ON \( P-I \)-OPEN SETS

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Abstract. The notions of pre-local function, semi-local functions and \( \alpha \)-local functions with respect to a topology and an ideal are introduced, and several properties are investigated. Also, the concept of \( P-I \)-open sets and \( P-I \)-closed sets in ideal topological spaces are discussed. Relations between \( I \)-open sets and \( P-I \)-open sets are provided, and several properties related to \( P-I \)-open sets, pre-local functions, semi-local functions and \( \alpha \)-local functions with respect to a topology and an ideal are investigated.

1. Introduction

In 1990, D. Janković, and T.R. Hamlett have introduced the notion of \( I \)-open sets in topological spaces. Since then, several kinds of \( I \)-openness, that is, (weakly) semi-\( I \)-open set, \( \delta \)-\( I \)-open sets, \( \beta \)-\( I \)-open sets, \( \alpha \)-\( I \)-open sets, \( b \)-\( I \)-open sets, (weakly) pre-\( I \)-open sets, etc. are introduced, and several properties and relations are investigated (see [1, 2, 6, 8, 9, 10, 7, 19, 23]).

In this paper, we first introduce the notions of pre-local function, semi-local function and \( \alpha \)-local function with respect to a topology and an ideal, and several properties are investigated. We next introduce the concept of \( P-I \)-open set and \( P-I \)-closed set in ideal topological spaces, and investigates related properties. We discuss relations between \( I \)-open sets and \( P-I \)-open sets. Finally we introduce the notion of \( P-* \)-closure, and investigates many properties related to \( P-I \)-open set, pre-local function, semi-local function and \( \alpha \)-local function with respect to a topology and an ideal.
2. Preliminaries

Through this paper, \((X, \tau)\) and \((Y, \kappa)\) (simply \(X\) and \(Y\)) always mean topological spaces. A subset \(A\) of \(X\) is said to be semi-open [20] (respectively, \(\alpha\)-open [21] and pre-open [18]) if \(A \subset \text{Cl}((\text{Int}(A)))\) (respectively, \(A \subset \text{Int}((\text{Cl}(\text{Int}(A))))\) and \(A \subset \text{Int}(\text{Cl}(A))\)). The complement of a pre-open set (respectively, an \(\alpha\)-open set and a semi-open set) is called a pre-closed set (respectively, an \(\alpha\)-closed set and a semi-closed set). The intersection of all pre-closed sets (respectively, \(\alpha\)-closed sets and semi-closed sets) containing \(A\) is called the pre-closure (respectively, \(\alpha\)-closure and semi-closure) of \(A\), denoted by \(p\text{Cl}(A)\) (respectively, \(\alpha\text{Cl}(A)\) and \(s\text{Cl}(A)\)). A subset \(A\) is also pre-closed (respectively, \(\alpha\)-closed and semi-closed) if and only if \(A = p\text{Cl}(A)\) (respectively, \(A = \alpha\text{Cl}(A)\) and \(A = s\text{Cl}(A)\)). We denote the family of all pre-open sets (respectively, \(\alpha\)-open sets and semi-open sets) of \((X, \tau)\) by \(\tau^p\) (respectively, \(\tau^\alpha\) and \(\tau^s\)).

An ideal is defined as a nonempty collection \(I\) of subsets of \(X\) satisfying the following two conditions.

1. If \(A \in I\) and \(B \subset A\), then \(B \in I\). (heredity)
2. If \(A \in I\) and \(B \in I\), then \(A \cup B \in I\). (finite additivity)

An ideal topological space is a topological space \((X, \tau)\) with an ideal \(I\) on \(X\), and it is denoted by \((X, \tau, I)\). For a subset \(A \subset X\), the set \(A^*(\tau, I) = \{x \in X : U \cap A \not\in I \text{ for each } U \in \tau(x)\}\) is called the local function of \(A\) with respect to \(\tau\) and \(I\), where \(\tau(x) = \{U \in \tau : x \in U\}\).

We will use \(A^*\) and/or \(A^*(I)\) instead of \(A^*(\tau, I)\). Given ideal topological space \((X, \tau, I)\), the \(*\)-topology, written as \(\tau^*(I)\) or written simply as \(\tau^*\), on \(X\) is defined to be a topology with a basis \(\beta(I, \tau) = \{U \setminus E \mid U \in \tau, E \in I\}\).

Note that \(\tau^*\) is finer than \(\tau\). For a subset \(A \subset X\), \(\text{Cl}^*(A) = A \cup A^*\) defined a Kuratowski closure operator for a topology \(\tau^*\).

**Lemma 2.1.** [25] Let \((X, \tau)\) be a topological space with ideals \(I\) and \(J\) on \(X\). For subsets \(A\) and \(B\) of \(X\), we have the following assertions.

1. \(A \subset B \Rightarrow A^* \subset B^*\).
2. \(I \subset J \Rightarrow A^*(J) \subset A^*(I)\).
3. \(A^* = \text{Cl}(A^*) \subset \text{Cl}(A)\) (\(A^*\) is a closed subset of \(\text{Cl}(A)\)).
4. \((A^*)^* \subset A^*\).
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(v) $(A \cup B)^* = A^* \cup B^*$.
(vii) $U \in \tau \Rightarrow U \cap A^* = U \cap (U \cap A)^* \subset (U \cap A)^*$.
(viii) $B \in \mathcal{I} \Rightarrow (A \cup B)^* = A^* = (A \setminus B)^*$.

**Definition 2.2.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset $A$ of $X$ is said to be $\mathcal{I}$-open [26] if $A \subset \text{Int}(A^*)$.

The set of all $\mathcal{I}$-open sets in ideal topological space $(X, \tau, \mathcal{I})$ is denoted by $\mathcal{IO}(X, \tau, \mathcal{I})$ or written simply as $\mathcal{IO}(X)$ when there is no chance for confusion.

**Remark 2.3.** [26] One can deduce that $\mathcal{I}$-open set $\Rightarrow$ pre-open set, and the converse is not true, in general.

3. **Pre(resp. semi and $\alpha$)-local functions in ideal topological spaces**

**Definition 3.1.** Let $A$ be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. Then the set

$$A_p^*(\tau, \mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for each } U \in \tau^p(x)\}$$

is called the pre-local function with respect to $\tau$ and $\mathcal{I}$, where

$$\tau^p(x) = \{U \in \tau^p : x \in U\}.$$ We will use $A_p^*$ and/or $A_p^*(\mathcal{I})$ instead of $A_p^*(\tau, \mathcal{I})$.

**Definition 3.2.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the set

$$A_s^*(\tau, \mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for each } U \in \tau^s(x)\}$$

is called the semi-local function with respect to $\tau$ and $\mathcal{I}$, where

$$\tau^s(x) = \{U \in \tau^s : x \in U\}.$$ We will use $A_s^*$ and/or $A_s^*(\mathcal{I})$ instead of $A_s^*(\tau, \mathcal{I})$.

**Definition 3.3.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the set

$$A_\alpha^*(\tau, \mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for each } U \in \tau^\alpha(x)\}$$

is called the $\alpha$-local function with respect to $\tau$ and $\mathcal{I}$, where

$$\tau^\alpha(x) = \{U \in \tau^\alpha : x \in U\}.$$ We will use $A_\alpha^*$ and/or $A_\alpha^*(\mathcal{I})$ instead of $A_\alpha^*(\tau, \mathcal{I})$. 
Example 3.4. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $I = \{\emptyset, \{a\}\}$. Then $(X, \tau, I)$ is an ideal topological space. We know that $\tau^\alpha = \tau$, $\tau^\beta = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$ and $\tau^* = \{X, \emptyset, \{c\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}\}$. If $A = \{a, b\}$ then $A^*_p = \{a, b, d\}$, $A^*_s = \{a, b\}$ and $A^*_p = \{b\}$. If $B = \{a, c\}$, then $B^*_p = \{c, d\} = B^*_p$ and $A^*_s = \{c\}$.

Theorem 3.5. Let $(X, \tau, I)$ be an ideal topological space and let $A$ be a subset of $X$. Then

(i) $A^*_p \subset A^*_s \subset A^*$.

(ii) $A^*_s \subset pCl(A)$.

(iii) $A^*_p \subset pCl(A)$, $A^*_s \subset sCl(A)$ and $A^*_s \subset \alpha Cl(A)$.

Proof. (i) and (ii) are straightforward.

(iii) Let $x \in A^*_p$. Then $x \in \{y \in X : U \cap A \notin I \text{ for every } U \in \tau^p(y)\}$ where $\tau^p(y) = \{U \in \tau^p : y \in U\}$. If $x \in A$ then $x \in A \cup D_p(A) = pCl(A)$ where $D_p(A)$ is the pre-derived set of $A$. If $x \notin A$ then $U \cap A \notin I$ because $x \notin A^*_p$ for every $U \in \tau^p(x)$. Since $x \notin A$, $(U \setminus \{x\}) \cap A \notin I$. It follows that $(U \setminus \{x\}) \cap A \neq \emptyset$ because $\emptyset \subset I$ so that $x \in D_p(A) \subset A \cup D_p(A) = pCl(A)$. Hence $A^*_p \subset pCl(A)$. By the similar way, we can obtain the other results.

Theorem 3.6. Let $(X, \tau, I)$ be an ideal topological space and let $A$ be a subset of $X$. Then

(i) If $I = \{\emptyset\}$, then $A^*_p = pCl(A)$, $A^*_s = sCl(A)$ and $A^*_s = \alpha Cl(A)$.

(ii) If $I = \mathcal{P}(X)$, then $A^*_p = A^*_s = A^*_s = \emptyset$.

Proof. (i) Let $I = \{\emptyset\}$. We know that $pCl(A) = A \cup D_p(A)$ where $D_p(A)$ is a pre-derived set of $A$. Let $x \in A \cup D_p(A)$ and let $G_x \in \tau^p$ containing $x$. Then $x \in A$ or $x \in D_p(A)$. If $x \in A$ then $x \in G_x \cap A$, and so $G_x \cap A \neq \emptyset$. If $x \in D_p(A)$ then $\emptyset \neq (G_x \setminus \{x\}) \cap A \subset G_x \cap A$ and thus $G_x \cap A \neq \emptyset$. Hence $pCl(A) = A \cup D_p(A) \subset A^*_p$. We know from Theorem 3.5 that $A^*_p \subset pCl(A)$. Therefore $A^*_p = pCl(A)$. Similarly, we have $A^*_s = sCl(A)$ and $A^*_s = \alpha Cl(A)$.

(ii) Straightforward.

Lemma 3.7. [19] Let $(X, \tau)$ be a topological space and let $A, B$ be subsets of $X$. If $A \in \tau^\alpha$ and $B \in \tau^p$ then $A \cap B \in \tau^p$.

Lemma 3.8. Let $(X, \tau)$ be a topological space with ideals $I$ and $J$ on $X$, and let $A, B$ be subsets of $X$. Then

(i) $A \subset B \Rightarrow A^*_p \subset B^*_p$.

(ii) $I \subset J \Rightarrow A^*_p(J) \subset A^*_p(I)$.
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(iii) \(A_p^* = pCl(A_p^*) \subset pCl(A)\) (\(A_p^*\) is a pre-closed subset of \(pCl(A)\)).
(iv) \((A_p^*)_p^* \subset A_p^*\).
(v) \(B \in I \Rightarrow B_p^* = \emptyset\).
(vi) \(U \in \tau^\alpha \Rightarrow U \cap A_p^* = U \cap (U \cap A)^*_p \subset (U \cap A)^*_p\).
(vii) \(B \in I \Rightarrow (A \cup B)^*_p = A_p^* = (A \setminus B)^*_p\).
(viii) \(A_p^*(I \cup J) \supset A_p^*(I) \cup A_p^*(J)\).

Proof. (i) Suppose that \(A \subset B\). Let \(x \in A_p^*\). Then \(G_x \cap A \not\in I\) for every pre-open \(G_x\) containing \(x\). Since \(A \subset B\), we have \(G_x \cap A \subset G_x \cap B\). Then \(G_x \cap B \not\in I\) by the heredity of ideal. Hence \(x \in B_p^*\), and therefore \(A_p^* \cap B_p^*\).

(ii) Suppose that \(I \subset J\). Let \(x \in A_p^*(J)\). Then \(G_x \cap A \not\in J\) for every pre-open \(G_x\) containing \(x\). Since \(I \subset J\), we get \(G_x \cap A \not\in I\). Hence \(x \in A_p^*(I)\), and so \(A_p^*(I) \subset A_p^*(J)\).

(iii) We will show that \(A_p^*\) is a pre-closed set. Let \(x \in (A_p^*)^c\). Then \(x \not\in A_p^*\) and so there exist a pre-open set \(U_x\) containing \(x\) such that \(U_x \cap A \in I\). It follows that \(y \not\in A_p^*\) for all \(y \in U_x\) so that \(y \in (A_p^*)^c\). Then \(x \in U_x \subset (A_p^*)^c\). Hence \((A_p^*)^c\) is pre-open. Finally, we know that \(A_p^*\) is pre-closed. Therefore \(A_p^* = pCl(A_p^*) \subset pCl(A)\) by Theorem 3.5.

(iv) By (iii), we obtain \((A_p^*)_p^* \subset pCl(A_p^*) = A_p^*\).

(v) Suppose that \(B \in I\). Since \(G \cap B \subset B\) for every subset \(G\) of \(X\), \(G \cap B \in I\) by the heredity of ideal. Hence \(B_p^* = \emptyset\).

(vi) Since \(U \cap A \subset A\), (i) implies that \((U \cap A)^*_p \subset A_p^*\). Hence \(U \cap (U \cap A)^*_p \subset U \cap A_p^*\). Let \(x \in U \cap A_p^*\). Then \(x \in U\) and \(x \in A_p^*\). It follows that \(G \cap A \not\in I\) for every \(G \in \tau^\alpha(x) = \{H \in \tau^\alpha : x \in H\}\). Since \(U\) is an \(\alpha\)-open set, \(U \cap G\) is pre-open containing \(x\) by Lemma 3.7 and so \(U \cap G \in \tau^\alpha(x)\). Thus \((G \cap U) \cap A \not\in I\), and hence \(G \cap (U \cap A) \not\in I\). Then \(x \in (U \cap A)^*_p\). Hence \(x \in U \cap (U \cap A)^*_p\). Finally, we have \(U \cap A_p^* = U \cap (U \cap A)^*_p \subset (U \cap A)^*_p\).

(vii) By (i), we have that \((A \cup B)^*_p \subset A_p^*\). Let \(x \in (A \cup B)^*_p\). Then \(U \cap (A \cup B) \not\in I\) for every \(U \in \tau^\alpha(x)\). This implies that \((U \cap A) \cup (U \cap B) \not\in I\). Since \(U \cap B \not\in I\), \(U \in I\) by the heredity of ideal and so \(U \cap A \not\in I\). Hence \(x \in A_p^*\). Therefore \((A \cup B)^*_p \subset A_p^*, \) Consequently \((A \cup B)^*_p = A_p^*, \) Since \(A_p^* = ((A \setminus B) \cup (A \cap B))^*_p\), we obtain \(A_p^* = ((A \setminus B) \cup (A \cap B))^*_p = (A \setminus B)^*_p\) by the first result of (vii).

(viii) It is straightforward by (ii). \(\square\)

In Lemma 3.8, the reverse inclusions of (iii) and (viii) are not valid as seen in the following example.
Example 3.9. Consider ideal topological spaces \((X, \tau, I)\), \((X, \tau, J)\) and \((X, \kappa, I)\) where \(X = \{a, b, c, d\}\), \(\tau = \{\emptyset, \{a, b\}, \{a, b, c\}\}\),
\[
\kappa = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\},
\]
\(I = \{\emptyset \{a\}\} \) and \(J = \{\emptyset , \{b\}\}\). If \(A = \{a, c\}\) in \((X, \kappa, I)\) then
\[
A^*_p(\kappa, I) = \{c, d\} \not\supseteq \{a, c, d\} = pCl(A).
\]
If \(A = \{a, b\}\) then \(A^*_p(\tau, I) = \{b\}\), \(A^*_p(\tau, J) = \{a\}\) and \(A^*_p(\tau, I \cap J) = \{a, b, c\}\). Hence we know that \(A^*_p(\tau, I \cap J) \not\subset A^*_p(\tau, I) \cup A^*_p(\tau, J)\).

The following example shows that the identity \(A^*_p \setminus B^*_p = (A \setminus B)^*_p \setminus B^*_p\) and the inclusions \(A^*_p \setminus B^*_p \subset (A \setminus B)^*_p\) and \(A^*_p \setminus B^*_p \supset (A \setminus B)^*_p\) are not valid.

Example 3.10. Let \(X = \{a, b, c, d\}\), \(\tau = \{X, \emptyset, \{a, b\}, \{a, b, c\}\}\) and \(I = \{\emptyset \}\). Then \((X, \tau, I)\) is an ideal topological space. Take \(A = \{a, b\}\), \(B = \{a, d\}\). Then \(A^*_p = \{a, b, c, d\}\), \(B^*_p = \{a, d\}\) and \((A \setminus B)^*_p = \{b\}\). Thus \(A^*_p \setminus B^*_p = \{b, c\} \not= \{b\} = (A \setminus B)^*_p \setminus B^*_p\).

If \(A = \{a, b\}, B = \{a\}\) then \(A^*_p = \{a, b, c, d\}\), \(B^*_p = \{a\}\). Hence \(A^*_p \setminus B^*_p = \{b, c, d\} \not\subset \{b\} = (A \setminus B)^*_p \).

If \(A = \{a, b, c, d\}, B = \{a, b, c\}\) then \(A^*_p = \{a, b, c, d\}\), \(B^*_p = \{a, b, c, d\}\). Hence \(A^*_p \setminus B^*_p = \emptyset \not\supset \{d\} = (A \setminus B)^*_p\).

The converses of (i), (ii), (vi) and (vii) in Lemma 3.8 may not be true as seen in the following example.

Example 3.11. Let \(X = \{a, b, c, d\}\), \(\tau = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}\), \(I = \{\emptyset, \{a\}\}\) and \(J = \{\emptyset, \{c\}\}\). Then \((X, \tau, I)\) and \((X, \tau, J)\) are ideal topological spaces.

(i) If \(A = \{b, c\}\) and \(B = \{a, b, d\}\) then \(A^*_p(I) = \{b, c, d\}\) and \(B^*_p(I) = \{b, d\}\). Hence we know that \(A^*_p(I) \supset B^*_p(I)\) but \(A \not\supset B\).

(ii) If \(A = \{b, c\}\) then \(A^*_p(J) = \{b, c, d\}\) and \(A^*_p(J) = \{b\}\). Thus \(A^*_p(J) \subset A^*_p(I)\) but \(I \not\subset J\).

(vi) If \(A = \{a, b, c\}\) and \(U = \{a, c, d\}\) then \((U \cap A)^*_p(I) = U \cap (U \cap A)^*_p(I) = \{c, d\}\). Thus \(U \cap A^*_p(I) = \{c, d\}\) and \(U \cap A^*_p(I) = \{c, d\}\). Thus \(U \cap A^*_p(I) = \{c, d\}\) but \(U \not\in \tau^a\).

(vii) If \(A = \{b, c, d\}\) and \(B = \{d\}\) then \((A \cup B)^*_p(I) = A^*_p(I) = \{b, c, d\}\) and \((A \setminus B)^*_p(I) = \{b, c, d\}\). Thus \((A \cup B)^*_p(I) = A^*_p(I) = (A \setminus B)^*_p(I)\) but \(B \not\subset I\).
Lemma 3.12. For a class \( \{ U_i \mid i \in \Lambda \} \) of subsets of an ideal topological space \((X, \tau, \mathcal{I})\), we have

\[
\begin{align*}
(\text{i}) \quad & \bigcup_{i \in \Lambda} ((U_i)_p^*) \subset \bigcup_{i \in \Lambda} (U_i)_p^*, \\
(\text{ii}) \quad & \bigcap_{i \in \Lambda} ((U_i)_p^*) \supset \bigcap_{i \in \Lambda} (U_i)_p^*.
\end{align*}
\]


The reverse inclusion of (i) in Lemma 3.12 is not valid as seen in the following example.

Example 3.13. Let \( X = \{a, b, c, d\} \), \( \tau = \{X, \emptyset, \{a, b\}, \{a, b, c\}\} \) and \( \mathcal{I} = \{\emptyset\} \). Then \((X, \tau, \mathcal{I})\) is an ideal topological space. We note that \( \tau^p = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\) if \( A = \{a\} \) and \( B = \{b\} \) then \( A_p^* = \{a\}, B_p^* = \{b\} \) and \( (A \cup B)_p^* \neq (A \cup B)^p \).

Lemma 3.14. For any subset \( A \) of an ideal topological space \((X, \tau, \mathcal{I})\), if \( A \subset A_p^* \) then \( A_p^* = pCl(A_p^*) = pCl(A) \).

Proof. Straightforward.

Theorem 3.15. Let \((X, \tau, \mathcal{I})\) be an ideal topological space. Then

\[ A_p^* \supset A \setminus \{U \subset X : U \in \mathcal{I}\} \]

for all \( A \subset X \).

Proof. Put \( B := \bigcup \{U \subset X : U \in \mathcal{I}\} \) and let \( x \in A \setminus B \). Then \( x \in A \) and \( x \notin B \). This implies that \( x \notin U \) for all \( U \in \mathcal{I} \) so that \( \{x\} = \{x\} \cap A \notin \mathcal{I} \) because \( x \in A \). For every \( G \in \tau^p(x) \), we have \( \{x\} \cap A \subset G \cap A \notin \mathcal{I} \) by the heredity of ideal. Hence \( x \in A_p^* \). This completes the proof.

The reverse inclusion of Theorem 3.15 is not valid as seen in the following example.

Example 3.16. Consider an ideal topological space \((X, \tau, \mathcal{I})\) where \( X = \{a, b, c, d\} \), \( \tau = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\} \) and \( \mathcal{I} = \{\emptyset, \{a\}\} \). Then \( B := \bigcup \{U \subset X : U \in \mathcal{I}\} = \{a\} \). If \( A = \{a, c\} \) then \( A_p^* = \{c, d\} \). Hence \( A_p^* = \{c, d\} \notin \{c\} = A \setminus B \).

Theorem 3.17. Let \((X, \tau, \mathcal{I})\) be an ideal topological space. Put

\[ B := \bigcup \{U \subset X : U \in \mathcal{I}\}. \]

If \( B \in \mathcal{I} \) then \( (A_p^*)_p^* = A_p^* \) for all \( A \subset X \).
Proof. Let $A$ be a subset of $X$. Then $(A^*_p)_p \subset A^*_p$ by Lemma 3.8(iv). Furthermore $A^*_p \supset A \setminus B$ by Theorem 3.15. It follows from Lemma 3.8(i) that $(A^*_p)_p \supset (A \setminus B)_p^*$. Since $B \in \mathcal{I}$, Lemma 3.8(vii) implies that $(A^*_p)_p \supset (A \setminus B)_p^* = A^*_p$. Therefore $(A^*_p)_p = A^*_p$.

**Theorem 3.18.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space in which the cardinality of $\mathcal{I}$ is finite. Then $(A^*_p)_p = A^*_p$ for all $A \subset X$.

**Proof.** Let $B := \bigcup\{U \subset X : U \in \mathcal{I}\}$. Since the cardinality of $\mathcal{I}$ is finite, $B \in \mathcal{I}$ by the finite additivity of ideal. Therefore $(A^*_p)_p = A^*_p$ for all $A \subset X$ by Theorem 3.17.

**Corollary 3.19.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space in which the cardinality of $X$ is finite. Then $(A^*_p)_p = A^*_p$ for all $A \subset X$.

**Proof.** Straightforward.

**Theorem 3.20.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space in which $\tau^p = \mathcal{P}(X)$. Then
\[
A^*_p = A \setminus \{U \subset X : U \in \mathcal{I}\}
\]
for all $A \subset X$.

**Proof.** Put $B := \bigcup\{U \subset X : U \in \mathcal{I}\}$ and let $x \in A^*_p$. Then $\{x\} \cap A \notin \mathcal{I}$ because $\{x\} \in \tau^p = \mathcal{P}(X)$. Since ideal $\mathcal{I}$ always contain $\emptyset$, $\{x\} \cap A \neq \emptyset$ and so $x \in A$. It follows that $\{x\} = \{x\} \cap A \notin \mathcal{I}$ so that $x \notin U$ for all $U \in \mathcal{I}$. Hence $x \notin B$, and therefore $x \in A \setminus B$. Hence $A^*_p \subset A \setminus B$. The reverse inclusion is obvious by Theorem 3.15.

**Corollary 3.21.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space in which every member of $\tau$ is clopen. Then
\[
A^*_p = A \setminus \{U \subset X : U \in \mathcal{I}\}
\]
for all $A \subset X$ or all $A \subset X$.

**Proof.** Put $B := \bigcup\{U \subset X : U \in \mathcal{I}\}$ and let $A \in \mathcal{P}(X)$. Then $A \subset \text{Cl}(A) = \text{Int}(\text{Cl}(A))$ because every member of $\tau$ is clopen. Hence $A \in \tau^p$, which means that $\mathcal{P}(X) \subset \tau^p$ so that $\mathcal{P}(X) = \tau^p$. By Theorem 3.20, $A^*_p = A \setminus B$.

**Lemma 3.22.** Let $(X, \tau)$ be a topological space. Then the set
\[
\{U \subset X : U \cap G \neq \emptyset \text{ for all } G \in \tau \setminus \{\emptyset\}\}
\]
is contained in $\tau^p$.
Proof. Let \( H \in \{ U \subset X : U \cap G \neq \emptyset \text{ for all } G \in \tau \setminus \{ \emptyset \} \} \). For any \( a \in X \), if \( a \in H \) then clearly \( a \in \text{Cl}(H) \). If \( a \not\in H \) then \( H \cap (F \setminus \{ a \}) = H \cap F \neq \emptyset \) for any \( F \in \tau \setminus \{ \emptyset \} \) containing \( a \) because \( H \in \{ U \subset X : U \cap G \neq \emptyset \text{ for all } G \in \tau \setminus \{ \emptyset \} \} \). Thus \( a \in D(H) \subset \text{Cl}(H) \) where \( D(H) \) is the derived set of \( H \). It follows that \( \text{Cl}(H) = X \) so that \( H \subset X = \text{Int}(\text{Cl}(H)) \). Hence \( H \in \tau^p \). Therefore the result is valid. \( \square \)

**Lemma 3.23.** Let \((X, \tau)\) be a topological space. If \( \tau \) is a chain under the set inclusion then
\[
\tau^p = \{ U \subset X : U \cap G \neq \emptyset \text{ for all } G \in \tau \setminus \{ \emptyset \} \} \cup \{ \emptyset \}.
\]

*Proof.* Since \( \emptyset \in \tau^p \), \( \tau^p \supset \{ U \subset X : U \cap G \neq \emptyset \text{ for all } G \in \tau \setminus \{ \emptyset \} \} \cup \{ \emptyset \} \) by Lemma 3.22.

Conversely, let \( H \in \tau^p \). Then \( H \subseteq \text{Int}(\text{Cl}(H)) \). If \( H = \emptyset \) then clearly \( H \in \{ U \subset X : U \cap G \neq \emptyset \text{ for all } G \in \tau \setminus \{ \emptyset \} \} \cup \{ \emptyset \} \). Suppose that \( H \neq \emptyset \). If there exist \( F \in \tau \setminus \{ \emptyset \} \) such that \( H \cap F = \emptyset \), then \( H \subset F^c \) and so \( \text{Int}(\text{Cl}(H)) \subset \text{Int}(\text{Cl}(F^c)) = \text{Int}(F^c) \). Since \( \tau \) is a chain under the set inclusion, \( I \cap F \neq \emptyset \) for all \( I \in \tau \setminus \{ \emptyset \} \). It follows that \( I \not\subset F^c \) so that \( \text{Int}(F^c) = \emptyset \). Thus \( H \subseteq \text{Int}(\text{Cl}(H)) \subset \text{Int}(\text{Cl}(F^c)) = \text{Int}(F^c) = \emptyset \). This is a contradiction. Hence \( H \cap F = \emptyset \) for all \( F \in \tau \setminus \{ \emptyset \} \). Therefore \( \tau^p = \{ U \subset X : U \cap G \neq \emptyset \text{ for all } G \in \tau \setminus \{ \emptyset \} \} \cup \{ \emptyset \} \). \( \square \)

**Theorem 3.24.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space. If \( \tau \) is a chain under the set inclusion, then either \( G^*_p = X \) or \( G^*_p \subset G \) for \( G \subset X \).

*Proof.* Let \( G \) be a subset of \( X \). Suppose that \( G^*_p \neq X \). We will show that \( G \subseteq G^*_p \). Let \( x \not\in G \). Since \( G^*_p \neq X \), there exists \( a \in X \) such that \( a \not\in G^*_p \). This implies that there exists \( H \in \tau^p(a) \) such that \( H \cap G \in \mathcal{I} \). Since \( \tau \) is a chain under the set inclusion, \( H \cap F \neq \emptyset \) for each \( F \in \tau \setminus \{ \emptyset \} \) by Lemma 3.23. Then \( (H \cup \{ x \}) \cap F \neq \emptyset \) for each \( F \in \tau \setminus \{ \emptyset \} \) and so, \( x \in H \cup \{ x \} \in \tau^p \) by Lemma 3.23. Furthermore \( (H \cup \{ x \}) \cap G = H \cap G \in \mathcal{I} \). Hence \( x \not\in G^*_p \). Therefore we obtain the desired result \( G^*_p \subseteq G \). \( \square \)

**Theorem 3.25.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space. Assume that \( \tau \) is a chain under the set inclusion in which there exists the smallest element \( A \) of \( \tau \setminus \{ \emptyset \} \). If \( A \cap B = \emptyset \) for every \( B \in \mathcal{I} \), then \( G^*_p = X \) for \( G \subset X \) containing \( A \).

*Proof.* Let \( A \subset G \). Assume that there exists \( x \in X \) such that \( x \not\in G^*_p \). Then there exists \( H \in \tau^p(x) = \{ U \in \tau^p : x \in U \} \) such that \( H \cap G \in \mathcal{I} \). Since \( H \in \tau^p(x) \), \( H \cap A \neq \emptyset \) by Lemma 3.23. Since \( H \cap G \in \mathcal{I} \), \( A \cap (H \cap G) = \emptyset \) by the hypothesis. It follows that \( A \cap H = A \cap (H \cap G) = \emptyset \) because \( A \subset G \). This is a contradiction. Therefore \( x \in G^*_p \). \( \square \)
4. $P-I$-open sets and $P-I$-closed sets

**Definition 4.1.** Let $(X, \tau, I)$ be an ideal topological space. A subset $A$ of $X$ is said to be $P-I$-open if $A \subset p \text{Int}(A^*_p)$. A subset $B$ of $X$ is said to be $P-I$-closed if the complement of $B$ is $P-I$-open.

The set of all $P-I$-open sets in $(X, \tau, I)$ is denoted by $P_{IO}(X, \tau, I)$.

Simply $P_{IO}(X, \tau, I)$ is written as $P_{IO}(X)$ or $P_{IO}(X, \tau)$ when there is no chance for confusion.

**Lemma 4.2.** [24] Let $S$ be a set in $(X, \tau)$. Then

(i) $p \text{Cl}(S) = S \cup \text{Cl}(\text{Int}(S))$, $p \text{Int}(S) = S \cap \text{Int}(\text{Cl}(S))$.

(ii) $\text{Int}(s \text{Cl}(S)) = p \text{Int}(\text{Cl}(S)) = s \text{Cl}(p \text{Int}(S)) = \text{Int}(\text{Cl}(S))$.

(iii) $\text{Int}(p \text{Cl}(S)) = s \text{Cl}(\text{Int}(S)) = \text{Int}(\text{Cl}(\text{Int}(S)))$.

**Proposition 4.3.** For any set $A$ in $(X, \tau, I)$, we have

$p \text{Int}(A^*_p) \subseteq \text{Int}(A^*)$.

**Proof.** We have

\[
p \text{Int}(A^*_p) = A^*_p \cap \text{Int}(\text{Cl}(A^*_p)) \quad (\text{by Lemma 4.2})
\]

\[
\subseteq \text{Int}(\text{Cl}(A^*_p))
\]

\[
\subseteq \text{Int}(\text{Cl}(A^*)) \quad (\text{by Theorem 3.5})
\]

\[
= \text{Int}(A^*) \quad (\text{by Lemma 2.1})
\]

This completes the proof. \(\square\)

**Theorem 4.4.** Let $A \in P_{IO}(X)$. Then $A$ is $I$-open.

**Proof.** It is straightforward by Proposition 4.3. \(\square\)

The converse of Theorem 4.4 may not be true as seen in the following example.

**Example 4.5.** Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $I = \{\emptyset, \{a\}\}$. Then $(X, \tau, I)$ be an ideal topological space. We have that

$\tau^p = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$.

Since $\text{Int}(\{a, b\})^* = \text{Int}(\{a, b, d\}) = \{a, b\}$, we have $\{a, b\} \in I_{O}(X)$. But $\{a, b\} \notin P_{IO}(X)$ because $p \text{Int}(\{a, b\})^*_p = p \text{Int}(\{b\}) = \{b\}$.

**Lemma 4.6.** [27] Let $A$ be a subset of a topological space $(X, \tau)$. Then the following assertions are satisfied.

(i) $(p \text{Int}(A))^c = p \text{Cl}(A^c)$. 

(ii) \((p\text{Cl}(A))^c = p\text{Int}(A^c)\).

**Theorem 4.7.** If \(A\) is \(P-I\)-closed in an ideal topological space \((X, \tau, I)\), then \(A \supset (p\text{Int}(A))^*\).

**Proof.** Since \(A\) is \(P-I\)-closed, \(A^c\) is \(P-I\)-open. Thus
\[
A^c \subseteq p\text{Int}((A^c)^*_p) \\
\subseteq p\text{Int}(p\text{Cl}(A^c)) \\
= (p\text{Cl}(p\text{Int}(A))^c) (\text{by Lemma 4.6}).
\]
Hence \(A \supset p\text{Cl}(p\text{Int}(A)) \supset (p\text{Int}(A))^*_p\) by Theorem 3.5.

**Theorem 4.8.** Let \((X, \tau, I)\) be an ideal topological space. Let \(A\) be a subset of \(X\) such that \(((p\text{Int}(A))^*_p)^c = p\text{Int}((A^c)^*_p)\). Then \(A\) is \(P-I\)-closed if and only if \(A \supset (p\text{Int}(A))^*_p\).

**Proof.** Let \(A\) be a subset of \(X\) such that \(((p\text{Int}(A))^*_p)^c = p\text{Int}((A^c)^*_p)\). Suppose that \(A\) is \(P-I\)-closed. Then \(A \supset (p\text{Int}(A))^*_p\) by Theorem 4.7.

Conversely, suppose that \(A \supset (p\text{Int}(A))^*_p\). Then \(A^c \subset ((p\text{Int}(A))^*_p)^c = p\text{Int}((A^c)^*_p)\). Hence \(A\) is \(P-I\)-closed.

**Theorem 4.9.** For any \(P-I\)-open set \(A\) in an ideal topological space \((X, \tau, I)\), we have \(A^*_p = (p\text{Int}(A^*_p))^*_p\).

**Proof.** Let \(A\) be a \(P-I\)-open set. Then \(A \subset p\text{Int}(A^*_p)\). By Lemma 3.8, we know that \(A^*_p \subset (p\text{Int}(A^*_p))^*_p\). In general, since \(A^*_p \supset p\text{Int}(A^*_p)\), it follows from Lemma 3.8 that \(A^*_p \supset (A^*_p)^*_p \supset (p\text{Int}(A^*_p))^*_p\). Therefor we obtain the desired result.

**Theorem 4.10.** Let \(\{U_i \in P\text{TO}(X) : i \in \Lambda\}\) be a class of \(P-I\)-open sets in an ideal topological space \((X, \tau, I)\). Then \(\bigcup_{i \in \Lambda} \{U_i \in P\text{TO}(X) : i \in \Lambda\}\) is \(P-I\)-open.

**Proof.** Note that \(U_i \subset p\text{Int}((U_i)^*_p)\) for every \(i \in \Lambda\). It follows that
\[
\bigcup_{i \in \Lambda} U_i \subset \bigcup_{i \in \Lambda} p\text{Int}((U_i)^*_p) \\
\subset p\text{Int}(\bigcup_{i \in \Lambda} (U_i)^*_p) \\
\subset p\text{Int}((\bigcup_{i \in \Lambda} (U_i))^*_p) (\text{by Lemma 3.12})
\]
so that \(\bigcup_{i \in \Lambda} \{U_i \in P\text{TO}(X) : i \in \Lambda\}\) is \(P-I\)-open.
The intersection of two $P$-$\mathcal{I}$-open sets need not be $P$-$\mathcal{I}$-open as seen in following example.

**Example 4.11.** Consider an ideal topological space $(X, \tau, \mathcal{I})$ where $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a, b\}, \{a, b, c\}\}$, $\mathcal{I} = \{\emptyset\}$. Then $\tau^p = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Note that $\{a, c\}, \{b, c, d\} \in P\text{TO}(X)$, but $\{a, c\} \cap \{b, c, d\} = \{c\} \notin P\text{TO}(X)$.

**Theorem 4.12.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $W$ is a $P$-$\mathcal{I}$-open and $\tau^p$ is a topology, then $p\text{Cl}(V) \cap W \subset (V \cap W)^*_p$ for every $V \in \tau^\alpha$.

**Proof.** Suppose that $W$ is a $P$-$\mathcal{I}$-open and $\tau^p$ is a topology. Let $V \in \tau^\alpha$. Then $V \subset \text{Int}(\text{Cl}(\text{Int}(V)))$. It follows that

$$p\text{Cl}(V) \cap W \subset p\text{Cl}(V) \cap p\text{Int}(W^*_p)$$
$$\subset p\text{Cl}(V \cap p\text{Int}(W^*_p)) \quad \text{(by [27, Theorem 4.17])}$$
$$\subset p\text{Cl}(V \cap W^*_p)$$
$$\subset p\text{Cl}(V \cap W)^*_p \quad \text{(by Lemma 3.8)}$$
$$= (V \cap W)^*_p.$$ 

This completes the proof. \qed

**Lemma 4.13.** Let $(X, \tau)$ be a topological space. Let $A$ be a subset of $X$ and let $B \in \tau^\alpha$. Then $p\text{Int}(A) \cap B = p\text{Int}(A \cap B)$.

**Proof.** Since $p\text{Int}(A \cap B) \subset p\text{Int}(A)$ and $p\text{Int}(A \cap B) \subset B$, we have $p\text{Int}(A \cap B) \subset p\text{Int}(A) \cap B$.

Conversely, let $x \in p\text{Int}(A) \cap B$. Then $x \in p\text{Int}(A)$ and $x \in B$. Since $x \in p\text{Int}(A)$, there exists pre-open set $G_x$ containing $x$ such that $G_x \subset A$. Since $x \in B$ and $B \in \tau^\alpha$, $G_x \cap B \in \tau^p$ by Lemma 3.7 and $x \in G_x \cap B \subset A \cap B$. Hence $x \in p\text{Int}(A \cap B)$. Therefore $p\text{Int}(A) \cap B = p\text{Int}(A \cap B)$. \qed

**Theorem 4.14.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let $A, B$ be subsets of $X$. If $A \in P\text{TO}(X, \tau)$ and $B \in \tau^\alpha$ then $A \cap B \subset p\text{Int}(B \cap (A \cap B)^*_p)$.

**Proof.** Let $A \in P\text{TO}(X, \tau)$ and $B \in \tau^\alpha$. Then

$$A \cap B \subset p\text{Int}(A^*_p) \cap B$$
$$= p\text{Int}(A^*_p \cap B) \quad \text{(by Lemma 4.13)}$$
$$= p\text{Int}(B \cap (A \cap B)^*_p). \quad \text{(by Lemma 3.8)}$$

This completes the proof. \qed
If $A$ is not $P$-$\mathcal{I}$-open but $\mathcal{I}$-open in an ideal topological space $(X, \tau, \mathcal{I})$ then is the above theorem valid? The answer is negative as seen in the following example.

**Example 4.15.** Consider the ideal topological space $(X, \tau, \mathcal{I})$ which is presented in Example 3.4. Let $A = \{a, b\}$ and $B = \{a, b, c\}$. Then $A$ is $\mathcal{I}$-open which is not $P$-$\mathcal{I}$-open, and $B \in \tau^\alpha$. But $A \cap B = \{a, b\} \not\subset \{b\} = \text{pInt}(B \cap (A \cap B)_p)$.

**Theorem 4.16.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let $A, B$ be subsets of $X$. If $A \in P\mathcal{I}O(X, \tau)$ and $B \in \tau^\alpha$ then $A \cap B \in P\mathcal{I}O(X, \tau)$.

**Proof.** Let $A \in P\mathcal{I}O(X, \tau)$ and $B \in \tau^\alpha$. Then
\[
A \cap B \supset \text{pInt}(A^*_p) \cap B
= \text{pInt}(A^*_p \cap B) \quad (\text{by Lemma 4.13})
\subset \text{pInt}((A \cap B)^*_p), \quad (\text{by Lemma 3.8})
\]
and therefore $A \cap B \in P\mathcal{I}O(X, \tau)$.

If $A \in P\mathcal{I}O(X, \tau)$ and $B \in \tau^p$ then is $A \cap B$ $P$-$\mathcal{I}$-open? The answer is negative as seen in the following example.

**Example 4.17.** Consider an ideal topological space $(X, \tau, \mathcal{I})$ where $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $\tau^p = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, X\}$. Let $A = \{b, c, d\}$ and $B = \{a, c, d\}$. Then $A \in P\mathcal{I}O(X, \tau)$ and $B \in \tau^p$. Since $(A \cap B)^*_p = \{c, d\}$, $\text{pInt}((A \cap B)^*_p) = \text{pInt}((\{c, d\}) = \{c\}$. Hence $A \cap B = \{c, d\} \not\subset \text{pInt}((A \cap B)^*_p) = \{c\}$. Therefore $A \cap B \notin P\mathcal{I}O(X, \tau)$.

If $A \in \mathcal{I}O(X, \tau)$ and $B \in \tau^\alpha$ then is $A \cap B$ $P$-$\mathcal{I}$-open? The answer is negative as seen in the following example.

**Example 4.18.** Consider the ideal topological space $(X, \tau, \mathcal{I})$ in Example 4.17. Take $A = \{a, b, c\} \in \mathcal{I}O(X) \setminus P\mathcal{I}O(X)$ and $B = \{a, b\} \in \tau \subset \tau^\alpha$. Then $A \cap B = \{a, b\} \not\subset \{b\} = \text{pInt}((a, b)^*_p) = \text{pInt}((A \cap B)^*_p)$. Hence $A \cap B$ is not $P$-$\mathcal{I}$-open.

**Corollary 4.19.** The union of $P$-$\mathcal{I}$-closed set and $\alpha$-closed set is $P$-$\mathcal{I}$-closed set.

**Proof.** Let $A$ be a $P$-$\mathcal{I}$-closed set and $B$ be an $\alpha$-closed set. Then $A^c$ is a $P$-$\mathcal{I}$-open set and $B^c$ is an $\alpha$-open set. By Theorem 4.16, $(A \cup B)^c = A^c \cap B^c \in P\mathcal{I}O(X, \tau)$. Hence $A \cup B$ is $P$-$\mathcal{I}$-closed.

\[\blacksquare\]
**Theorem 4.20.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space and let \(A\) be a subset of \(X\). If \(A\) is \(P\mathcal{I}\)-open and semi-closed, then \(A = p\text{Int}(A_p^*)\).

**Proof.** Since \(A\) is \(P\mathcal{I}\)-open, \(A \subset p\text{Int}(A_p^*)\). Since \(A\) is semi-closed, \(A^c \subset \text{Cl}(\text{Int}(A^c))\).

Thus
\[
A^c \subset \text{Cl}(\text{Int}(A^c))
= p\text{Cl}(\text{Int}(A^c)) \quad \text{(by Lemma 4.2)}
\subset p\text{Cl}(p\text{Int}(A^c))
= (p\text{Int}(p\text{Cl}(A)))^c. \quad \text{(by Lemma 4.6)}.
\]

Hence \(A \supset p\text{Int}(p\text{Cl}(A)) \supset p\text{Int}(A_p^*)\), and therefore \(A = p\text{Int}(A_p^*)\).

If \(A \in \mathcal{IO}(X, \tau)\) is semi-closed then is \(A = p\text{Int}(A_p^*)\) valid? The answer is negative as seen in the following example.

**Example 4.21.** Consider the ideal topological space \((X, \tau, \mathcal{I})\) in Example 4.17. Take \(A = \{a, b\}\). Then \(A\) is both semi-closed and \(\mathcal{I}\)-open but not \(P\mathcal{I}\)-open. We know that \(p\text{Int}(A_p^*) = \{b\} \neq A\).

**Definition 4.22.** A set \(S\) in \((X, \tau)\) is called \(b\)-open [24] if \(S \subset \text{Cl}(\text{Int}(S)) \cup \text{Int}(\text{Cl}(S))\).

**Theorem 4.23.** If \(A \subset W \subset p\text{Cl}(A)\) and \(A \in \mathcal{IO}(X, \tau)\), then \(W\) is \(b\)-open.

**Proof.** Let \(A \subset W \subset p\text{Cl}(A)\) and \(A \in \mathcal{IO}(X, \tau)\). Then \(A \subset \text{Int}(A^*)\) and \(\text{Cl}(W) = \text{Cl}(A)\). This implies that
\[
W \subset p\text{Cl}(A)
= A \cup \text{Cl}(\text{Int}(A)) \quad \text{(by Lemma 4.2)}
\subset \text{Int}(A^*) \cup \text{Cl}(\text{Int}(A))
\subset \text{Int}(\text{Cl}(A)) \cup \text{Cl}(\text{Int}(A))
\subset \text{Int}(\text{Cl}(W)) \cup \text{Cl}(\text{Int}(W)).
\]

Therefore \(W\) is \(b\)-open.

**Corollary 4.24.** If \(A \subset W \subset p\text{Cl}(A)\) and \(A \in \mathcal{PIO}(X, \tau)\), then \(W\) is \(b\)-open.

**Proof.** Let \(A \in \mathcal{PIO}(X, \tau)\). Then by Theorem 4.4, \(A \in \mathcal{IO}(X, \tau)\). Hence \(W\) is \(b\)-open by Theorem 4.23.

Is every superset of a \(P\mathcal{I}\)-open set \(P\mathcal{I}\)-open? The answer is negative as seen in the example.
Example 4.25. Consider the ideal topological space \((X, \tau, I)\) in Example 3.4. Let \(A = \{b, c\}\) and \(W = \{a, b, c\}\). Then \(A\) is a \(P-I\)-open set and \(A \subset W\). But \(W\) is not a \(P-I\)-open set.

Theorem 4.26. Let \(A\) be a subset of an ideal topological space \((X, \tau, I)\). If \(A \subset W \subset p\text{Int}(A^*_p)\) then \(W\) is \(P-I\)-open.

Proof. Let \(A\) be a subset of \(X\). Suppose that \(A \subset W \subset p\text{Int}(A^*_p)\), then \(W \subset p\text{Int}(A^*_p) \subset p\text{Int}(W^*_p)\) by Lemma 3.8. Hence \(W\) is \(P-I\)-open. \(\square\)

5. \(P\)-\(s\)-closure in the ideal topological spaces

Throughout this chapter \(D(A)\) (resp. \(D_\alpha(A), D_p(A), D_s(A)\)) always mean a derived set (resp. \(\alpha\)-derived set, pre-derived set, semi-derived set) of \(A\).

Definition 5.1. Let \((X, \tau, I)\) be an ideal topological space. Let \(A\) be a subset of \(X\). Then \(\text{Cl}^*_{p}(A, \tau, I)\) is defined as \(\text{Cl}^*_{p}(A, \tau, I) = A \cup A^*_p(\tau, I)\). \(\text{Cl}^*_{p}(A, \tau, I)\) is written simply as \(\text{Cl}^*_{p}(A)\) when there is no chance for confusion.

In an ideal topological space \((X, \tau, I)\), \(\tau^*_p(I)\) is the collection of the complement of all sets satisfying the following condition :

\[\text{Cl}^*_{p}(A) = A \text{ for } A \in \mathcal{P}(X)\]

That is, \(\tau^*_p(I) := \{A^c : \text{Cl}^*_{p}(A) = A \text{ for } A \in \mathcal{P}(X)\}\). \(\tau^*_p(I)\) is written simply as \(\tau^*_p\) when there is no chance for confusion.

Example 5.2. Consider the ideal topological space \((X, \tau, I)\) in Example 3.4. If \(A = \{a, c\}\) then \(A^*_p = \{c, d\}\). Hence \(\text{Cl}^*_{p}(A, \tau, I) = A \cup A^*_p = \{a, c, d\}\).

Theorem 5.3. Let \((X, \tau, I)\) be an ideal topological space. Then the following assertions are valid.

(i) If \(I = \{\emptyset\}\) then \(\text{Cl}^*_{p}(A) = p\text{Cl}(A)\).
(ii) If \(I = \mathcal{P}(X)\) then \(\text{Cl}^*_{p}(A) = A\).
(iii) If \(A\) is \(P-I\)-open then \(\text{Cl}^*_{p}(A) = A^*_p\).
(iv) If \(A \in I\) then \(\text{Cl}^*_{p}(A) = A\).

Proof. Straightforward. \(\square\)

Let \((X, \tau, I)\) be an ideal topological space. If \(A \subset W \subset \text{Cl}^*_{p}(A)\) and \(A \in P-I\text{O}(X, \tau)\), then is \(W\) a \(P-I\)-open set? The answer to this question is negative as seen in the following example.
Example 5.4. Let \( X = \{a, b, c, d\}, \tau = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\} \) and \( I = \{\emptyset\} \). Then \( (X, \tau, I) \) is an ideal topological space. Let \( A = \{c\} \) and \( W = \{c, d\} \). Then \( A^*_p = \{c, d\}, \text{Int}(A^*_p) = \{c\} \). We know that \( A \) is \( P\)-\( I \)-open and \( A \subset W \subset \text{Cl}^*_p(A) \). But \( W \not\subset \text{Int}(W^*_p) = \{c\} \). Hence \( W \notin \text{PIO}(X, \tau) \).

Lemma 5.5. For any subsets \( A \) and \( B \) of an ideal topological space \((X, \tau, I)\), we have the following assertions:

(i) \( \text{Cl}^*_p(\emptyset) = \emptyset, \text{Cl}^*_p(X) = X \).
(ii) If \( A \subset B \) then \( \text{Cl}^*_p(A) \subset \text{Cl}^*_p(B) \).
(iii) \( \text{Cl}^*_p(\text{Cl}^*_p(A)) \supset \text{Cl}^*_p(A) \).
(iv) \( \text{Cl}^*_p(A \cup B) \supset \text{Cl}^*_p(A) \cup \text{Cl}^*_p(B) \).
(v) \( \text{Cl}^*_p(A \cap B) \supset \text{Cl}^*_p(A) \cap \text{Cl}^*_p(B) \).
(vi) \( \text{Cl}^*_p(A) \subset \text{Cl}^*(A) \).
(vii) \( \text{Cl}^*_p(A) \subset p\text{Cl}(A) \).
(viii) Let \( J \) be a superset of \( I \). If \( J \) is an ideal, then \( \text{Cl}^*_p(A, \tau, J) \supset \text{Cl}^*_p(A, \tau, I) \).

Proof. It is easy to verify the desired results by using Lemma 3.8, Theorem 3.5 and Definition 5.1. \( \square \)

In Lemma 5.5, the reverse inclusions of (iv), (v), (vi), (vii) and the converse of (ii) and (viii) are not valid as seen in the following example.

Example 5.6. Consider an ideal topological space \((X, \tau, I)\) where \( X = \{a, b, c, d\}, \tau = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\} \) and \( I = \{\emptyset, \{c\}\} \). Let \( A = \{a, d\} \) and \( B = \{a, b\} \). Then \( \text{Cl}^*_p(A) = \{a, d\} \subset \{a, b, d\} = \text{Cl}^*_p(B) \).

But \( A \not\subset B \). Let \( A = \{a\} \) and \( B = \{b\} \). Then \( \text{Cl}^*_p(A \cup B) = \{a, b, d\} \not\subset \{a, b\} = \text{Cl}^*_p(A) \cup \text{Cl}^*_p(B) \).

Let \( A = \{a, d\} \) and \( B = \{a, b\} \). Then \( \text{Cl}^*_p(A \cap B) = \text{Cl}^*_p(\{a\}) = \{a\} \not\subset \{a, d\} = \text{Cl}^*_p(A) \cap \text{Cl}^*_p(B) \).

Let \( A = \{a, c\} \).

Then \( \text{Cl}^*_p(A) = \{a, c\} \not\subset X = \text{Cl}^*(A) \).

Let \( A = \{a, c\} \).

Then \( \text{Cl}^*_p(A) = \{a, c\} \not\subset \{a, c, d\} = \text{pCl}(A) \).

Let \( J = \{\emptyset, \{a\}\} \). Then \( J \) is an ideal. Let \( A = \{a, b\} \). Then \( \text{Cl}^*_p(A, \tau, J) = \{a, b, d\} \not\subset \{a, b\} = \text{Cl}^*_p(A, \tau, J) \).

But \( I \not\subset J \).

In an ideal topological space \((X, \tau, I)\), since
\[
A \cup A^*_p = \text{Cl}^*_p(A) \subset \text{pCl}(A) = A \cup D_p(A),
\]
we can guess that \( A^*_p \subset D_p(A) \). But this is wrong as seen in the following example.

Example 5.7. Let \( X = \{a, b, c, d\}, \tau = \{\emptyset, X, \{a, b\}, \{a, b, c\}\} \) and \( I = \{\emptyset, \{a\}\} \). Then \( (X, \tau, I) \) is an ideal topological space. If \( A = \)
\{a, b, c\}, then \(A_p^* = \{b, c\}\) and \(D_p(A) = \{c, d\}\). Hence \(A_p^* \not\subset D_p(A)\) and \(A_p^* \not\supset D_p(A)\).

**Theorem 5.8.** Let \((X, \tau, I)\) be an ideal topological space. For any subset \(A\) of \(X\), the following assertions are valid:

(i) \(A_p^* \setminus A \subset p\text{Cl}(A) \setminus A \subset D_p(A)\).
(ii) \(\text{If } I = \{\emptyset\}, \text{then } A_p^* \setminus A = p\text{Cl}(A) \setminus A \subset D_p(A)\).
(iii) \(\text{If } I = \mathcal{P}(X), \text{then } A_p^* \subset D_p(A)\).

Proof. (i) By Theorem 3.5, \(A_p^* \setminus A \subset p\text{Cl}(A) \setminus A\). Since \(p\text{Cl}(A) = A \cup D_p(A), p\text{Cl}(A) \setminus A \subset D_p(A)\). It follows that \(A_p^* \setminus A \subset p\text{Cl}(A) \setminus A \subset D_p(A)\).
(ii) and (iii) are straightforward by (i) and Theorem 3.6.

**Theorem 5.9.** Let \((X, \tau, I)\) be an ideal topological space. If \(\tau\) is a chain under the set inclusion, then either \(\text{Cl}_{p}^*(G) = X\) or \(\text{Cl}_{p}^*(G) = G\) for \(G \subset X\).

Proof. It follows from Theorem 3.24.

**Theorem 5.10.** Let \((X, \tau, I)\) be an ideal topological space. Then \(\tau^p \subset \tau^*_p\).

Proof. Let \(A \in \tau^p\). Then \(A^c\) is pre-closed (i.e. \(A^c = p\text{Cl}(A^c)\)). This implies that \(A^c \subset \text{Cl}_{p}^*(A^c) \subset p\text{Cl}(A^c) = A^c\). Hence \(A^c = \text{Cl}_{p}^*(A^c)\) (i.e. \(A \in \tau^*_p\)) Therefore \(\tau^p \subset \tau^*_p\).

**Remark 5.11.** The above theorem shows that every open set, pre-open set, \(P-I\)-open set and \(I\)-open set in an ideal topological space \((X, \tau, I)\) is a member of \(\tau^*_p\).

The reverse inclusion of Theorem 5.10 may not hold as seen in the following example

**Example 5.12.** Let \(X = \{a, b, c, d\}\), \(\tau = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\) and \(I = \{\emptyset, \{c\}\}\). Then \((X, \tau, I)\) is an ideal topological space and \(\tau^p = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}\)
\(\tau^*_p = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}\. We know that \(\{a, d\} \in \tau^*_p\) but \(\{a, d\} \notin \tau^p\).

**Theorem 5.13.** Let \((X, \tau, I)\) be an ideal topological space. Then

(i) \(\text{If } I = \{\emptyset\} \text{ then } \tau^*_p = \tau^p\).
(ii) \(\text{If } I = \mathcal{P}(X) \text{ then } \tau^*_p = \mathcal{P}(X)\).
Proof. (i) Since $I = \{\emptyset\}$, $A_p^* = p\text{Cl}(A)$ for all $A \in \mathcal{P}(X)$ by Theorem 3.6. Thus
\[
\tau_p^* = \{B^c : \text{Cl}_p^*(B) = B \text{ for } B \in \mathcal{P}(X)\} = \{B^c : B \cup B_p^* = B \text{ for } B \in \mathcal{P}(X)\} = \{B^c : \text{pCl}(B) = B \text{ for } B \in \mathcal{P}(X)\} = \tau^p.
\]

(ii) Suppose that $I = \mathcal{P}(X)$. Let $A \in \mathcal{P}(X)$. Then $\text{Cl}_p^*(A^c) = A^c \cup (A^c)_p^* = A^c$ by Theorem 3.6. Hence $A \in \tau_p^*$. Therefore $\mathcal{P}(X) \subset \tau_p^*$, and hence $\mathcal{P}(X) = \tau_p^*$. \hfill \Box

Theorem 5.14. Let $(X, \tau, I)$ be an ideal topological space. Then $\tau^* \subset \tau_p^*$.

Proof. Let $A \in \tau^*$. Then $\text{Cl}_p^*(A^c) = A^c$, which implies that $A^c \subset \text{Cl}_p^*(A^c) \subset \text{Cl}^*(A^c) = A^c$. Hence $\text{Cl}_p^*(A^c) = A^c$ (i.e. $A \in \tau_p^*$). Therefore $\tau^* \subset \tau_p^*$. \hfill \Box

The reverse inclusion of Theorem 5.14 does not hold as seen in the following example

Example 5.15. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $I = \{\emptyset\}$. Then $(X, \tau, I)$ is an ideal topological space and
\[
\tau_p^* = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}
\]
and
\[
\tau^* = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}.
\]
We know that $\{b\} \in \tau_p^*$ but $\{b\} \not\in \tau^*$.

Note that $\tau_p^*$ contains both $\tau^p$ and $\tau^*$. The following example shows that $\tau^p$ and $\tau^*$ are independent.

Example 5.16. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $I = \{\emptyset, \{c\}\}$. Then $(X, \tau, I)$ is an ideal topological space and
\[
\tau_p^* = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}
\]
and
\[
\tau^* = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}.
\]
We know that $\tau^p$ and $\tau^*$ are independent.

Theorem 5.17. Let $(X, \tau, I)$ be an ideal topological space. If every member of $\tau$ is clopen then $\tau_p^* = \mathcal{P}(X)$.
Proof. Let $A \in \mathcal{P}(X)$. Since every member of $\tau$ is clopen, $A \subset \overline{\text{Cl}}(A) = \text{Int}(\overline{\text{Cl}}(A))$ and so $A \in \tau^p$. This means that $\mathcal{P}(X) = \tau^p$. By Theorem 5.10, $\mathcal{P}(X) = \tau^p \subset \tau^*_p \subset \mathcal{P}(X)$. Hence $\tau^*_p = \mathcal{P}(X)$. □

In an ideal topological space $(X, \tau, \mathcal{I})$, is $\tau^*_p$ a topology? The answer to this question is negative as seen in the following example.

**Example 5.18.** Consider the ideal topological space $(X, \tau, \mathcal{I})$ in Example 5.15. Then $\{a, c, d\}, \{b, c, d\} \in \tau^*_p$. But $\{a, c, d\} \cap \{b, c, d\} = \{c, d\} \notin \tau^*_p$.

**Theorem 5.19.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If

$$\text{Cl}^*_p(B \cup C) \subset \text{Cl}^*_p(B) \cup \text{Cl}^*_p(C)$$

for any subsets $B$ and $C$ of $X$, then $\tau^*_p$ is a topology.

**Proof.** Obviously, $\emptyset, X \in \tau^*_p$ by Lemma 5.5(i). Suppose that $G_i \in \tau^*_p$ for every $i \in \Lambda$. Then $\text{Cl}^*_p(G_i^c) = G_i^c$ for every $i \in \Lambda$ and so

$$\text{Cl}^*_p((\bigcup_{i \in \Lambda} (G_i))^c) = \text{Cl}^*_p(\bigcap_{i \in \Lambda} G_i^c) \subset \bigcap_{i \in \Lambda} \text{Cl}^*_p(G_i^c) = \bigcap_{i \in \Lambda} G_i^c = (\bigcup_{i \in \Lambda} G_i)^c.$$

Furthermore, $(\bigcup_{i \in \Lambda} G_i)^c \subset \text{Cl}^*_p((\bigcup_{i \in \Lambda} G_i)^c)$ by Definition 5.1. Hence

$$(\bigcup_{i \in \Lambda} G_i)^c = \text{Cl}^*_p((\bigcup_{i \in \Lambda} G_i)^c),$$

and so $\bigcup_{i \in \Lambda} G_i \in \tau^*_p$. Let $G, H \in \tau^*_p$. Then $\text{Cl}^*_p(G^c) = G^c$ and $\text{Cl}^*_p(H^c) = H^c$. It follows from Lemma 5.5(iv) and hypothesis that

$$\text{Cl}^*_p((G \cap H)^c) = \text{Cl}^*_p(G^c \cup H^c) = \text{Cl}^*_p(G^c) \cup \text{Cl}^*_p(H^c) = G^c \cup H^c = (G \cap H)^c$$

so that $G \cap H \in \tau^*_p$. □

**Corollary 5.20.** Let $(X, \tau)$ be a topological space with ideal $\mathcal{I} = \mathcal{P}(X)$. Then $\tau^*_p$ is a topology.

**Proof.** By Theorem 3.6(ii), $A^*_p = \emptyset$ for all subset $A$ of $X$. This implies that $\text{Cl}^*_p(B \cup C) = (B \cup C) \cup (B \cup C)^p = B \cup C = (B \cup B^*_p) \cup (C \cup C^*_p) = \text{Cl}^*_p(B) \cup \text{Cl}^*_p(C)$ for any subsets $B$ and $C$ of $X$. Hence by Theorem 5.19, $\tau^*_p$ is a topology. □

**Corollary 5.21.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If

$$(B \cup C)^* \subset B^* \cup C^*$$

for any subsets $B$ and $C$ of $X$, then $\tau^*_p$ is a topology.
For any subsets $B$ and $C$ of $X$, $(B \cup C)^* \supset B^*_p \cup C^*_p$ by Lemma 3.8. Using hypothesis, we get $(B \cup C)^* \subset B^*_p \cup C^*_p$. Hence $(B \cup C)^* = B^*_p \cup C^*_p$. we can easily check that $\text{Cl}^*_p(B \cup C) = \text{Cl}^*_p(B) \cup \text{Cl}^*_p(C)$. It follows from Theorem 5.19 that $\tau^*_p$ is a topology.

**Theorem 5.22.** Let $(X, \tau, I)$ and $(X, \tau, J)$ be ideal topological spaces. If $I \subset J$ then $\tau^*_p(I) \subset \tau^*_p(J)$.

**Proof.** Suppose that $I \subset J$ and let $A \in \tau^*_p(I)$. Using Lemma 5.5(viii), we have $A^c = \text{Cl}^*(A^c, \tau, I) \supset \text{Cl}^*(A^c, \tau, J) \supset A^c$. Hence $\text{Cl}^*(A^c, \tau, J) = A^c$, i.e., $A \in \tau^*_p(J)$. Therefore $\tau^*_p(I) \subset \tau^*_p(J)$.

Let $(X, \tau, I)$ and $(X, \kappa, I)$ be ideal topological spaces. If $\tau \subset \kappa$ then is the inclusion $\tau^*_p \subset \kappa^*_p$ valid? The answer is negative as seen in the following example.

**Example 5.23.** Consider two ideal topological spaces $(X, \tau, I)$ and $(X, \kappa, I)$ where $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}$, $\kappa = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$ and $\tau^*_p = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Then $\tau^*_p = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$ and $\kappa^*_p = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. It follows that $\tau^*_p \not\subset \kappa^*_p$ and $\tau^*_p \not\supset \kappa^*_p$.

Let $(X, \tau, I)$ be an ideal topological space. If $I \subset \tau$ then is $\tau^p$ equal to $\tau^*_p$? The answer is negative as seen in the following example.

**Example 5.24.** Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $I = \{\emptyset, \{c\}\}$. Then $(X, \tau, I)$ is an ideal topological spaces with $I \subset \tau$, and $\tau^p = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$ and $\tau^*_p = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Clearly $\{a, d\} \in \tau^*_p$ but $\{a, d\} \not\in \tau^p$. Hence $\tau^p \neq \tau^*_p$.

**References**


On $P$-$I$-open sets


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