A NOTE ON END PROPERTIES OF MARCINKIEWICZ INTEGRAL

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ABSTRACT. In this note we give the mapping properties of the Marcinkiewicz integral μ_Ω at some end spaces. More precisely, we first prove that μ_Ω is a bounded operator from $H^{1,\infty}(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. As a corollary of the results above, we obtain again the weak type $(1,1)$ boundedness of μ_Ω , but the condition assumed on Ω is weaker than Stein's condition. Finally, we show that μ_Ω is bounded from $\text{BMO}(\mathbb{R}^n)$ to $\text{BMO}(\mathbb{R}^n)$. The results in this note are the extensions of the results obtained by Lee and Rim recently.

1. Introduction

Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n$ ($n \geq 2$) equipped with the Lebesgue measure $d\sigma$. Let Ω be a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying $\Omega \in L^1(S^{n-1})$ and

$$(1.1) \quad \int_{S^{n-1}} \Omega(x') d\sigma(x') = 0,$$

where $x' = x/|x|$ for any $x \neq 0$. Then the Marcinkiewicz integral operator μ_Ω of higher dimension is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy.$$

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It is well known that the operator μ_Ω was first defined by Stein in [9]. He proved that if Ω satisfies a Lip_α ($0 < \alpha \leq 1$) condition on S^{n-1} , then μ_Ω is of type (p,p) for $1 < p \leq 2$ and of weak type $(1,1)$. Subsequently Benedek, Calderón and Panzone showed in [1] that if $\Omega \in \mathcal{C}^1(S^{n-1})$ then μ_Ω is of type (p,p) for $1 < p < \infty$. In 2000, Ding, Fan and Pan[4] proved the following result:

THEOREM A. *Suppose that $\Omega \in H^1(S^{n-1})$ and satisfies (1.1). Then for $1 < p < \infty$, there exists a constant $C > 0$, independent of f , such that $\|\mu_\Omega(f)\|_{L^p} \leq C\|f\|_{L^p}$.*

Here $H^1(S^{n-1})$ denotes the Hardy space on the unit sphere (see [2] and [3] for the definition and properties of $H^1(S^{n-1})$). Note that the following relationship between $H^1(S^{n-1})$ and the other function spaces on S^{n-1} :

$$(1.2) \quad \begin{aligned} \text{Lip}_\alpha(S^{n-1}) &\subset L^q(S^{n-1}) \subset L \log^+ L(S^{n-1}) \\ &\subset H^1(S^{n-1}) \subset L^1(S^{n-1}), \end{aligned}$$

where all inclusions are proper for any $0 < \alpha \leq 1$ and $q > 1$. Hence, Theorem A improved the results in [9] (for $1 < p \leq 2$) and [1] mentioned above.

Recently, Lee and Rim[8] discussed the (H^1, L^1) , $(L^2 \cap L^\infty, BMO)$ and (L^p, L^p) ($1 < p < \infty$) boundedness of μ_Ω when there exist constants $C > 0$ and $\rho > 1$ such that

$$(1.3) \quad |\Omega(x') - \Omega(y')| \leq \frac{C}{(\log \frac{1}{|x'-y'|})^\rho},$$

holds uniformly in $x', y' \in S^{n-1}$.

It is easy to see that the condition (1.3) is weaker than the Lip_α ($0 < \alpha \leq 1$) condition on S^{n-1} . In addition, it is also obvious if Ω satisfies (1.3) for some $\rho > 1$, then by (1.2)

$$(1.4) \quad \Omega \in L^\infty(S^{n-1}) \subset L^q(S^{n-1}) (1 < q < \infty) \subset H^1(S^{n-1}).$$

In this note we will consider the mapping properties of the Marcinkiewicz integral μ_Ω on some end function spaces when Ω satisfies (1.3) for $\rho > 2$. More precise, we will first prove that μ_Ω is bounded from $H^{1,\infty}(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. Here $H^{1,\infty}(\mathbb{R}^n)$ denotes the weak Hardy space, which was introduced first by Fefferman and Soria in [6]. To state our results, let us recall the definition of $H^{1,\infty}(\mathbb{R}^n)$.

DEFINITION 1. Suppose that $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\int \phi \neq 0$. Denote $f_+^*(x) = \sup_{t>0} |(\phi_t * f)(x)|$, where $\phi_t(x) = t^{-n}\phi(x/t)$. A function f is said to belong to the weak Hardy space $H^{1,\infty}(\mathbb{R}^n)$ if $f_+^* \in L^{1,\infty}(\mathbb{R}^n)$, i.e., there exists a constant $C > 0$ such that for any $\beta > 0$

$$\sup_{\beta>0} \beta |\{x \in \mathbb{R}^n : f_+^*(x) > \beta\}| \leq C.$$

The smallest constant C satisfying the above inequality is called the $H^{1,\infty}(\mathbb{R}^n)$ norm of f , which is denoted by $\|f\|_{H^{1,\infty}}$.

We have the following conclusions.

THEOREM 1. Let Ω satisfy (1.1) and (1.3) for some $\rho > 2$. Then μ_Ω is bounded operator from $H^{1,\infty}(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. That is, there exists a constant $C > 0$ such that for any $f \in H^{1,\infty}(\mathbb{R}^n)$ and $\beta > 0$,

$$(1.5) \quad |\{x : \mu_\Omega f(x) > \beta\}| \leq C \|f\|_{H^{1,\infty}} / \beta.$$

REMARK 1. Let M be the Hardy-Littlewood maximal operator. It is known that if $\phi \in C_0^\infty(\mathbb{R}^n)$ and $f \in L^1(\mathbb{R}^n)$, then $f_+^*(x) \leq CM(f)(x)$ (see [10, pp. 62, Theorem 2]). Hence, by the weak (1,1) boundedness of M , it is easy to see that the space $L^1(\mathbb{R}^n)$ is continuously embedded as a subspace of the space $H^{1,\infty}(\mathbb{R}^n)$, and $\|f\|_{H^{1,\infty}} \leq C \|f\|_{L^1}$ for any $f \in L^1(\mathbb{R}^n)$. Thus we get immediately the following corollary of Theorem 1.

COROLLARY. If Ω satisfy (1.1) and (1.3) for some $\rho > 2$, then μ_Ω is of weak type (1,1). That is, there exists a constant $C > 0$ such that for any $f \in L^1(\mathbb{R}^n)$ and $\beta > 0$, $|\{x : \mu_\Omega f(x) > \beta\}| \leq C \|f\|_{L^1} / \beta$.

REMARK 2. In [9], Stein proved that μ_Ω is of weak (1,1) if $\Omega \in \text{Lip}_\alpha(S^{n-1})$ for $0 < \alpha \leq 1$. In [5], Fan and Sato showed that if $\Omega \in L \log^+ L(S^{n-1})$, then μ_Ω is also of weak (1,1). Hence, the condition (1.3) of Corollary is weaker than Stein's condition but stronger than Fan-Sato's condition.

The second result in this note shows that μ_Ω is also bounded from $\text{BMO}(\mathbb{R}^n)$ to $\text{BMO}(\mathbb{R}^n)$.

DEFINITION 2. A locally integrable function $f(x)$ is said to belong to $\text{BMO}(\mathbb{R}^n)$ if

$$\|f\|_{\text{BMO}} := \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx < \infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes, and f_Q denotes the average of f over Q , i.e. $f_Q = \frac{1}{|Q|} \int_Q f(x) dx$.

A well-known important fact on $BMO(\mathbb{R}^n)$ is that for any $1 \leq p < \infty$,

$$\|f\|_{BMO} \sim \sup_Q \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p dx \right)^{1/p} < \infty.$$

THEOREM 2. *Let Ω satisfy (1.1) and (1.3) for some $\rho > 2$. Suppose that $f(x) \in BMO(\mathbb{R}^n)$ and there is a measurable set $E \subset \mathbb{R}^n$ with $|E| > 0$ such that $\mu_\Omega(f)(x) < \infty$ for any $x \in E$. Then $\mu_\Omega(f)(x) < \infty$ a.e. on $\mathbb{R}^n$ and*

$$(1.6) \quad \|\mu_\Omega(f)\|_{BMO} \leq C \|f\|_{BMO},$$

where the constant C is independent of f .

REMARK 3. In [7], Han gave an example to show that there exist an Ω and $f_0 \in L^\infty(\mathbb{R}^n)$ such that $\mu_\Omega(f_0)(x) = \infty$ for any $x \in \mathbb{R}^n$. Therefore, μ_Ω can not map an $L^\infty(\mathbb{R}^n)$ function into a $BMO(\mathbb{R}^n)$ function in general.

2. Proof of Theorem 1

We need the following Fefferman-Soria's decomposition theorem of function in $H^{1,\infty}(\mathbb{R}^n)$ (See [6]).

THEOREM B. *Given a function $f \in H^{1,\infty}(\mathbb{R}^n)$, there exists a sequence of bounded functions $\{f_k\}_{k=-\infty}^\infty$ with the following properties:*

- (a) $f - \sum_{|k| \leq N} f_k$ tends to zero in the sense of distributions;
- (b) Each f_k may be further decomposed as $f_k = \sum_i h_i^k$ in L^1 and $\{h_i^k\}$ satisfies
 - (i) $\text{supp}(h_i^k) \subset B_i^k := B(x_i^k, r_i^k)$, where $B(x, r)$ denotes the ball in $\mathbb{R}^n$ with the center at x and radius r . Moreover, $\sum_i |B_i^k| \leq C_1 2^{-k}$ and $\sum_i \chi_{B_i^k}(x) \leq C$, where $C_1 \sim \|f\|_{H^{1,\infty}}$;
 - (ii) $\|h_i^k\|_\infty \leq C 2^k$, where C is independent of i and k ;
 - (iii) $\int h_i^k(x) dx = 0$, for every i and k .

Now let us turn to the proof of Theorem 1. We need to show that there exists a constant $C > 0$ such that (1.5) holds for any $f \in H^{1,\infty}(\mathbb{R}^n)$

and $\beta > 0$. To do this, for any given $\beta > 0$, we take k_0 satisfying $2^{k_0} \leq \beta < 2^{k_0+1}$, then by Theorem B we may write

$$f = \sum_{k=-\infty}^{k_0} f_k + \sum_{k=k_0+1}^{\infty} f_k := F_1 + F_2 \quad \text{and} \quad f_k = \sum_i h_i^k,$$

where h_i^k satisfies (i), (ii) and (iii). Now denote $A_k = \text{supp} f_k$, then $A_k = \cup_i B_i^k$ and $|A_k| \leq \sum_i |B_i^k| \leq C2^{-k} \|f\|_{H^{1,\infty}}$. Note that $\|f_k\|_{\infty} \leq C2^k$, we have

$$\begin{aligned} \|F_1\|_2 &\leq \sum_{k=-\infty}^{k_0} \|f_k\|_2 \leq C \sum_{k=-\infty}^{k_0} 2^k |A_k|^{1/2} \\ &\leq C \sum_{k=-\infty}^{k_0} 2^{k/2} \|f\|_{H^{1,\infty}}^{1/2} \leq C \|f\|_{H^{1,\infty}}^{1/2} \beta^{1/2}. \end{aligned}$$

Using (1.4) and Theorem A, it is easy to see that

$$\begin{aligned} (2.1) \quad |\{x : \mu_{\Omega}(F_1)(x) > \beta\}| &\leq \|\mu_{\Omega}(F_1)\|_2^2 / \beta^2 \\ &\leq C \|F_1\|_2^2 / \beta^2 \\ &\leq C \|f\|_{H^{1,\infty}} / \beta. \end{aligned}$$

On the other hand, let

$$\bar{B}_i^k = B(x_i^k, 2(3/2)^{(k-k_0)/n} r_i^k) \quad \text{and} \quad B_{k_0} = \bigcup_{k=k_0+1}^{\infty} \bigcup_i \bar{B}_i^k.$$

We have

$$\begin{aligned} (2.2) \quad |B_{k_0}| &\leq \sum_{k=k_0+1}^{\infty} \sum_i |\bar{B}_i^k| \\ &= \sum_{k=k_0+1}^{\infty} \sum_i 2^n (3/2)^{k-k_0} |B_i^k| \\ &\leq C \sum_{k=k_0+1}^{\infty} (3/2)^{k-k_0} 2^{-k} \|f\|_{H^{1,\infty}} \\ &\leq C \|f\|_{H^{1,\infty}} / \beta. \end{aligned}$$

Thus, to prove (1.5) it suffices to show

$$(2.3) \quad |\{x \in (B_{k_0})^c : \mu_\Omega(F_2)(x) > \beta\}| \leq C\|f\|_{H^{1,\infty}}/\beta.$$

Using the Minkowski inequality, we get

$$\begin{aligned} & \int_{(B_{k_0})^c} \mu_\Omega(F_2)(x) dx \\ &= \int_{(B_{k_0})^c} \left(\int_0^\infty \left| \sum_{k=k_0+1}^\infty \sum_i \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} h_i^k(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} dx \\ &\leq \int_{(B_{k_0})^c} \sum_{k=k_0+1}^\infty \sum_i \left(\int_0^\infty \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} h_i^k(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} dx \\ &\leq C \sum_{k=k_0+1}^\infty \sum_i (I_1 + I_2), \end{aligned}$$

where

$$I_1 = \int_{(B_{k_0})^c} \left(\int_0^{|x-x_i^k|+2r_i^k} \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} h_i^k(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} dx$$

and

$$I_2 = \int_{(B_{k_0})^c} \left(\int_{|x-x_i^k|+2r_i^k}^\infty \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} h_i^k(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} dx.$$

It is easy to check that when $y \in B_i^k$, for any $x \in (B_{k_0})^c$, $|x - x_i^k| \sim |x - y| \sim |x - x_i^k| + 2r_i^k$, and

$$(2.4) \quad \left| \frac{1}{(|x - x_i^k| + 2r_i^k)^2} - \frac{1}{|x - y|^2} \right| \leq C \frac{r_i^k}{|x - y|^3}.$$

Hence by $\Omega \in L^\infty(S^{n-1})$ and (2.4), we have

$$\begin{aligned} I_1 &\leq \int_{(B_{k_0})^c} \int_{B_i^k} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |h_i^k(y)| \left(\int_{|x-y|\leq t \leq |x-x_i^k|+2r_i^k} \frac{1}{t^3} dt \right)^{1/2} dy dx \\ &\leq C 2^k \|\Omega\|_{L^\infty(S^{n-1})} \\ &\quad \times \int_{(B_{k_0})^c} \int_{B_i^k} \frac{1}{|x-y|^{n-1}} \left| \frac{1}{(|x-x_i^k| + 2r_i^k)^2} - \frac{1}{|x-y|^2} \right|^{1/2} dy dx \\ &\leq C 2^k (r_i^k)^{1/2} \int_{B_i^k} \int_{2(3/2)^{(k-k_0)/n} r_i^k}^\infty \frac{1}{s^{1+1/2}} ds dy \\ &\leq C 2^k (r_i^k)^{1/2} |B_i^k| \left(\frac{2}{3}\right)^{(k-k_0)/(2n)} (r_i^k)^{-1/2}. \end{aligned}$$

Thus

$$(2.5) \quad \sum_{k=k_0+1}^{\infty} \sum_i I_1 \leq C \sum_{k=k_0+1}^{\infty} \left(\frac{2}{3}\right)^{(k-k_0)/(2n)} \|f\|_{H^{1,\infty}} \leq C \|f\|_{H^{1,\infty}}.$$

Now let us consider I_2 . As above we know that if $x \in (B_{k_0})^c$, $y \in B_i^k$, then $|x - x_i^k| \sim |x - y| \sim |x - x_i^k| + 2r_i^k$, and

$$(2.6) \quad \left| \frac{x - y}{|x - y|} - \frac{x - x_i^k}{|x - x_i^k|} \right| \leq C \frac{r_i^k}{|x - x_i^k|}.$$

Thus by (1.3) and (2.6), we get

$$(2.7) \quad \begin{aligned} |\Omega(x - y) - \Omega(x - x_i^k)| &= \left| \Omega\left(\frac{x - y}{|x - y|}\right) - \Omega\left(\frac{x - x_i^k}{|x - x_i^k|}\right) \right| \\ &\leq \frac{C}{\left(\log \frac{|x - x_i^k|}{r_i^k}\right)^\rho}. \end{aligned}$$

Applying (2.7) we have

$$(2.8) \quad \begin{aligned} &\left| \frac{\Omega(x - y)}{|x - y|^{n-1}} - \frac{\Omega(x - x_i^k)}{|x - x_i^k|^{n-1}} \right| \\ &\leq \frac{|\Omega(x - y) - \Omega(x - x_i^k)|}{|x - x_i^k|^{n-1}} + |\Omega(x - y)| \left| \frac{1}{|x - y|^{n-1}} - \frac{1}{|x - x_i^k|^{n-1}} \right| \\ &\leq \frac{C}{|x - x_i^k|^{n-1} \left(\log \frac{|x - x_i^k|}{r_i^k}\right)^\rho} + \frac{r_i^k}{|x - x_i^k|^n} \\ &\leq \frac{C}{|x - x_i^k|^{n-1} \left(\log \frac{|x - x_i^k|}{r_i^k}\right)^\rho}. \end{aligned}$$

Now let us return to the estimate of I_2 . Note that when $y \in B_i^k$, for any $x \in (B_{k_0})^c$ and $t > |x - x_i^k| + 2r_i^k$, we have $B_i^k \subset \{y \in \mathbb{R}^n : |x - y| \leq t\}$. By the cancellation property of $h_i^k(y)$ and (2.8), we get

$$\begin{aligned} I_2 &= \int_{(B_{k_0})^c} \left(\int_{|x - x_i^k| + 2r_i^k}^{\infty} \left| \int_{|x - y| \leq t} \left(\frac{\Omega(x - y)}{|x - y|^{n-1}} - \frac{\Omega(x - x_i^k)}{|x - x_i^k|^{n-1}} \right) \right. \right. \\ &\quad \left. \left. \times h_i^k(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} dx \end{aligned}$$

$$\begin{aligned}
&\leq C2^k \int_{(B_{k_0})^c} \int_{B_i^k} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_i^k)}{|x-x_i^k|^{n-1}} \right| \\
&\quad \times \left(\int_{|x-y| \leq t, |x-x_i^k| < t} \frac{1}{t^3} dt \right)^{1/2} dy dx \\
&\leq C2^k \int_{B_i^k} \int_{|x-x_i^k| > (3/2)^{(k-k_0)/n} r_i^k} \frac{1}{|x-x_i^k|^{n-1} \left(\log \frac{|x-x_i^k|}{r_i^k} \right)^\rho} \\
&\quad \cdot \frac{1}{|x-x_i^k|} dx dy \\
&\leq C2^k |B_i^k| \int_{(3/2)^{(k-k_0)/n} r_i^k}^\infty \frac{ds}{s \left(\log \frac{s}{r_i^k} \right)^\rho} ds.
\end{aligned}$$

Note that $\rho > 2$

$$\begin{aligned}
&\int_{(3/2)^{(k-k_0)/n} r_i^k}^\infty \frac{ds}{s \left(\log \frac{s}{r_i^k} \right)^\rho} ds \\
&= \sum_{j=1}^\infty \int_{(3/2)^{j(k-k_0)/n}}^{(3/2)^{(j+1)(k-k_0)/n}} \frac{ds}{s (\log s)^\rho} ds \\
(2.9) \quad &\leq \sum_{j=1}^\infty \frac{1}{\left(\log(3/2)^{j(k-k_0)/n} \right)^\rho} \int_{(3/2)^{j(k-k_0)/n}}^{(3/2)^{(j+1)(k-k_0)/n}} \frac{ds}{s} \\
&= \sum_{j=1}^\infty \frac{\log(3/2)^{(k-k_0)/n}}{\left(\log(3/2)^{j(k-k_0)/n} \right)^\rho} \\
&\leq C(k-k_0)^{1-\rho}.
\end{aligned}$$

We therefore get by (2.9)

$$(2.10) \quad I_2 \leq C2^k |B_i^k| (k-k_0)^{1-\rho}.$$

Thus

$$(2.11) \quad \sum_{k=k_0+1}^\infty \sum_i I_2 \leq C \sum_{k=k_0+1}^\infty (k-k_0)^{1-\rho} \|f\|_{H^{1,\infty}} \leq C \|f\|_{H^{1,\infty}}.$$

By (2.5) and (2.11), we have

$$(2.12) \quad \int_{(B_{k_0})^c} \mu_\Omega(F_2)(x) dx \leq C \|f\|_{H^{1,\infty}}.$$

From (2.12) we get (2.3). Hence we finish the proof of Theorem 1.

3. Proof of Theorem 2

Let us begin by recalling a known result.

LEMMA 1. Suppose that $1 \leq p < \infty$, $\eta > 0$ and $f \in BMO(\mathbb{R}^n)$. Then there exists a $C = C(n, p, \eta) > 0$ such that for any cube Q with its center at x_0 and side length d ,

$$\int_{\mathbb{R}^n} \frac{d^\eta |f(x) - f_Q|^p}{d^{n+\eta} + |x - x_0|^{n+\eta}} dx \leq C \|f\|_{BMO}^p.$$

Now let us turn to the proof of Theorem 2. Take any density point $\bar{x}$ of E and any cube Q with center at $\bar{x}$, and denote by d the side length of Q . First we show that $\mu_\Omega(f)(x) < \infty$ a.e. on Q . To do this, we denote $Q^* = 16Q$ to be the sixteen times extension of Q with its center at $\bar{x}$. Decompose $f(x)$ as

$$\begin{aligned} f(x) &= f_{Q^*} + (f(x) - f_{Q^*})\chi_{Q^*} + (f(x) - f_{Q^*})(1 - \chi_{Q^*}) \\ (3.1) \quad &:= f_1(x) + f_2(x) + f_3(x). \end{aligned}$$

By (1.1) we get $\mu_\Omega(f_1)(x) \equiv 0$. From (1.4) and Theorem A, we obtain

$$\begin{aligned} \int_Q |\mu_\Omega(f_2)(x)|^2 dx &\leq \int_{\mathbb{R}^n} |\mu_\Omega(f_2)(x)|^2 dx \\ &\leq C \int_{\mathbb{R}^n} |f_2(x)|^2 dx \\ &\leq C|Q| \|f\|_{BMO}^2. \end{aligned}$$

Hence

$$\begin{aligned} (3.2) \quad \int_Q |\mu_\Omega(f_2)(x)| dx &\leq |Q|^{1/2} \left(\int_Q |\mu_\Omega(f_2)(x)|^2 dx \right)^{1/2} \\ &\leq C|Q| \|f\|_{BMO}. \end{aligned}$$

This shows that $\mu_\Omega(f_2)(x) < \infty$ a.e. on Q . Since $|E| > 0$, we have $|Q \cap E| > 0$. Hence there exists an $x_0 \in Q \cap E$ such that $\mu_\Omega(f)(x_0) < \infty$ and $\mu_\Omega(f_2)(x_0) < \infty$ hold at the same time, and we may chose x_0 to close to $\bar{x}$ enough, say, $|x_0 - \bar{x}| < d/4$. Thus

$$(3.3) \quad \mu_\Omega(f_3)(x_0) \leq \mu_\Omega(f)(x_0) + \mu_\Omega(f_2)(x_0) < \infty.$$

On the other hand, for any $x \in Q$ we have

$$\begin{aligned}
 (3.4) \quad & |\mu_\Omega(f_3)(x) - \mu_\Omega(f_3)(x_0)| \\
 & \leq \left(\int_0^\infty \left| \int_{|x-y| \leq t} \frac{\Omega(x-y)f_3(y)}{|x-y|^{n-1}} dy \right. \right. \\
 & \quad \left. \left. - \int_{|x_0-y| \leq t} \frac{\Omega(x_0-y)f_3(y)}{|x_0-y|^{n-1}} dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
 & \leq \left(\int_0^\infty \left(\int_{\substack{|x-y| \leq t \\ |x_0-y| > t}} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} f_3(y) \right| dy \right)^2 \frac{dt}{t^3} \right)^{1/2} \\
 & \quad + \left(\int_0^\infty \left(\int_{\substack{|x-y| > t \\ |x_0-y| \leq t}} \left| \frac{\Omega(x_0-y)}{|x_0-y|^{n-1}} f_3(y) \right| dy \right)^2 \frac{dt}{t^3} \right)^{1/2} \\
 & \quad + \left(\int_0^\infty \left(\int_{|x_0-y| \leq t} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x_0-y)}{|x_0-y|^{n-1}} \right| |f_3(y)| dy \right)^2 \frac{dt}{t^3} \right)^{1/2} \\
 & := I_1(x) + I_2(x) + I_3(x).
 \end{aligned}$$

Below we shall give the estimates of I_1, I_2 and I_3 , respectively. For I_1 , we have

$$\begin{aligned}
 & I_1(x) \\
 & \leq \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f_3(y)| \left(\int_{|x_0-y| > t} \frac{dt}{t^3} \right)^{1/2} dy \\
 & \leq C \|\Omega\|_{L^\infty(S^{n-1})} \int_{(Q^*)^c} \frac{1}{|x-y|^{n-1}} |f_3(y)| \left| \frac{1}{|x-y|^2} - \frac{1}{|x_0-y|^2} \right|^{1/2} dy.
 \end{aligned}$$

Note that $y \in (Q^*)^c$ and $x, x_0 \in Q$, we have $|x-y| \sim |x_0-y| \sim |\bar{x}-y|$ and

$$\left| \frac{1}{|x-y|^2} - \frac{1}{|x_0-y|^2} \right| \leq \frac{Cd}{|x-y|^3}.$$

From this and applying Hölder's inequality, we get

$$\begin{aligned}
 (3.5) \quad & I_1(x) \leq C \|\Omega\|_{L^\infty(S^{n-1})} \int_{(Q^*)^c} \frac{d^{1/2} |f_3(y)|}{|x-y|^{n+1/2}} dy \\
 & \leq C \|\Omega\|_{L^\infty(S^{n-1})} \left(\int_{(Q^*)^c} \frac{d^{1/2}}{|x-y|^{n+1/2}} dy \right)^{1/2} \\
 & \quad \times \left(\int_{(Q^*)^c} \frac{d^{1/2} |f_3(y)|^2}{|x-y|^{n+1/2}} dy \right)^{1/2}.
 \end{aligned}$$

It is easy to check that

$$d^{n+1/2} + |x - y|^{n+1/2} \leq 2|x - y|^{n+1/2}$$

for $x \in Q$ and $y \in (Q^*)^c$. Hence by (3.5) and Lemma 1

$$(3.6) \quad \begin{aligned} I_1(x) &\leq C \left(\int_{(Q^*)^c} \frac{d^{1/2} |f_3(y)|^2}{d^{n+1/2} + |x - y|^{n+1/2}} dy \right)^{1/2} \\ &\leq C \|f\|_{BMO}. \end{aligned}$$

Similarly, we have $I_2(x) \leq C \|f\|_{BMO}$ in the same way of estimating $I_1(x)$. Now let us consider $I_3(x)$. Using Hölder's inequality again, we get

$$\begin{aligned} I_3(x) &\leq C \int_{(Q^*)^c} \left| \frac{\Omega(x - y)}{|x - y|^{n-1}} - \frac{\Omega(x_0 - y)}{|x_0 - y|^{n-1}} \right| |f_3(y)| \\ &\quad \times \left(\int_{|x-y| \leq t, |x_0-y| \leq t} \frac{dt}{t^3} \right)^{1/2} dy \\ &\leq C \int_{(Q^*)^c} \left| \frac{\Omega(x - y)}{|x - y|^{n-1}} - \frac{\Omega(x_0 - y)}{|x_0 - y|^{n-1}} \right| \frac{|f_3(y)|}{|x_0 - y|} dy. \end{aligned}$$

Similar to the estimates of (2.6)–(2.8), we get

$$I_3(x) \leq C \int_{(Q^*)^c} \frac{|f_3(y)| dy}{|y - x_0|^n \left(\log \frac{|y - x_0|}{d} \right)^\rho}.$$

Note that $y \in (Q^*)^c$ and $x_0, \bar{x} \in Q$, and hence

$$|y - x_0| \geq |y - \bar{x}| - |\bar{x} - x_0| \geq 8d - d/4 > 4d.$$

Thus

$$\begin{aligned} I_3(x) &\leq C \sum_{j=2}^{\infty} \int_{2^j d \leq |y - x_0| < 2^{j+1} d} \frac{|f_3(y)| dy}{|y - x_0|^n \left(\log \frac{|y - x_0|}{d} \right)^\rho} \\ &\leq C \sum_{j=2}^{\infty} \left(\int_{2^j d \leq |y - x_0| < 2^{j+1} d} \frac{dy}{|y - x_0|^n \left(\log \frac{|y - x_0|}{d} \right)^{2\rho}} \right)^{1/2} \\ &\quad \times \left(\int_{2^j d \leq |y - x_0| < 2^{j+1} d} \frac{|f_3(y)|^2 dy}{|y - x_0|^n} \right)^{1/2} \end{aligned}$$

Observe

$$\begin{aligned}
 (3.7) \quad & \left(\int_{2^j d \leq |y-x_0| < 2^{j+1} d} \frac{dy}{|y-x_0|^n \left(\log \frac{|y-x_0|}{d} \right)^{2\rho}} \right)^{1/2} \\
 & \leq C \left(\int_{2^j d}^{2^{j+1} d} \frac{ds}{s (\log(s/d))^{2\rho}} \right)^{1/2} \\
 & \leq \frac{C}{(\log 2^j)^\rho} \left(\int_{2^j}^{2^{j+1}} \frac{ds}{s} \right)^{1/2} \\
 & \leq \frac{C}{j^\rho}.
 \end{aligned}$$

On the other hand, denote by Q_j the 2^j times extension of Q , then we have

$$\begin{aligned}
 (3.8) \quad & \left(\int_{2^j d \leq |y-x_0| < 2^{j+1} d} \frac{|f_3(y)|^2 dy}{|y-x_0|^n} \right)^{1/2} \\
 & \leq \left(\frac{1}{(2^j d)^n} \int_{2^j d \leq |y-x_0| < 2^{j+1} d} |f_3(y)|^2 dy \right)^{1/2} \\
 & \leq \left(\frac{1}{(2^j d)^n} \int_{2^j d \leq |y-x_0| < 2^{j+1} d} (|f(y) - f_{Q_{j+1}}|^2 + |f_{Q_{j+1}} - f_{Q^\bullet}|^2) dy \right)^{1/2} \\
 & \leq C \left(\frac{1}{|Q_{j+1}|} \int_{Q_{j+1}} |f(y) - f_{Q_{j+1}}|^2 dy + (j+1)^2 2^{2n} \|f\|_{BMO}^2 \right)^{1/2}.
 \end{aligned}$$

Thus by (3.7) and (3.8) and note that $\rho > 2$

$$(3.9) \quad I_3(x) \leq C \sum_{j=2}^{\infty} \frac{1}{j^\rho} [1 + (j+1)2^n] \|f\|_{BMO} \leq C \|f\|_{BMO}.$$

Summing up (3.4), (3.6) and (3.9), we get

$$(3.10) \quad |\mu_\Omega(f_3)(x) - \mu_\Omega(f_3)(x_0)| \leq C \|f\|_{BMO}.$$

Thus we prove that

$$\begin{aligned}
 \mu_\Omega(f)(x) & \leq \mu_\Omega(f_1)(x) + \mu_\Omega(f_2)(x) \\
 & \quad + |\mu_\Omega(f_3)(x) - \mu_\Omega(f_3)(x_0)| + \mu_\Omega(f_3)(x_0) \\
 & < \infty \quad \text{a.e. on } Q.
 \end{aligned}$$

Because Q is any cube with center at $\bar{x} \in E$, we get actually $\mu_\Omega(f)(x) < \infty$ a.e. on $\mathbb{R}^n$.

Finally, let us show that (1.6) holds. In fact, from the process of the above proof we know that for any cube $Q \subset \mathbb{R}^n$, there exists an $x_0 \in Q$ such that $\mu_\Omega(f_3)(x_0) < \infty$. Let $Q^* = 16Q$ and write $f = f_1 + f_2 + f_3$ as in (3.1). From the process of the above proof, we know that $\mu_\Omega(f_1)(x) = 0$ on Q , and $\int_Q |\mu_\Omega(f_2)(x)| dx \leq C|Q| \|f\|_{BMO}$. And by (3.2) and (3.10) we get

$$\begin{aligned}
 (3.11) \quad & \int_Q |\mu_\Omega(f)(x) - \mu_\Omega(f_3)(x_0)| dx \\
 & \leq \int_Q |\mu_\Omega(f_2)(x)| dx + \int_Q |\mu_\Omega(f_3)(x) - \mu_\Omega(f_3)(x_0)| dx \\
 & \leq C|Q| \|f\|_{BMO}.
 \end{aligned}$$

Using (3.10) and (3.11) again, we have

$$\begin{aligned}
 & \int_Q |\mu_\Omega(f)(x) - (\mu_\Omega(f))_Q| dx \\
 & \leq \int_Q |\mu_\Omega(f)(x) - \mu_\Omega(f_3)(x_0)| dx + \int_Q |\mu_\Omega(f_3)(x_0) - (\mu_\Omega(f))_Q| dx \\
 & \leq C|Q| \|f\|_{BMO} + \int_Q |\mu_\Omega(f)(x) - \mu_\Omega(f_3)(x_0)| dx \\
 & \leq C|Q| \|f\|_{BMO}.
 \end{aligned}$$

Thus we obtain (1.6) and complete the proof of Theorem 2.

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