GROUND STATE MASS CONCENTRATION IN THE L^2 -CRITICAL NONLINEAR HARTREE EQUATION BELOW H^1

MYEONGJU CHAE

Abstract. We consider finite time blowup solutions of the L^2 -critical focusing Hartree equation on $\mathbb{R}^n$, $n \geq 3$ below H^1 .

1. Introduction

In this paper we study the initial value problem of the L^2 -critical focusing Hartree equation,

$$(1.1) \quad \begin{cases} i\partial_t \phi + \frac{1}{2}\Delta \phi = -(|x|^{-2} * |\phi|^2)\phi, & x \in \mathbb{R}^n, t > 0, \\ \phi(x, 0) = \phi_0(x) \in H^s(\mathbb{R}^n). \end{cases}$$

Here $H^s(\mathbb{R}^n)$ denotes the usual inhomogeneous Sobolev space. (1.1) is meaningful in dimension $n \geq 3$, where the Hartree potential is locally integrable. Hartree type equation arises in atomic and nuclear physics and is related to the mean-field theory with respect to wave functions describing boson systems. ([15], [26])

The local well-posedness results for $s \geq 0$ can be shown by the Strichartz estimates similarly as in the mass critical NLS with the polynomial nonlinearity $|\phi|^{\frac{4}{n}}\phi$. For $s > 0$ (1.1), which is the case we consider, is locally well-posed in the subcritical sense. More precisely, for any $\phi_0 \in H^s(\mathbb{R}^n)$, the lifetime span of the solution depends on the norm of the initial data, $\|\phi_0\|_{H^s}$. Whereas, for $s = 0$ the lifetime depends on the profile of the initial data as well.

The classical solutions to (1.1) enjoy the mass conservation law,

$$\|\phi(\cdot, t)\|_{L^2(\mathbb{R}^n)} = \|\phi_0(\cdot)\|_{L^2(\mathbb{R}^n)},$$

Received February 9, 2009. Accepted February 11, 2009.

2000 Mathematics Subject Classification. 35Q55

keywords and phrases : mass concentration, Hartree equation; I-method

The author is supported by KRF-2007-C00020

and the energy conservation law,

$$(1.2) \quad E[t] := \int_{\mathbb{R}^n} |\nabla \phi|^2 + (|x|^{-2} * |\phi|^2) |\phi|^2 dx.$$

When $s \geq 1$, the energy conservation law (1.2) together with the sub-critical local theory immediately yields the global well-posedness for defocusing case. When $0 \leq s < 1$, the energy could be infinite and the mass conservation law cannot imply the global well-posedness. In this case the gwp results is extended to $\frac{2(n-2)}{3n-4} < s < 1$ in [4] employing the I -method and the interaction Morawetz inequality developed in [10], and later in [8, 13] modified for mass critical situation. Also the small data theory asserts that sufficiently small initial L^2 data leads to a unique global L^2 solution regardless with the focusing/defocusing signs. However, blowup may occur in focusing case for large data.

The equation (1.1) is known to have a ground state solution Q , which solves

$$\Delta Q - Q = -(|x|^{-2} * |Q|^2)Q.$$

The existence of Q is proven in [23] with the decisive property of being the sharp constant of the Gagliardo-Nirenberg inequality such as

$$\int_{\mathbb{R}^n} (|x|^{-2} * |u|^2) |u|^2(x) dx \leq \frac{2}{\|Q\|_{L^2}} \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2.$$

The uniqueness is open except $n=4$ [29]. The mass concentration phenomenon of H^s - blow up solution is considered for the L^2 -critical cubic NLS on $\mathbb{R}^2$ in [12] and [20]. Combining the argument in [12, 20] and the almost conservation law established in [4], we can reach the similar conclusion for Hartree type (1.1). We state the main theorem as follows.

Theorem 1.1. *There exist $0 < s_Q < 1$ such that the following is true for any $s_Q < s < 1$. Let $\phi_0 \in H^s(\mathbb{R}^n)$ be such that the corresponding solution ϕ of (1.1) blows up in finite time $T^* > 0$ and let $\lambda(t) > 0$ such that $\frac{(T^*-t)^{\frac{s}{2}}}{\lambda(t)} \rightarrow 0$ as $t \rightarrow T^*$. Then there exists $x(t) \in \mathbb{R}^n$ such that*

$$\limsup_{t \rightarrow T^*} \int_{|x-x(t)| \leq \lambda(t)} |\phi(t, x)|^2 dx \geq \int Q^2.$$

The number $s_Q = \frac{1}{4} + \frac{\sqrt{185}}{20} = 0.9 \dots$ arises from the almost conservation law for the modified energy. For L^2 critical cubic case, the almost conservation law ((3.25) in [12]) reads

$$\sup_{t \in [0, \tilde{T}_{lwp}]} |E[I_N \phi(t)]| \leq |E[I_N \phi_0]| + CN^{-\alpha_4} \|I_N \langle D \rangle \phi_0\|_{L^2}^4 + CN^{-\alpha_6} \|I_N \langle D \rangle \phi_0\|_{L^2}^6$$

with $\alpha_4 = \frac{3}{2}-$ and $\alpha_6 = 2-$. In Hartree case, we have the similar estimate with $\alpha_4 = \alpha_6 = 1-$ as seen in Proposition 3.1 in Section 3. The slow decay rate in Hartree case is partly due that the bilinear estimate used in [12] seems not to make a meaningful gain in higher dimension greater than 2, while L^2 -critical focusing Hartree equation makes sense in $n \geq 3$. The paper is organized as follows.

Notations

Given A, B , we write $A \lesssim B$ to mean that for some universal constant $K > 2$, $A \leq K \cdot B$. We write $A \sim B$ when both $A \lesssim B$ and $B \lesssim A$. The notation $A \ll B$ denotes $B > 3 \cdot A$. We write $\langle A \rangle \equiv (1 + A^2)^{\frac{1}{2}}$, and $\langle \nabla \rangle$ for the operator with Fourier multiplier $(1 + |\xi|^2)^{\frac{1}{2}}$. The symbol ∇ denote the spatial gradient. We will often use the notation $\frac{1}{2}+ \equiv \frac{1}{2} + \epsilon$ for some universal $0 < \epsilon \ll 1$. Similarly, we write $\frac{1}{2}- \equiv \frac{1}{2} - \epsilon$. We use the function space $L_t^q L_x^r$ and $H^{s,p}$ given norms by

$$\|F\|_{L_t^q L_x^r(\mathbb{R}^{n+1})} \equiv \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |F(x, t)|^r dx \right)^{\frac{q}{r}} dt \right)^{\frac{1}{q}},$$

$$\|u\|_{H^{s,p}(\mathbb{R}^n)} \equiv \|\mathcal{F}^{-1}[(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}u]\|_{L^p(\mathbb{R}^n)},$$

where $\mathcal{F}$ is a fourier transform, $1 \leq p, q, r \leq \infty$.

2. The local well-posedness

We refer (q, r) the admissible pair when $2 \leq q \leq \infty$, $2 \leq r \leq \frac{2n}{n-2}$ and

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$$

and state the Strichartz inequality in dimension n .

Proposition 2.1. *Suppose that $(q, r), (\lambda, \eta)$ are any two admissible pairs. Suppose that $u(x, t)$ is a solution of the problem*

$$(2.3) \quad i\partial_t u(x, t) + \Delta u(x, t) = F(x, t), \quad (x, t) \in \mathbb{R}^n \times [0, T],$$

for a data $u(0) \in H^s$, $F \in L_t^{\lambda'} H_x^{s, \eta'}([0, T] \times \mathbb{R}^n)([0, T] \times \mathbb{R}^n)$ where λ' and η' are the Hölder conjugates of λ and η , respectively. Then u belongs to $L_t^q H_x^{s, r}([0, T] \times \mathbb{R}^n) \cap C_t H_x^{s, r}([0, T] \times \mathbb{R}^3)$ and we have the estimate

$$\|u\|_{L_t^q H_x^{s, r}([0, T] \times \mathbb{R}^n)} \lesssim \|u(0)\|_{H^s(\mathbb{R}^n)} + \|F\|_{L_t^{\lambda'} H_x^{s, \eta'}([0, T] \times \mathbb{R}^n)}.$$

For the pure power nonlinearity $\lambda|u|^\alpha u$, the local well-posedness of $i\partial_t u + \frac{1}{2}\Delta u = \lambda|u|^\alpha u$ with the rough data $u(0) \in H^s$, $0 < s < 1$ was proven in [2]. See also [3, 27].

We define the Strichartz norm of functions $\phi : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{C}$ by

$$\|\phi\|_{S_T^0} = \sup_{(q,r) \text{ admissible}} \|\phi\|_{L_t^q L_x^r([0,T] \times \mathbb{R}^n)}.$$

In particular $S_T^0 \subset C_t L_x^2([0, T] \times \mathbb{R}^n)$. Then the Strichartz estimates may be written as

$$\|\phi\|_{S_T^0} \leq \|\phi\|_{L^2} + \|(i\partial_t + \Delta)\phi\|_{L_t^{q'} L_x^{r'}([0,T] \times \mathbb{R}^n)},$$

where (q, r) is any admissible pair. We define

$$\|\phi\|_{S_T^s} = \sup_{(q,r) \text{ admissible}} \|\langle \nabla \rangle^s \phi\|_{L_t^q L_x^r([0,T] \times \mathbb{R}^n)}$$

The local existence theorem of (1.1) is as follows.

Theorem 2.1. *For a given $\phi_0 \in H^s(\mathbb{R}^n)$, $0 < s$, there exists a positive time $T_{lwp} = T(\|\phi_0\|_{H^s})$ and the unique solution ϕ of (1.1), in $\phi \in C_t H_x^s([0, T_{lwp}] \times \mathbb{R}^n) \cap S_T^s$ for every admissible pair (q, r) with*

$$(2.4) \quad T_{lwp} = c_0 \|\phi_0\|_{H^s}^{-\frac{2}{s}}$$

for a constant c_0 and

$$(2.5) \quad \|\phi\|_{S_{T_{lwp}}^s} \leq 2\|\phi_0\|_{H^s}.$$

Proof. Let $S^L(t)$ be the flow map $e^{it\Delta}$ corresponding to the linear Schrödinger equation. Then the integral formulation of (1.1) is

$$\phi(t) = S^L(t)\phi_0 - i \int_0^t S^L(t-\tau) V * |\phi|^2 \phi(\tau) d\tau.$$

We will show that the map $A : \phi \longrightarrow S^L(t)\phi_0 + \int_0^t S^L(t-\tau)[(|x|^{-2} * |\phi|^2)\phi](\tau) d\tau$ is a contraction mapping on the ball $\|\phi\|_{S_T^s} \leq 2M$ when T is chosen later and $\|\phi_0\|_{H^s} < M$.

Let us show A is well defined on X . Applying the linear and the dual Strichartz estimates, we have

$$(2.6) \quad \|A\phi\|_{S_T^s} \lesssim \|\phi_0\|_{H^s} + \| |x|^{-2} * |\phi|^2 \phi \|_{L_t^{\lambda'} H_x^{s, \eta'}([0, T] \times \mathbb{R}^n)}$$

for any admissible (λ, η) . We recall the Leibnitz rule for fractional Sobolev spaces [7]: For $s > 0$, $1 < p < \infty$,

$$\|fg\|_{H^{s,p}} \lesssim \|f\|_{L^{q_1}} \|g\|_{H^{s, q_2}} + \|f\|_{L^{r_1}} \|g\|_{H^{s, r_2}}$$

provided $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{r_1} + \frac{1}{r_2}$, with $q_2, r_2 \in (1, \infty)$ and $q_1, r_1 \in (1, \infty]$.

Let us choose $(\lambda', \eta') = (\frac{4}{3+s}, \frac{2n}{n-s+1})$. The fractional Leibnitz rule, Hardy-Sobolev and Hölder's inequalities lead to

$$\begin{aligned}
 (2.7) \quad & \langle (|x|^{-2} * |\phi|^2) \phi \rangle_{H^{s, \frac{2n}{n-s+1}}} \\
 & \leq \| |x|^{-2} * |\phi|^2 \|_{H^{s, n}} \|\phi\|_{L^{\frac{2n}{n-s-1}}} + \| |x|^{-2} * |\phi|^2 \|_{L^{\frac{n}{1-s}}} \|\phi\|_{H^{s, \frac{2n}{n+s-1}}} \\
 & \lesssim \| |\phi|^2 \|_{H^{s, \frac{n}{n-1}}} \|\phi\|_{L^{\frac{2n}{n-s-1}}} + \|\phi\|_{L^{\frac{2n}{n-s-1}}}^2 \|\phi\|_{H^{s, \frac{2n}{n+s-1}}} \\
 & \lesssim 2 \|\phi\|_{L^{\frac{2n}{n-s-1}}}^2 \|\phi\|_{H^{s, \frac{2n}{n+s-1}}}.
 \end{aligned}$$

By use of the Sobolev embedding $\|\phi\|_{L^{\frac{2n}{n-s-1}}} \lesssim \|\phi\|_{H^{s, \frac{2n}{n+s-1}}}$ we have

$$\| (|x|^{-2} * |\phi|^2) \phi \|_{H^{s, \frac{2n}{n-s+1}}} \lesssim \|\phi\|_{H^{s, \frac{2n}{n+s-1}}}^3.$$

Combining this with (2.6) we find

$$\begin{aligned}
 \|A\phi\|_{S_T^s} & \lesssim \|\phi_0\|_{H^s} + \left(\int_0^T \|\phi\|_{H^{s, \frac{2n}{n+s-1}}}^{\frac{12}{3+s}} dt \right)^{\frac{3+s}{4}} \\
 & \lesssim \|\phi_0\|_{H^s} + T^s \|\phi\|_{L_t^{\frac{4}{1-s}} H_x^{s, \frac{2n}{n+s-1}}([0, T] \times \mathbb{R}^n)}^3 \\
 & \lesssim \|\phi_0\|_{H^s} + T^s \|\phi\|_{S_T^s}^3.
 \end{aligned}$$

We choose T as $T \lesssim \|\phi_0\|_{H^s}^{-\frac{2}{s}}$ so as to $\|A\phi\|_{S_T^s} \lesssim \|\phi_0\|_{H^s}$. It can be similarly argued that A is a contraction. In particular the local solution ϕ satisfies as long as $T \leq T_{lp} = c_0 \|\phi_0\|_{H^s}^{-\frac{2}{s}}$,

$$\|\phi\|_{S_T^s} \leq 2 \|\phi_0\|_{H^s}$$

The uniqueness assertion follows in the similar way as Proposition 4.2 of [2]. $\square$

Let us define the smoothing operator I_N , which sends an H^s function to an H^1 function.

$$(2.8) \quad \widehat{I_N f}(\xi) \equiv m(\xi) \widehat{f}(\xi),$$

where the multiplier $m(\xi)$ is smooth, radially symmetric, nonincreasing in $|\xi|$ and

$$(2.9) \quad m_N(\xi) = \begin{cases} 1 & |\xi| \leq N \\ \left(\frac{N}{|\xi|}\right)^{1-s} & |\xi| \geq 2N. \end{cases}$$

A parameter $N \gg 1$ is sometimes dropped when there will not be a confusion. The operator I_N is the same smoothing operator as in [10] introduced to replace the energy conservation law below H^1 space. As intended, the definition of $m(\xi)$ gives the following relations between $\|I\phi\|_{H^1}$ and $\|\phi\|_{H^s}$ for $0 < s < 1$;

$$(2.10) \quad \begin{aligned} \|I_N\phi\|_{L^2} &\leq \|\phi\|_{L^2} \\ \|\phi\|_{H^s} &\leq \|I_N\langle\nabla\rangle\phi\|_{L^2} \leq N^{1-s}\|\phi\|_{H^s}. \end{aligned}$$

Now we state the modified local well posedness as follows.

Proposition 2.2. *For $\phi_0 \in H^s$, $s > 0$ the (1.1) evolution $\phi_0 \mapsto \phi(t)$ is well-posed on the time interval $[0, \tilde{T}_{lwp}]$ with*

$$(2.11) \quad \tilde{T}_{lwp} = c_0 \|I_N \nabla \phi_0\|_{L^2}^{-\frac{2}{s}},$$

$$(2.12) \quad \|I_N \langle \nabla \rangle \phi\|_{S_{\tilde{T}_{lwp}}^0} \leq 2 \|I_N \langle \nabla \rangle \phi_0\|_{L^2}.$$

Proof. The proof is a direct modification of the arguments used to prove Theorem . We use that the Leibnitz rule holds for the operator $I_N \langle \nabla \rangle$ for $s > 0$. $\square$

3. Almost conservation law of the modified energy

Let us define the iteration space $Z_I(t)$ as

$$Z_I(t) := \|\langle \nabla \rangle I\phi\|_{S_{lwp}^0} = \sup_{(q,r) \text{ admissible}} \|\langle \nabla \rangle I\phi\|_{L_t^q L_x^r([0,t] \times \mathbb{R}^n)}$$

We show the *almost conservation law* of the modified energy. The usual energy (1.2) is shown to be conserved by differentiating in

time

$$\begin{aligned}
& \frac{d}{dt}E(\phi)(t) \\
&= \int_{\mathbb{R}^n} 2\operatorname{Re}\overline{\partial_t\phi}(2(|x|^{-2} * |\phi|^2)\phi - \Delta\phi - 2\partial_t\phi) + (|x|^{-2} * \partial_t|\phi|^2)|\phi|^2 \\
&\quad - (|x|^{-2} * |\phi|^2)\partial_t|\phi|^2 dx \\
&= \int_{\mathbb{R}^n} (|x|^{-2} * \partial_t|\phi|^2)|\phi|^2 - (|x|^{-2} * |\phi|^2)\partial_t|\phi|^2 dx \\
&= 0,
\end{aligned}$$

using the equation (1.1). Since $I\phi$ is not a solution to the equation (1.1), $E(I\phi)(t)$ is not conserved. But still we have a control of the time increment of the modified energy $E(I\phi)(t)$. Differentiating $E(I\phi)(t)$ in time, we obtain

$$\frac{d}{dt}E(I\phi)(t) = \int_{\mathbb{R}^n} 2\operatorname{Re}\overline{\partial_t I\phi} [2(I(|x|^{-2} * |\phi|^2)\phi) - \Delta I\phi - 2i\partial_t I\phi] dx.$$

Then we have

$$\begin{aligned}
& E(I\phi(t)) - E(I\phi(0)) \\
&= 4\operatorname{Re} \int_0^T \int_{\mathbb{R}^n} \overline{\partial_t I\phi} [(|x|^{-2} * |I\phi|^2)I\phi - I(|x|^{-2} * |\phi|^2)\phi] dx dt \\
(3.13) \quad & := E_1(t)
\end{aligned}$$

The following proposition shows that $E(I\phi)$ is an "almost" conserved quantity.

Proposition 3.1. *Assume we have $s > 0$, $N \gg 1$, $\phi_0 \in C_0^\infty(\mathbb{R}^n)$, and a solution of (1.1) on a time interval $[0, T]$, $T \leq \tilde{T}_{lwp}$. Then we have*

$$\begin{aligned}
(3.14) \quad & \sup_{t \in [0, \tilde{T}_{lwp}]} |E(I_N\phi)(t)| \leq E(I_N\phi_0) + CN^{-1+} \|I_N \langle \nabla \rangle \phi_0\|_{L^2}^4 + CN^{-1+} \|I_N \langle \nabla \rangle \phi_0\|_{L^2}^6.
\end{aligned}$$

Proof of Proposition 3.1.

The proof is same as in the defocusing case ([4]) apart that here we can bound $Z_I(t) \leq 2\|\langle \nabla \rangle I\phi_0\|_{L^2}$ by Proposition 2.2. We compute in the frequency space. Applying the Parseval formula to E_1 in (3.13), we

obtain

$$\begin{aligned}
 E_1 &= \operatorname{Re} \int_0^T \int_{\sum_{j=1}^4 \xi_j = 0} \left(1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right) \\
 (3.15) \quad &\times |\xi_2 + \xi_3|^{-(n-2)} \widehat{\partial_t I \phi}(\xi_1) \widehat{I \phi}(\xi_2) \widehat{I \phi}(\xi_3) \widehat{I \phi}(\xi_4) d\xi_1 d\xi_2 d\xi_3 d\xi_4 dt.
 \end{aligned}$$

Now if we use equation (1.1) to substitute for $\partial_t I \phi$ in (3.15), then it is split into two terms as follows:

$$\begin{aligned}
 E_{1a} &\equiv \left| \int_0^T \int_{\sum_{j=1}^4 \xi_j = 0} \left(1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right) |\xi_2 + \xi_3|^{-(n-2)} \right. \\
 &\quad \times \widehat{\Delta I \phi}(\xi_1) \widehat{I \phi}(\xi_2) \widehat{I \phi}(\xi_3) \widehat{I \phi}(\xi_4) d\xi_1 d\xi_2 d\xi_3 d\xi_4 dt \Big|, \\
 E_{1b} &\equiv \left| \int_0^T \int_{\sum_{j=1}^4 \xi_j = 0} \left(1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right) |\xi_2 + \xi_3|^{-(n-2)} \right. \\
 &\quad \times \widehat{(I(|x|^{-2} * |\phi|^2 \phi))}(\xi_1) \widehat{I \phi}(\xi_2) \widehat{I \phi}(\xi_3) \widehat{I \phi}(\xi_4) d\xi_1 d\xi_2 d\xi_3 d\xi_4 dt \Big|.
 \end{aligned}$$

In both cases, we break down ϕ into Littlewood-Paley pieces ϕ_j , each localized in 2^{K_j} in frequency, $\langle \xi_j \rangle \sim 2^{k_j} = N_j$, $k_j = 0, 1, 2, \dots$, and then use a version of Coifman-Meyer estimate for a class of multiplier operators.

Proposition 3.2 (Proposition 6.1 in [6]). *Let $\sigma(\xi)$ be infinitely differentiable so that for all $\alpha \in \mathbb{N}^{nk}$ and all $\xi = (\xi_1, \dots, \xi_k) \in \mathbb{R}^{3k}$. Then there is a constant $c(\alpha)$ with*

$$(3.16) \quad |\partial_\xi^\alpha \sigma(\xi)| \leq c(\alpha) (1 + |\xi|)^{-|\alpha|}.$$

Let the multi-linear operator Λ be given

$$\begin{aligned}
 &[\Lambda(f_1, \dots, f_k)](x) \\
 &= \int_{\mathbb{R}^{nk}} e^{ix(\xi_1 + \dots + \xi_k)} \sigma(\xi_1, \dots, \xi_k) |\xi_2 + \xi_3|^{-(n-2)} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \\
 &\quad \dots \hat{f}_k(\xi_k) d\xi_1 \dots d\xi_k
 \end{aligned}$$

for $k \geq 2$. Then we have

$$\|\Lambda(f_1, \dots, f_k)\|_{L^p} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \dots \|f_k\|_{L^{p_k}}$$

where (p, p_i) is related by $\frac{1}{p} + 1 = \frac{2}{n} + \sum_{i=1}^k \frac{1}{p_i}$.

We first estimate a pointwise bound on the symbol

$$\left| 1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right| \leq B(N_2, N_3, N_4)$$

Factoring $B(N_1, N_2, N_3)$ out of the integral in E_1 , it leaves a symbol σ_1 , which satisfies the condition of Proposition 3.2, as the following:

$$\begin{aligned} & \sum_{N_1, N_2, N_3, N_4} B(N_2, N_3, N_4) \int_0^T \int_{\mathbb{R}^n} [\Lambda(\Delta I\phi_1, I\phi_2, I\phi_3)](\xi_4) \widehat{I\phi_4}(\xi_4) d\xi_4 dt \\ & + \sum_{N_1, N_2, N_3, N_4} B(N_2, N_3, N_4) \int_0^T \int_{\mathbb{R}^n} [\Lambda(I(|x|^{-2} * |\phi_1|^2 \phi_1), I\phi_2, I\phi_3)] \\ & \quad \widehat{I\phi_4}(\xi_4) d\xi_4 dt \end{aligned}$$

where

$$\begin{aligned} & [\Lambda(f, g, h)](x) \\ & = \int_{\mathbb{R}^{3n}} e^{ix(\xi_1 + \xi_2 + \xi_3)} \sigma_1(\xi_1, \xi_2, \xi_3) |\xi_2 + \xi_3|^{-(n-2)} \hat{f}(\xi_1) \hat{g}(\xi_2) \hat{h}(\xi_3) d\xi_1 d\xi_2 d\xi_3. \end{aligned}$$

We shall show that

$$E_{1a} + E_{1b} \lesssim N^{-1+} (Z_I(T))^P$$

for some $P > 0$.

For this aim, we claim that

(3.17)

$$\sum_{N_1, N_2, N_3, N_4} \int_0^T \int_{\mathbb{R}^n} B(N_2, N_3, N_4) [\Lambda(\Delta I\phi_1, I\phi_2, I\phi_3)](x, t) I\phi_4(x, t) dx dt$$

(3.18)

$$\begin{aligned} & + \sum_{N_1, N_2, N_3, N_4} \int_0^T \int_{\mathbb{R}^n} B(N_2, N_3, N_4) [\Lambda(I(|x|^{-2} * |\phi_1|^2 \phi_1), I\phi_2, I\phi_3)](x, t) \\ & I\phi_4(x, t) dx dt \\ & \lesssim N^{-1+} (Z_I(T)^4 + Z_I(T)^6). \end{aligned}$$

From Proposition 3.2, we have

$$(3.19) \quad \|\Lambda(f, g, h)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \|h\|_{L^{p_3}}$$

where $\frac{1}{p} = \frac{2}{n} - 1 + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$.

For the first term E_{1a} , we use (3.19) and Hölder inequality to get

(3.20)

$$\left| \int_0^T \int_{\mathbb{R}^n} B(N_2, N_3, N_4) [\Lambda(\Delta I\phi_1, I\phi_2, I\phi_3)](x, t) I\phi_4(x, t) dx dt \right|$$

$$\lesssim \|\Delta I\phi_1\|_{L_t^{q_1} L_x^{p_1}} \|I\phi_2\|_{L_t^{q_2} L_x^{p_2}} \|I\phi_3\|_{L_t^{q_3} L_x^{p_3}} \|I\phi_4\|_{L_t^{q_4} L_x^{p_4}}$$

where $\sum \frac{1}{p_i} + \frac{2}{n} - 1 = 1$, $\sum \frac{1}{q_i} = 1$. Choosing $\frac{1}{p_i} = \frac{n-1}{2n}$, $\frac{1}{q_i} = \frac{1}{4}$, and using Bernstein inequality, we obtain

$$\text{LHS of (3.20)} \lesssim B(N_2, N_3, N_4) \frac{N_1}{N_2 N_3 N_4} (Z_I(T))^4.$$

We reduce to show

$$(3.21) \quad \sum_{N_1, N_2, N_3, N_4} B(N_2, N_3, N_4) \frac{N_1}{N_2 N_3 N_4} \lesssim N^{-1+\epsilon}$$

By symmetry we may assume $N_2 \geq N_3 \geq N_4$. Then it suffices to consider the following three cases.

Case 1: $N \gg N_2$. We have $m(\xi_i) = 1$ since $\sum_i \xi_i = 0$. So, the symbol

$$\left| 1 - \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right| = 0.$$

Case 2: $N_2 \geq N \gg N_3 \geq N_4$. Since $\sum_i \xi_i = 0$, we have $N_1 \sim N_2$. By the mean value theorem,

$$\left| 1 - \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right| = \left| \frac{m(\xi_2) - m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)} \right|$$

$$\lesssim \frac{|\nabla m(\xi_2) \cdot (\xi_3 + \xi_4)|}{m(\xi_2)} \lesssim \frac{N_3}{N_2}.$$

Thus,

$$\text{LHS of (3.20)} \lesssim \frac{1}{N_2 N_4} (Z_I(T))^4$$

$$\lesssim N^{-1+\epsilon} N_2^{-\epsilon} (Z_I(T))^4$$

Summing up with N_4, N_3, N_2 , we have (3.21).

Case 3: $N_2 \geq N_3 \gtrsim N$. In this case we need to consider two subcases $N_1 \sim N_2$ and $N_2 \gg N_1$ since by $\sum_i \xi_i = 0$ the case $N_1 \gg N_2$ cannot happen.

For the first case, $N_1 \sim N_2$, we estimate

$$\begin{aligned} \left| 1 - \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right| \frac{N_1}{N_2 N_3 N_4} &\lesssim \frac{1}{N_3 m(\xi_3) N_4 m(\xi_4)} \\ &= \frac{1}{N_3 (\frac{N}{\xi_3})^{1-s} N_4 m(\xi_4)} \sim \frac{N^s}{N_3^s} \cdot \frac{1}{N} \frac{1}{N_4 m(\xi_4)} \\ &\lesssim N^{-1+\epsilon} N_3^{-\epsilon} \end{aligned}$$

since $xm(x) \geq 1$ for $x \geq 1$. We can sum up N_4, N_3 directly. But when summing up N_2 , we use the Cauchy-Schwartz inequality with $\phi_i = P_{N_i} I \phi$ as follows:

$$\sum_N P_N \nabla I \phi \cdot P_N \nabla I \phi \leq \left(\sum_N (P_N \nabla I \phi)^2 \right).$$

In the second case, $N_2 \gg N_1$, again by $\sum_i \xi_i = 0$, we have $N_2 \sim N_3$.

$$\begin{aligned} \left| 1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right| \frac{N_1}{N_2 N_3 N_4} &\lesssim \frac{m(\xi_1)}{m(\xi_2)^2 m(\xi_4)} \frac{N_1}{N_2 N_3 N_4} \\ &\sim N_1 m(\xi_1) \frac{1}{N^2} \frac{N^{2s}}{N_2^{2s}} \cdot \frac{1}{N_4 m(\xi_4)} \end{aligned}$$

For our purpose, we want to show

$$\left(\frac{N}{N_2} \right)^{2s} N_1 m(\xi_1) \lesssim N.$$

If $N_1 \leq N$, then $m(\xi_1) = 1$ and this is true. If $N_1 \gtrsim N$, then

$$\left(\frac{N}{N_2} \right)^{2s} N_1 m(\xi_1) = \frac{N^{1+s}}{N_2^s} \cdot \left(\frac{N_1}{N_2} \right)^s = N \frac{(N N_1)^s}{N_2^{2s}} \leq N.$$

This conclude the proof of (3.17). Now we turn to the estimate of E_{1b} . The above analysis is applied to E_{1b} , once we show the following lemma.

Lemma 3.1.

$$(3.22) \quad \|P_M I(|x|^{-2} * (\phi_1 \phi_2)) \phi_3\|_{L_t^4 L_x^{\frac{2n}{n-1}}} \lesssim M^{\frac{1}{2}}(Z_I(T))^3$$

Proof. We divide ϕ into $\phi = \phi_{lo} + \phi_{hi}$ where

$$\begin{aligned} \text{supp } \widehat{\phi}_{lo}(\xi, t) &\subseteq \{|\xi| < 2\} \\ \text{supp } \widehat{\phi}_{hi}(\xi, t) &\subseteq \{|\xi| > 1\} \end{aligned}$$

In the case that all ϕ 's are ϕ_{lo} we simply estimate

$$\begin{aligned}
& \|P_M(I(|x|^{-2} * \phi_{lo} \bar{\phi}_{lo}) \phi_{lo})\|_{L_t^4 L_x^{\frac{2n}{n-1}}} = \|(|x|^{-2} * \phi_{lo} \bar{\phi}_{lo}) \phi_{lo}\|_{L_t^4 L_x^{\frac{2n}{n-1}}} \\
& \lesssim \|\phi_{lo}\|_{L_t^{12} L_x^{\frac{6n}{3n-5}}}^3 \\
& \lesssim \|\langle \nabla \rangle I \phi_{lo}\|_{L_t^{12} L_x^{\frac{6n}{3n-1}}}^3 \\
& \lesssim (Z_I(T))^3.
\end{aligned}$$

When all ϕ 's are ϕ_{hi} , we use Bernstein inequality, Sobolev embedding and the Leibniz rule as following:

$$\begin{aligned}
& \|M^{-\frac{1}{2}} P_M(I(|x|^{-2} * \phi_{hi} \bar{\phi}_{hi}) \phi_{hi})\|_{L_t^4 L_x^{\frac{2n}{n-1}}} \\
& \lesssim \|\nabla^{-1} P_M(I(|x|^{-2} * \phi_{hi} \bar{\phi}_{hi}) \phi_{hi})\|_{L_t^4 L_x^{\frac{2n}{n-1}}} \\
& \lesssim \|\langle \nabla \rangle^{\frac{3}{4}} I(|x|^{-2} * \phi_{hi} \bar{\phi}_{hi}) \phi_{hi}\|_{L_t^4 L_x^{\frac{4n}{2n+3}}} \\
& \lesssim \|\langle \nabla \rangle^{\frac{n}{3n-4}} I \phi_{hi}\|_{L_t^{12} L_x^p}^3 \\
& \text{where } \frac{3}{p} = -\frac{2}{n} + 1 + \frac{2n+3}{4n} \\
& \|\langle \nabla \rangle I \phi_{hi}\|_{L_t^{12} L_x^{\frac{6n}{3n-1}}}^3 \\
& \lesssim (Z_I(T))^3
\end{aligned}$$

where note that $\frac{3n-1}{6n} - \frac{3}{4n} = -\frac{1}{p}$.

The remaining $lo - hi$ cases are controlled in a similar manner to the $hi - hi$ case. We omit the detail here. $\square$

Hence, we have shown (3.17), (3.18) and so conclude the proof. $\square$

From the proposition above we induce that H^s kinetic energy dominates modified total energy, which corresponds to Proposition 2.1 in [12]. Before we state the proposition, let us define the *blowup parameter* and *blowup rate* as follows:

$$(3.23) \quad \Lambda(t) = \sup_{0 \leq \tau \leq t} \|u(\tau)\|_{H^s},$$

$$(3.24) \quad \rho(t) = \frac{\|\nabla Q\|_{L^2}}{\|\nabla I_N \phi(t, \cdot)\|_{L^2}}.$$

Proposition 3.3. *There exists $s_Q \leq \frac{9+\sqrt{721}}{40}$ such that for all $s_Q < s < 1$ there exists $p(s) < 2$ with the following holding true: if $H^s \in$*

$\phi_0 \mapsto \phi(t)$ solves (1.1) on a forward maximal finite existence $[0, T^*)$, for all $T < T^*$ there exists $N = N(T)$ such that

$$(3.25) \quad |E(I_N(T)\phi(T))| \leq C_0(\Lambda(T))^{p(s)}$$

with $C_0 = C_0(s, T^*, \|\phi_0\|_{H^s})$. More precisely $N(T) = C(\Lambda(T))^{\frac{p(s)}{2(1-s)}}$, $p(s) = \frac{6+\frac{2}{s}}{\frac{3}{2}-(1-s)(4+\frac{2}{s})}2(1-s)$.

The proof of the above proposition is identical with the proof of Proposition 2.1 in [12] once we have the modified local well posedness (2.12) and the almost conservation law (3.14). The choice of $p(s)$ is relevant to how fast the increment of the modified energy approximates zero, which is calculated in Proposition 3.1. The number s_Q is a consequence of $p(s) < 2$.

4. Proof of Theorem 1.1

Let us now prove Theorem 1.1. The proof is parallel with that of the corollary 1.10 in [20]. There T. Himidi- S. Kerrani nicely combined ideas in [12] with a profile decomposition theorem following the work by P.Gerard which we state in the below.

Proposition 4.1. *Let $\mathbf{v} = \{v_n\}_{n=1}^\infty$ be a bounded sequence in $H^1(\mathbb{R}^n)$. Then there exist a subsequence of $\{v_n\}_{n=1}^\infty$ (still denoted $\{v_n\}_{n=1}^\infty$), a family $\{\mathbf{x}^j\}_{j=1}^\infty$ of sequences in $\mathbb{R}^d$, and a sequence $\{V^j\}_{j=1}^\infty$ of H^1 functions such that*

1. for every $k \neq j$, $|x_n^k - x_n^j| \rightarrow \infty$ as $n \rightarrow \infty$;
2. for every $l \geq 1$ and every $x \in \mathbb{R}^n$,

$$v_n(x) = \sum_{j=1}^l V^j(x - x_n^j) + v_n^l(x),$$

with

$$\limsup_{n \rightarrow \infty} \|v_n^l\|_{L^p(\mathbb{R}^n)} \rightarrow 0 \text{ as } l \rightarrow \infty,$$

for every $p \in (2, 2^*)$.

Moreover, we have, as $n \rightarrow \infty$,

$$\|v_n\|_{L^2}^2 = \sum_{j=1}^l \|V^j\|_{L^2}^2 + \|v_n^l\|_{L^2}^2 + o(1)$$

and

$$\|\nabla v_n\|_{L^2}^2 = \sum_{j=1}^l \|\nabla V^j\|_{L^2}^2 + \|\nabla v_n^l\|_{L^2}^2 + o(1).$$

As in [12], we choose $\{t_n\}_{n=1}^\infty$ to be a sequence such that $t_n \uparrow T^*$ and for each t_n

$$\|\phi(t_n)\|_{H^s} = \Lambda(t_n).$$

We set

$$\psi_n = \rho_n^{\frac{n}{2}} I_N \phi(t_n, \rho_n x),$$

where

$$\rho_n = \frac{\|\nabla Q\|_{L^2}}{\|\nabla_N \phi(t_n, \cdot)\|_{L^2}} \lesssim \frac{1}{\|\phi(t_n, \cdot)\|_{H^s}} = \frac{1}{\Lambda(t_n)}$$

by (2.10). Also from the modified local well posedness, it holds that

$$(4.26) \quad \rho_n \leq A(T^* - t_n)^{\frac{s}{2}}$$

for some constant $A > 0$. Hence $\|\psi_n\|_{H^1}$ is normalized as

$$\|\psi_n\|_{L^2} \leq \|\phi_0\|_{L^2}, \quad \|\nabla \psi_n\|_{L^2} = \|\nabla Q\|_{L^2}.$$

On the other hands, the energy of ψ_n satisfies

$$E(\psi_n) = \rho_n^2 E(I_{N(t_n)} \phi(t_n)) \leq \rho_n^2 (\Lambda(t_n))^{p(s)-2}$$

by Proposition 3.3. Since $\Lambda(t_n) \rightarrow \infty$ and $p(s) < 2$, it holds that

$$E(\psi_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which yields

$$\int |x|^{-2} * |\psi_n|^2(x) |\psi_n(x)|^2 dx \rightarrow 2 \|\nabla Q\|_{L^2}.$$

Note that the sequence $\{\psi_n\}$ satisfies the condition of Proposition 4.1, thus it admits the decomposition as

$$\psi_n(x) = \sum_{j=1}^l U^{(j)}(x - x_n^j) + r_n^l(x)$$

up to subsequence, where $U^{(j)}$, x_n^j , $r_n^l(x)$ meet the conditions in Proposition 4.1. Under the circumstance it is shown in [23] that there exists j_0 such that

$$(4.27) \quad \psi_n(\cdot + x_n) \rightharpoonup U^{j_0} \in H^1 \text{ with } \|U^{j_0}\|_{L^2} \geq \|Q\|_{L^2}.$$

This is a replacement of Theorem 5 in [20] for the L^2 critical Hartree equation. Let us denote U^{j_0} by V . Coming back to $\{\phi_n\}_{n=1}^\infty$ one has

$$\rho_n^{\frac{n}{2}} I_N \phi(t_n, \rho_n x \cdot + x_n^{j_0}) \rightharpoonup V \text{ in } H^1.$$

It turns into

$$(4.28) \quad \rho_n^{\frac{n}{2}} \phi(t_n, \rho_n x + x_n) \rightharpoonup V \text{ in } H^{\bar{s}-} \text{ for } \bar{s} = \frac{s+1}{4-2s},$$

since

$$\begin{aligned} \|\rho_n^{\frac{n}{2}} (I_N \phi(t_n) - \phi(t_n))(\rho \cdot + x_n)\|_{H^{\bar{s}}}^2 &\leq \rho_n^{2\bar{s}} \int_{|\xi| \geq N} (N^{1-s} |\xi|^{s-1} - 1)^2 |\xi|^{2\bar{s}} |\hat{\phi}(t_n)|^2 d\xi \\ &\leq \rho_n^{2\bar{s}} N^{2(\bar{s}-s)} \|\phi(t_n)\|_{H^s}^2 \\ &\leq (\Lambda(t_n))^{\frac{p(s)(\bar{s}-s)}{2(1-s)} + 1 - \bar{s}} \end{aligned}$$

by using $\rho_n \lesssim \Lambda^{-1}(t_n)$ and $N(t_n) = C(\Lambda(t_n))^{\frac{p(s)}{2(1-s)}}$ in Proposition 3.3. Note that $\frac{p(s)(\bar{s}-s)}{2(1-s)} + 1 - \bar{s} < 0$ if $\bar{s} < \tilde{s}$. Thus we have

$$\|\rho_n^{\frac{n}{2}} (I_N \phi(t_n) - \phi(t_n))(\rho \cdot + x_n)\|_{H^{\bar{s}-}}^2 \rightarrow 0, \text{ as } n \rightarrow \infty,$$

which implies (4.28).

Now we are ready to prove Theorem 1.1. From (4.28) it follows that

$$\lim_{n \rightarrow \infty} \rho_n^n \int_{|x| \leq R} |\phi(t_n, \rho_n x + x_n)|^2 dx \geq \int_{|x| \leq R} |V|^2 dx$$

for every $R > 0$. By change of variable,

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^n} \int_{|x-y| \leq R \rho_n} |\phi(t_n, x)|^2 dx \geq \int_{|x| \leq R} |V|^2 dx.$$

Combining the assumption $\frac{(T^*-t)^{\frac{\bar{s}}{2}}}{\lambda(t)} \rightarrow 0$ as $t \rightarrow T^*$ and (4.26), we have $\frac{\rho_n}{\lambda(t_n)} \rightarrow 0$, and then

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^n} \int_{|x-y| \leq \lambda(t_n)} |\phi(t_n, x)|^2 dx \geq \int_{|x| \leq R} |V|^2 dx$$

for every $R > 0$. Letting R to go to infinity we obtain

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^n} \int_{|x-y| \leq \lambda(t_n)} |\phi(t_n, x)|^2 dx \geq \int_{\mathbb{R}^n} |V|^2 dx \geq \|Q\|_{L^2}^2.$$

This proves

$$\limsup_{t \uparrow T^*} \sup_{y \in \mathbb{R}^n} \int_{|x-y| \leq \lambda(t)} |\phi(t, x)|^2 dx \geq \|Q\|_{L^2}^2.$$

Since the map $y \rightarrow \int_{|x-y| \leq \lambda(t_n)} |\phi(t_n, x)|^2 dx$ is continuous and goes to 0 at infinity for each t , there exists a family $x(t)$ such that

$$\sup_{y \in \mathbb{R}^n} \int_{|x-y| \leq \lambda(t)} |\phi(t, x)|^2 dx = \int_{|x-x(t)| \leq \lambda(t)} |\phi(t, x)|^2 dx,$$

which concludes the proof of Theorem 1.1.

References

- [1] J. Bourgain, *Global solutions of nonlinear Schrödinger equations*, Amer. Math. Soc., Providence, RI, 1999.
- [2] T. Cazenave, F. Weissler, *The cauchy problem for the critical nonlinear Schrödinger equation in H^s* , Nonlinear Anal. T.M.A. **14** (1990), 807-836.
- [3] T. Cazenave, *Semilinear Schrödinger Equations*. Courant Lecture Notes **10**, New York (2003).
- [4] M. Chae and S. Kwon *Global well-posedness for the L^2 critical Hartree on $\mathbb{R}^n$, $n \geq 3$* , Comm. Pure. Appl. Analysis, accepted.
- [5] R. Coifman and Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque, Société Mathématique de France **57**, 1978.
- [6] M. Chae, S. Hong, J. Kim, C. Yang, *Scattering theory below energy for a class of Hartree type*, Comm. pde **33**(2008), 321-348
- [7] M. Christ, M. Weinstein, *Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation*, J. Funct. Anal. **100** (1991), 87-109.
- [8] J. Colliander, M. Grillakis, and N. Tzirakis *Improved interaction Morawetz inequalities for the cubic nonlinear Schrödinger equation in 2d*, To appear in International Mathematics Research Notices.
- [9] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Almost conservation laws and global rough solutions to a Nonlinear Schrödinger Equation*, Math. Res. Letters, **9** 1-24, 2002.
- [10] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on $\mathbb{R}^3$* , Comm. Pure Appl. Math. **57** (2004), 987-1014.
- [11] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Viriel, Morawetz, and interaction Morawetz inequalities*, preprint.
- [12] J. Colliander, S. Raynor, C. Sulem, and J.D. Wright, *Ground state mass concentration in the L^2 -critical nonlinear Schrödinger equation below H^1* , Math. Res. Lett. **12** (2005), 357-375.
- [13] D. de Silva, N. Pavlovic, G. Staffilani, and N. Tzirakis, *Global Well-Posedness for the L^2 -critical nonlinear Schrödinger equation in higher dimensions*, To appear in Comm. Pure. Appl. Anal.

- [14] D. de Silva, N. Pavlovic, G. Staffillani, and N. Tzirakis, *Global well-posedness and polynomial bounds for the defocusing nonlinear Schrödinger equation in 1d*, To appear in Comm. PDE.
- [15] L. Erdős, H.-T. Yau, *Derivation of the nonlinear Schrödinger equation from a many body Coulomb system*, Adv. Theor. Math. Phys. **5** no. 6 (2001), 1169-1205.
- [16] I. Gasser, P. A. Markowich, and B. Perthame, *Dispersion and Moment Lemmas Revisited*, J. Diff. Equations, **156** (1999), 254-281.
- [17] M. Grillakis, *On nonlinear Schrödinger equations*, Comm. Partial Differential Equations, **25** (2000), 1827-1844.
- [18] J. Ginibre and G. Velo, *Smoothing properties and retarded estimates for some dispersive evolution equations*, Comm. Math. Phys. **123** (1989), 535-573.
- [19] J. Ginibre and G. Velo, *Scattering theory in the energy space for a class of Hartree equations*, Contemp. Math. **263** (2000), 29-60.
- [20] T. Hmidi, S. Keraani, *Remarks on the blowup for the L^2 -critical nonlinear Schrödinger equations*, SIAM J. Math. Anal. **38** (2006), no. 4, 1035-1047.
- [21] J. Lin and W. Strauss, *Decay and scattering of solutions of a nonlinear Schrödinger equation*, J. Funct. Anal. **30** (1978), 245-263.
- [22] C. Miao, G. Xu, and L. Zhang, *Global well-posedness and scattering for the mass-critical Hartree equation with radial data*, preprint, arXiv:0801.3925.
- [23] C. Miao, G. Xu, and L. Zhang, *On the blow up phenomenon for the L^2 -critical focusing Hartree equation in $\mathbb{R}^3$* , preprint, arXiv:0708.2614.
- [24] C. Miao, G. Xu, and L. Zhang, *Global well-posedness and scattering for the defocusing $H^{\frac{1}{2}}$ -subcritical Hartree equation in $\mathbb{R}^d$* , preprint, arXiv:0805.3378.
- [25] K. Nakanishi, *Energy scattering for Hartree equations*, Math. Res. Lett. **6** (1999), 107-118.
- [26] H. Spohn, *On the Vlasov hierarchy*, Math. Methods Appl. Sci. **3** (1981), no. 4, 445-455.
- [27] T. Tao, *Nonlinear Dispersive equations, Local and global analysis.*, CBMS Regional Conference Series in Mathematics, 106. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006.
- [28] T. Tao, M. Visan, and X. Zhang, *The Schrödinger equation with combined power-type nonlinearities*, Comm. PDE **32** (2007), 1281-1343.
- [29] J. Krieger, E. Lenzmann, P. Raphaël, *On stability of pseudo-conformal blowup for L^2 -critical equation*, preprint, arXiv:0808.2324.

Myeongju Chae
 Department of Applied Mathematics,
 Hankyong National University,
 Ansong 456-749, Korea
E-mail: mchae@hknu.ac.kr