \mu_D(x), \gamma_D(x) \rangle$ is an intuitionistic fuzzy quick ideal of X , but not an intuitionistic fuzzy d -algebra of X since $\mu_D(a*c) = \mu_D(b) = 0.3 < \min\{\mu_D(a), \mu_D(c)\} = 0.8$ and $\gamma_D(a*c) = \gamma_D(b) = 0.08 > \max\{\gamma_D(a), \gamma_D(c)\} = 0.03$. Also $D = \langle x, \mu_D(x), \gamma_D(x) \rangle$ is not an intuitionistic fuzzy BCK -ideal of X , since $\mu_D(b) = 0.3 < \min\{\mu_D(b*c) = \mu_D(0) = 0.8, \mu_D(c) = 0.8\} = 0.8$ and $\gamma_D(b) = 0.08 > \max\{\gamma_D(b*c) = \gamma_D(0) = 0.03, \gamma_D(c) = 0.03\} = 0.03$.

Theorem 5.8. Let $D = \langle x, \mu_D(x), \gamma_D(x) \rangle$ be an intuitionistic fuzzy BCK-ideal of a d -algebra X . Then

- (i) μ_D is order reversing,
- (ii) $x * y \leq z$ implies $\gamma_D(x * z) \leq \gamma_D(z)$ for all $x, y, z \in X$.

Proof. (i) Let $x, y \in X$ with $x \leq y$. Then $x * y = 0$. Hence

$$\mu_D(x) \geq \min\{\mu_D(x * y) = \mu_D(0), \mu_D(y)\} = \mu_D(y).$$

Thus $\mu_D(x) \geq \mu_D(y)$.

(ii) For all $x, y, z \in X$, we have

$$\begin{aligned} \gamma_D(x * y) &\leq \max\{\gamma_D((x * y) * z), \gamma_D(z)\} \\ &= \max\{\gamma_D(0), \gamma_D(z)\} = \gamma_D(z). \end{aligned}$$

Therefore $\gamma_D(x * y) \leq \gamma_D(z)$. $\square$

Theorem 5.9. Let $D = \langle x, \mu_D(x), \gamma_D(x) \rangle$ be an intuitionistic fuzzy BCK-ideal of a d -algebra X . Then $x * y \leq z$ implies $\mu_D(x) \geq \min\{\mu_D(y), \mu_D(z)\}$ and $\gamma_D(x) \leq \max\{\gamma_D(y), \gamma_D(z)\}$ for all $x, y, z \in X$.

Proof. Let $x, y, z \in X$ such that $x * y \leq z$. Then $(x * y) * z = 0$. Hence

$$\begin{aligned} \mu_D(x * y) &\geq \min\{\mu_D((x * y) * z), \mu_D(z)\} \\ &= \min\{\mu_D(0), \mu_D(z)\} = \mu_D(z). \end{aligned}$$

Therefore we have

$$\begin{aligned} \mu_D(x) &\geq \min\{\mu_D(x * y), \mu_D(y)\} \\ &\geq \min\{\mu_D(z), \mu_D(y)\}. \end{aligned}$$

Thus $\mu_D(x) \geq \min\{\mu_D(z), \mu_D(y)\}$.

By a similar way, we obtain $\gamma_D(x) \leq \max\{\gamma_D(y), \gamma_D(z)\}$ for all $x, y, z \in X$. $\square$

Theorem 5.10. If $\{D_i | i \in \Lambda\}$ is an arbitrary family of intuitionistic fuzzy quick ideal of a d -algebra X , then $\cap D_i$ is an intuitionistic fuzzy quick ideal of X , where $\cap D_i = \{\langle x, \wedge \mu_{D_i}(x), \vee \gamma_{D_i}(x) \rangle | x \in X\}$.

Proof. Since $\mu_{D_i}(0) \geq \mu_{D_i}(x)$ and $\gamma_{D_i}(0) \leq \gamma_{D_i}(x)$ for all $x \in X$, we have $\wedge \mu_{D_i}(0) \geq \wedge \mu_{D_i}(x)$ and $\vee \gamma_{D_i}(0) \leq \vee \gamma_{D_i}(x)$ for all $x \in X$.

Let $x, y \in X$ with $x * y \neq 0$. Then

$$\begin{aligned} \wedge \mu_{D_i}(x * y) &\leq \wedge (\min\{\mu_{D_i}(x), \mu_{D_i}(y)\}) \\ &= \min\{\wedge \mu_{D_i}(x), \wedge \mu_{D_i}(y)\} \end{aligned}$$

and

$$\begin{aligned} \vee \gamma_{D_i}(x * y) &\geq \vee (\max\{\gamma_{D_i}(x), \gamma_{D_i}(y)\}) \\ &= \max\{\vee \gamma_{D_i}(x), \vee \gamma_{D_i}(y)\}. \end{aligned}$$

Hence $\cap D_i = \{ \langle x, \wedge \mu_{D_i}(x), \vee \gamma_{D_i}(x) \rangle \mid x \in X \}$ is an intuitionistic fuzzy quick ideal of X . $\square$

Theorem 5.11. *If $D = \langle x, \mu_D, \gamma_D \rangle$ is an intuitionistic fuzzy quick ideal of X , then so is $\square D$, where $\square D = \{ \langle x, \mu_D(x), 1 - \mu_D(x) \rangle \mid x \in X \}$.*

Proof. It is sufficient to show that $\bar{\mu}_D(x) := 1 - \mu_D(x)$ satisfies the second condition of (a_3) in Definition 5.2. Let $x, y \in X$ with $x * y \neq 0$. Then

$$\begin{aligned} \bar{\mu}_D(x * y) &= 1 - \mu_D(x * y) \\ &\geq 1 - \min\{\mu_D(x), \mu_D(y)\} \\ &= \max\{1 - \mu_D(x), 1 - \mu_D(y)\} \\ &= \max\{\bar{\mu}_D(x), \bar{\mu}_D(y)\}. \end{aligned}$$

Hence $\square D$ is an intuitionistic fuzzy quick ideal of X . $\square$

Theorem 5.12. *If an IFS $D = \langle x, \mu_D, \gamma_D \rangle$ in X is an intuitionistic fuzzy quick ideal of X , then the sets*

$$X_\mu := \{x \in X \mid \mu_D(x) = \mu_D(0)\} \text{ and } X_\gamma := \{x \in X \mid \gamma_D(x) = \gamma_D(0)\}$$

are quick ideals of X .

Proof. Clearly, $0 \in X_\mu$. Let $x, y \in X$ with $x * y \neq 0, x * y \in X_\mu$. Then $\mu_D(x * y) = \mu_D(0)$. Hence $\min\{\mu_D(x), \mu_D(y)\} \geq \mu_D(x * y) = \mu_D(0)$. By (a_0) , we have $\mu_D(x) = \mu_D(y) = \mu_D(0)$. Thus $x, y \in X_\mu$.

By the definition of X_γ , $0 \in X_\gamma$. Let $x, y \in X$ with $x * y \neq 0, x * y \in X_\gamma$. Then $\gamma_D(x * y) = \gamma_D(0)$. Hence $\gamma_D(0) = \gamma_D(x * y) \geq \max\{\gamma_D(x), \gamma_D(y)\}$. By (a_0) , we have $\gamma_D(x) = \gamma_D(y) = \gamma_D(0)$. Thus $x, y \in X_\gamma$. $\square$

Definition 5.13. Let $D = \langle x, \mu_D, \gamma_D \rangle$ be an IFS in X and let $t \in [0, 1]$. Then the set

$$U(\mu_D; t) := \{x \in X \mid \mu_D(x) \geq t\} \text{ (resp., } L(\gamma_D; t) := \{x \in X \mid \gamma_D(x) \leq t\})$$

is called a μ -level t -cut (resp., γ -level t -cut) of D .

Theorem 5.14. *If an IFS $D = \langle x, \mu_D, \gamma_D \rangle$ in X is an intuitionistic fuzzy quick ideal of X , then the μ -level t -cut and γ -level t -cut of D are quick ideals of X for every $t \in [0, 1]$ such that $t \in \text{Im}(\mu_D) \cap \text{Im}(\gamma_D)$, which are called a μ -level quick ideal and a γ -level quick ideal of X , respectively.*

Proof. By (a_0) , we have $0 \in U(\mu_D; t)$ and $0 \in L(\gamma_D; t)$.

Let $x, y \in X$ with $x * y \neq 0$, $x * y \in U(\mu_D; t)$. Then $\mu_D(x * y) \geq t$ and so $\min\{\mu_D(x), \mu_D(y)\} \geq \mu_D(x * y) \geq t$. Hence $x, y \in U(\mu_D; t)$.

Let $x, y \in X$ with $x * y \neq 0$, $x * y \in L(\gamma_D; t)$. Then $\gamma_D(x * y) \leq t$ and so $t \geq \gamma_D(x * y) \geq \max\{\gamma_D(x), \gamma_D(y)\}$. Hence $x, y \in L(\mu_D; t)$. $\square$

Theorem 5.15. *Let $D = \langle x, \mu_D, \gamma_D \rangle$ be an IFS in X such that the sets $U(\mu_D; t)$ and $L(\gamma_D; t)$ are quick ideals of X . Then $D = \langle x, \mu_D, \gamma_D \rangle$ is an intuitionistic fuzzy quick ideal of X*

Proof. Assume that there exists $x_0, y_0 \in X$ with $x_0 * y_0 \neq 0$ such that $\min\{\mu_D(x_0), \mu_D(y_0)\} < \mu_D(x_0 * y_0)$. Let $t_0 := \frac{1}{2}(\mu_D(x_0 * y_0) + \min\{\mu_D(x_0), \mu_D(y_0)\})$. Then $\min\{\mu_D(x_0), \mu_D(y_0)\} < t_0 < \mu_D(x_0 * y_0)$ and so $x_0, y_0 \notin U(\mu_D; t_0)$ and $x_0 * y_0 \in U(\mu_D; t_0)$. This is a contradiction, and therefore $\min\{\mu_D(x), \mu_D(y)\} \geq \mu_D(x * y)$ for all $x, y \in X$. Now suppose that there exists $x_0, y_0 \in X$ with $x_0 * y_0 \neq 0$ such that $\gamma_D(x_0 * y_0) < \max\{\gamma_D(x_0), \gamma_D(y_0)\}$. Taking $s_0 := \frac{1}{2}(\gamma_D(x_0 * y_0) + \max\{\gamma_D(x_0), \gamma_D(y_0)\})$, then $\gamma_D(x_0 * y_0) < s_0 < \max\{\gamma_D(x_0), \gamma_D(y_0)\}$. It follows that $x_0 * y_0 \in L(\gamma_D; s_0)$ and $x_0, y_0 \notin L(\gamma_D; s_0)$. This is a contradiction. Hence

$$\gamma_D(x * y) \geq \max\{\gamma_D(x), \gamma_D(y)\}$$

for all $x, y \in X$. This completes the proof. $\square$

Theorem 5.16. *Any quick ideal of a d -algebra X can be realized as both a μ -level quick ideal and γ -level quick ideal of some intuitionistic fuzzy quick ideal of X .*

Proof. Let S be a quick ideal of X and let μ_D and γ_D be fuzzy sets in X defined by

$$\mu_D(x) := \begin{cases} t_1, & \text{if } x \in S \\ t_2, & \text{otherwise} \end{cases}$$

and

$$\gamma_D(x) := \begin{cases} s_1, & \text{if } x \in S \\ s_2, & \text{otherwise} \end{cases}$$

for all $x \in X$ where t_i and s_i are fixed numbers in $(0, 1)$ such that $t_1 > t_2, s_1 < s_2$, and $t_i + s_i < 1$ for $i = 1, 2$. Since $0 \in S$, we have $\mu_D(0) = t_1$ and $\gamma_D(0) = s_1$. Hence $\mu_D(0) \geq \mu_D(x)$ and $\gamma_D(0) \leq \gamma_D(x), \forall x \in X$.

Let $x, y \in X$ with $x * y \neq 0$. If $x * y \in S$, then $x, y \in S$. Hence $\mu_D(x * y) = \min\{\mu_D(x), \mu_D(y)\}$ and $\gamma_D(x * y) = \max\{\gamma_D(x), \gamma_D(y)\}$. If $x * y \notin S$, then $\mu_D(x * y) = t_2$ and $\gamma_D(x * y) = s_2$. Hence $t_2 = \mu_D(x * y) \leq \min\{\mu_D(x), \mu_D(y)\}$ and $s_2 = \gamma_D(x * y) \geq \max\{\gamma_D(x), \gamma_D(y)\}$. Therefore

$D = \langle x, \mu_D, \gamma_D \rangle$ is an intuitionistic fuzzy quick ideal of X . Obviously, $U(\mu_D; t) = S = L(\gamma_D; s)$. This completes the proof. $\square$

Theorem 5.17. *Let α be a d -homomorphism of a d -algebra X into a d -algebra Y and B an intuitionistic quick ideal of Y , Then $\alpha^{-1}(B)$ is an intuitionistic fuzzy quick ideal of X .*

Proof. For all $x \in X$, we have

$$\begin{aligned}\mu_{\alpha^{-1}(B)}(0) &= \mu_B(\alpha(0)) \geq \mu_B(\alpha(x)) \\ &= \mu_{\alpha^{-1}(B)}(x)\end{aligned}$$

and

$$\begin{aligned}\gamma_{\alpha^{-1}(B)}(0) &= \gamma_B(\alpha(0)) \leq \gamma_B(\alpha(x)) \\ &= \gamma_{\alpha^{-1}(B)}(x).\end{aligned}$$

Let $x, y \in X$ with $x * y \neq 0$. Then

$$\begin{aligned}\mu_{\alpha^{-1}(B)}(x * y) &= \mu_B(\alpha(x * y)) = \mu_B(\alpha(x) * \alpha(y)) \\ &\leq \min\{\mu_B(\alpha(x)), \mu_B(\alpha(y))\} \\ &= \min\{\mu_{\alpha^{-1}(B)}(x), \mu_{\alpha^{-1}(B)}(y)\}\end{aligned}$$

and

$$\begin{aligned}\gamma_{\alpha^{-1}(B)}(x * y) &= \gamma_B(\alpha(x * y)) = \gamma_B(\alpha(x) * \alpha(y)) \\ &\geq \max\{\gamma_B(\alpha(x)), \gamma_B(\alpha(y))\} \\ &= \max\{\gamma_{\alpha^{-1}(B)}(x), \gamma_{\alpha^{-1}(B)}(y)\}.\end{aligned}$$

Hence $\alpha^{-1}(B)$ is an intuitionistic fuzzy quick ideal of X . $\square$

Theorem 5.18. *Let X be a d -algebra, Λ an non-empty set of $[0, 1]$, and $\{G_i | i \in \Lambda\}$ be a collection of quick ideals of X such that*

- (i) $X = \cup_{i \in \Lambda} G_i$,
- (ii) $s < t$ if and only if $G_t \subseteq G_s$ for all $s, t \in \Lambda$.

Then an IFS $D = \langle x, \mu_D, \gamma_D \rangle$ of X defined by $\mu_D(x) := \sup\{s \in \Lambda | x \in G_s\}$ and $\gamma_D(x) := \inf\{s \in \Lambda | x \in G_s\}$ is an intuitionistic fuzzy quick ideal of X .

Proof. Using Theorem 5.16, it is sufficient to show that $U(\mu_D; s)$ and $L(\gamma_D; t)$ are quick ideals of X for all $s \in [0, \mu_D(0)]$ and $t \in [\gamma_D(0), 1]$. In order to prove that $U(\mu_D; s)$ is a quick ideal of X , we divide into the following cases:

- (i) $s = \sup\{q \in \Lambda | q < s\}$, and (ii) $s \neq \sup\{q \in \Lambda | q < s\}$.

Case (i) implies that $x \in U(\mu_D; s) \Leftrightarrow x \in G_q$ for all $q < s \Leftrightarrow x \in \cap_{q < s} G_q$, so that $U(\mu_D; s) = \cap_{q < s} G_q$, which is a quick ideal of X .

For the case (ii), we claim that $U(\mu_D; s) = \cup_{q \geq s} G_q$. If $x \in \cup_{q \geq s} G_q$, then $x \in G_q$ for some $q \geq s$. Hence $\mu_D(x) \geq q \geq s$, so that $x \in U(\mu_D; s)$. This proves that $\cap_{q \geq s} G_q \subseteq U(\mu_D; s)$.

Now assume that $x \notin \cup_{q \geq s} G_q$. Then $x \notin G_q$ for all $q \geq s$. Since $s \neq \sup\{q \in \Lambda \mid q < s\}$, there exists $\epsilon > 0$ such that $(s - \epsilon, s) \cap \Lambda = \emptyset$. Hence $x \notin G_q$ for all $q > s - \epsilon$, which means that if $x \in G_q$, then $q \leq s - \epsilon$. Therefore $U(\mu_D; s) \subseteq \cup_{q \geq s} G_q$, and thus $U(\mu_D; s) = \cup_{q \geq s} G_q$, which is a quick ideal of X .

Next, we prove that $L(\gamma_D; t)$ is a quick ideal of X for all $t \in [\gamma_D(0), 1]$. We consider the following two cases:

(iii) $t = \inf\{q \in \Lambda \mid t < q\}$, and (iv) $t \neq \inf\{q \in \Lambda \mid t < q\}$.

For the case (iii) we have $x \in L(\gamma_D; t) \Leftrightarrow x \in G_q$ for all $t < q \Leftrightarrow x \in \cap_{t < q} G_q$ and hence $L(\gamma_D; t) = \cap_{t < q} G_q$, which is a quick ideal of X .

For the case (iv), there exists $\epsilon > 0$ such that $(t, t + \epsilon) \cap \Lambda = \emptyset$. We will show that $L(\gamma_D; t) = \cup_{t \geq q} G_q$.

If $x \in \cup_{t \geq q} G_q$, then $x \in G_q$ for some $t \geq q$. It follows that $\gamma_D(x) \leq q \leq t$ so that $x \in L(\gamma_D; t)$. Hence $\cup_{t \geq q} G_q \subseteq L(\gamma_D; t)$.

Conversely, if $x \notin \cup_{t \geq q} G_q$, then $x \notin G_q$ for all $q \geq t$, which implies that $x \notin G_q$ for all $q < t + \epsilon$, that is, if $x \in G_q$, then $q \geq t + \epsilon$. Thus $\gamma_D(x) \geq t + \epsilon > t$, i.e., $x \notin L(\gamma_D; t)$. Therefore $L(\gamma_D; t) \subseteq \cup_{t \geq q} G_q$ and consequently $L(\gamma_D; t) = \cup_{t \geq q} G_q$ which is a quick ideal of X . This completes the proof. $\square$

Let $IF(X)$ be the family of all intuitionistic fuzzy quick ideals of a d -algebra X and let $t \in [0, 1]$. Define binary relations U^t and L^t on $IF(X)$ as follows:

$$(A, B) \in U^t \iff U(\mu_A; t) = U(\mu_B; t)$$

and

$$(A, B) \in L^t \iff L(\gamma_A; t) = L(\gamma_B; t),$$

respectively, for $A = \langle x, \mu_A, \gamma_A \rangle$ and $B = \langle x, \mu_B, \gamma_B \rangle$ in $IF(X)$. Then clearly U^t and L^t are equivalence relations on $IF(X)$.

For any $A = \langle x, \mu_A, \gamma_A \rangle \in IF(X)$, let $[A]_{U^t}$ (resp., $[A]_{L^t}$) denote the equivalence class of A modulo U^t (resp., L^t), and denoted by $IF(X)/U^t$ (resp., $IF(X)/L^t$) the collection of all equivalence classes modulo U^t (resp., L^t); so

$$IF(X)/U^t := \{[A]_{U^t} \mid A = \langle x, \mu_A, \gamma_A \rangle \in IF(X)\}$$

$$(\text{resp., } IF(X)/L^t := \{[A]_{L^t} \mid A = \langle x, \mu_A, \gamma_A \rangle \in IF(X)\}).$$

Now let $I(X)$ denote the family of all quick ideals of X and let $t \in [0, 1]$. Define maps f_t and g_t from $IF(X)$ to $I(X) \cup \{\emptyset\}$ by $f_t(A) = U(\mu_A; t)$ and $g_t(A) = L(\gamma_A; t)$, respectively, for all $A = \langle x, \mu_A, \gamma_A \rangle \in IF(X)$. Then f_t and g_t are clearly well-defined.

Theorem 5.19. *For any $t \in (0, 1)$ the maps f_t and g_t are surjective from $IF(X)$ to $I(X) \cup \{\emptyset\}$.*

Proof. Let $t \in (0, 1)$. Note that $\mathbf{0}_\sim = \langle x, \mathbf{0}, \mathbf{1} \rangle$ is in $IF(X)$, where $\mathbf{0}$ and $\mathbf{1}$ are fuzzy sets in X defined by $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = x$ for all $x \in X$. Obviously $f_t(\mathbf{0}_\sim) = U(\mathbf{0}; t) = \emptyset = L(\mathbf{1}; t) = g_t(\mathbf{0}_\sim)$. Let $\emptyset \neq G \in I(X)$. For $G_\sim = \langle x, \chi_G, \bar{\chi}_G \rangle \in IF(X)$, we have $f_t(G_\sim) = U(\chi_G; t)$ and $g_t(G_\sim) = L(\bar{\chi}_G; t) = G$. Hence f_t and g_t are surjective. $\square$

Theorem 5.20. *The quotient sets $IF(X)/U^t$ and $IF(X)/L^t$ are equipotent to $I(X) \cup \{\emptyset\}$ for every $t \in (0, 1)$.*

Proof. For $t \in (0, 1)$ let f_t^* (resp., g_t^*) be a map from $IF(X)/U^t$ (resp., $IF(X)/L^t$) to $I(X) \cup \{\emptyset\}$ defined by $f_t^*([A]_{U^t}) = f_t(A)$ (resp., $g_t^*([A]_{L^t}) = g_t(A)$) for all $A = \langle x, \mu_A, \mu_A \rangle \in IF(X)$. If $U(\mu_A; t) = U(\mu_B; t)$ and $L(\gamma_A; t) = L(\gamma_B; t)$ for $A = \langle x, \mu_A, \gamma_A \rangle$ and $B = \langle x, \mu_B, \gamma_B \rangle$ in $IF(X)$, then $(A, B) \in U^t$ and $(A, B) \in L^t$; hence $[A]_{U^t} = [B]_{U^t}$ and $[A]_{L^t} = [B]_{L^t}$. Therefore the maps f_t^* and g_t^* are injective. Now let $\emptyset \neq G \in I(X)$. For $G_\sim = \langle x, \chi_G, \bar{\chi}_G \rangle \in IF(X)$, we have

$$f_t^*([G_\sim]_{U^t}) = f_t(G_\sim) = U(\chi_G; t) = G$$

and

$$g_t^*([G_\sim]_{L^t}) = g_t(G_\sim) = L(\bar{\chi}_G; t) = G.$$

Finally, for $\mathbf{0}_\sim = \langle x, \mathbf{0}, \mathbf{1} \rangle \in IF(X)$ we get

$$f_t^*([\mathbf{0}_\sim]_{U^t}) = f_t(\mathbf{0}_\sim) = U(\mathbf{0}; t) = \emptyset$$

and

$$g_t^*([\mathbf{0}_\sim]_{L^t}) = f_t(\mathbf{0}_\sim) = L(\mathbf{1}; t) = \emptyset.$$

This shows that f_t^* and g_t^* are surjective, and we are done. $\square$

For any $t \in [0, 1]$, we define another relation R^t on $IF(X)$ as follows:

$$(A, B) \in R^t \iff U(\mu_A; t) \cap L(\gamma_A; t) = U(\mu_B; t) \cap L(\gamma_B; t)$$

for any $A = \langle x, \mu_A, \gamma_A \rangle, B = \langle x, \mu_B, \gamma_B \rangle \in IF(X)$. Then the relation R^t is also an equivalence relation on $IF(X)$.

Theorem 5.21. *For any $t \in (0, 1)$, the map $\phi_t : IF(X) \rightarrow I(X) \cup \{\emptyset\}$ defined by $\phi_t(A) = f_t(A) \cap g_t(A)$ for each $A = \langle x, \mu_A, \gamma_A \rangle \in IF(X)$ is surjective.*

Proof. Let $t \in (0, 1)$. For $\mathbf{0}_\sim = \langle x, \mathbf{0}, \mathbf{1} \rangle \in IF(X)$, we get

$$\phi_t(\mathbf{0}_\sim) = f_t(\mathbf{0}_\sim) \cap g_t(\mathbf{0}_\sim) = U(\mathbf{0}; t) \cap L(\mathbf{1}; t) = \emptyset.$$

For any $H \in I(X)$, there exists $H_\sim = \langle x, \chi_H, \bar{\chi}_H \rangle \in IF(X)$ such that

$$\phi_t(H_\sim) = f_t(H_\sim) \cap g_t(H_\sim) = U(\chi_H; t) \cap L(\bar{\chi}_H; t) = H.$$

This completes the proof. $\square$

Theorem 5.22. *For any $t \in (0, 1)$, the quotient set $IF(X)/R^t$ is equipotent to $I(X) \cup \{\emptyset\}$.*

Proof. Let $t \in (0, 1)$ and let $\phi^* : IF(X)/R^t \rightarrow I(X) \cup \{\emptyset\}$ be defined by $\phi_t^*([A]_{R^t}) = \phi_t(A)$ for all $[A]_{R^t} \in IF(X)/R^t$. If $\phi_t^*([A]_{R^t}) = \phi_t^*([B]_{R^t})$ for any $[A]_{R^t}, [B]_{R^t} \in IF(X)/R^t$, then $f_t(A) \cap g_t(A) = f_t(B) \cap g_t(B)$, i.e., $U(\mu_A; t) \cap L(\gamma_A; t) = U(\mu_B; t) \cap L(\gamma_B; t)$, hence $(A, B) \in R^t$. It follows that $[A]_{R^t} = [B]_{R^t}$ so that ϕ_t^* is injective. For $\mathbf{0}_\sim = \langle x, \mathbf{0}, \mathbf{1} \rangle \in IF(X)$,

$$\phi_t^*([\mathbf{0}_\sim]_{R^t}) = \phi_t(\mathbf{0}_\sim) = f_t(\mathbf{0}_\sim) \cap g_t(\mathbf{0}_\sim) = U(\mathbf{0}; t) \cap L(\mathbf{1}; t) = \emptyset.$$

If $H \in I(X)$, then for $H_\sim = \langle x, \chi_H, \bar{\chi}_H \rangle \in IF(X)$, we have

$$\phi_t^*([H_\sim]_{R^t}) = \phi_t(H_\sim) = f_t(H_\sim) \cap g_t(H_\sim) = U(\chi_H; t) \cap L(\bar{\chi}_H; t) = H.$$

Hence ϕ_t^* is surjective, completing the proof. $\square$

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