STARLIKE FUNCTIONS ASSOCIATED WITH A PETAL SHAPED DOMAIN

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ABSTRACT. In this paper, we establish some radius results and inclusion relations for starlike functions associated with a petal-shaped domain.

1. Introduction

Let the open unit disk \( \{ z \in \mathbb{C} : |z| < 1 \} \) be represented by \( \mathbb{D} \) and \( \mathcal{H} \) be the class of all analytic functions in \( \mathbb{D} \). Consider \( \mathcal{A}_n \) as the class of analytic functions \( f \) in \( \mathbb{D} \) represented by

\[
 f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots.
\]

In particular, denote \( \mathcal{A}_1 := \mathcal{A} \) and let \( \mathcal{S} \) be the subclass of \( \mathcal{A} \) such that it involves all univalent functions \( f(z) \) in \( \mathbb{D} \). Let \( g, h \) be two analytic functions and \( \omega \) be a Schwarz function satisfying \( \omega(0) = 0 \) and \( |\omega(z)| \leq |z| \) such that \( g(z) = h(\omega(z)) \) then \( g \) is said to be subordinate to \( h \), or \( g \preceq h \). If \( h \) is univalent, then \( g \preceq h \) if and only if \( g(0) = h(0) \) and \( g(\mathbb{D}) \subseteq h(\mathbb{D}) \). Ma and Minda [11] introduced the univalent function \( \psi \) satisfying \( \text{Re} \psi(\mathbb{D}) > 0 \), \( \psi(\mathbb{D}) \) starlike with respect to \( \psi(0) = 1 \) and \( \psi'(0) > 0 \) and the domain \( \psi(\mathbb{D}) \) being symmetric about the real axis. Further, they gave the definitions for the general subclasses of starlike and convex functions, respectively, as follows:

\[
 S^*(\psi) := \{ f \in \mathcal{S} : zf'(z)/f(z) \prec \psi(z) \}
\]

and

\[
 K(\psi) := \{ f \in \mathcal{S} : 1 + zf''(z)/f'(z) \prec \psi(z) \}.
\]

For different choices of \( \psi \), many subclasses of \( S^* \) and \( K \) can be obtained. For example, the notable classes of Janowski starlike and convex functions [8] are represented by \( S^*[C, D] := S^*((1 + Cz)/(1 + Dz)) \) and \( K[C, D] := K((1 + Cz)/(1 + Dz)) \) for \(-1 \leq D < C \leq 1\), respectively. Further, \( S^*_\alpha := S^*[1 - 2\alpha, -1] \) and \( K_\alpha := K*[1 - 2\alpha, -1] \) represent the classes of starlike and convex functions.
of order $\alpha \in [0, 1]$, respectively. Note that $S^* := S_0^*$ and $K := K_0$ represent the well-known classes of starlike and convex functions, respectively. We denote $SS^*(\gamma) := S^*((1+z)/(1-z))^{\gamma}$ and $SK(\gamma) := K((1+z)/(1-z))^{\gamma}$ representing the class of strongly starlike and strongly convex functions of order $\gamma \in (0, 1]$, respectively.

Recall that for two subfamilies $G_1$ and $G_2$ of $A$, we say that $r_0$ is the $G_1$-radius for the class $G_2$ if $r_0 \ (0 < r \leq r_0)$ is the greatest number which satisfies $r^{-1}g(rz) \in G_1$, where $g \in G_2$. Moreover, starlike classes $S^*(\psi)$ for different $\psi(z)$ were considered by many authors, whose works examined the geometrical properties, radius results and coefficient estimates of the functions of their respective classes. Sokół and Stankiewicz [20, 21] considered the class $S_1^* := S^*(\sqrt{1+z})$ and Mendiratta et al. [13] worked on the class $S_0^* := S^*(\sqrt{2} - (\sqrt{2} - 1)((1-z)/(1+2(\sqrt{2}-1)z)))^{1/2}$. Sharma et al. [19] studied the class $S_2^* := S^*(1+4z/3 + 2z^2/3)$ while the class $S_3^* := S^*(1+ \sin(z))$ was examined by Cho et al. [6]. The classes $S_5^* := S^*(e^z)$ and $S_5^* := S^*(z + \sqrt{1+z})$ were considered by Mendiratta et al. [14] and Raina et al. [15], respectively. Kargar et al. [10] introduced and studied the class $SS^*(\alpha) := S^*(1+z/(1-\alpha z^2))$, $\alpha \in [0, 1]$, associated with the Booth lemniscate which was also investigated by Cho et al. [4]. Some more recent work on radius problems can be found in [1,3,5,7,23].

Motivated by the classes defined in [6, 10, 13–15, 19, 21], we consider the petal shaped region $\Omega_\rho := \{w \in C : |\sinh(w-1)| < 1\}$, which is characterised functionally as $\rho(z) = 1 + \sinh^{-1}(z)$ to define our class. Clearly, $\rho(z)$ is a Ma-Minda function. See Figure 2 for its boundary curve $\gamma_0$ which is petal shaped. Note that $\sinh^{-1}(z)$ is a multivalued function and has the branch cuts along the line segments $(-i\infty, -i) \cup (i, i\infty)$, on the imaginary axis and hence it is analytic in $D$. Now we introduce a new class of starlike functions

\begin{equation}
S_\rho^* := \left\{f \in A : \frac{zf'(z)}{f(z)} \prec 1 + \sinh^{-1}(z)\right\} \quad (z \in D),
\end{equation}

which is associated with the petal-shaped domain $\rho(D)$. From the above definition, we deduce that $f \in S_\rho^*$ if and only if there exists an analytic function $q(z) \prec \rho(z)$ such that

\begin{equation}
f(z) = z \exp\left(\int_0^z \frac{q(t) - 1}{t} dt\right).
\end{equation}

Table 1 presents some functions in the class $S_\rho^*$, where $q_j \prec \rho$. Since $\rho$ is univalent in $D$, $q_j(D) \subset \rho(D)$ and $q_j(0) = \rho(0)$ ($j = 1, 2, 3$), it follows that each $q_j \prec \rho$. Thus the functions $f_j(z)$ obtained from (5) are in the class $S_\rho^*$. In particular, if we choose

$$q(z) = 1 + \sinh^{-1}(z) = 1 + z - \frac{z^3}{6} + \frac{3z^5}{40} - \frac{5z^7}{112} + \cdots,$$
Table 1. Some functions in the class $S^*_\rho$

<table>
<thead>
<tr>
<th>$j$</th>
<th>$q_j(z)$</th>
<th>$f_j(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1 + z/5$</td>
<td>$z \exp(z/5)$</td>
</tr>
<tr>
<td>2</td>
<td>$(5 + 2z)/(5 + z)$</td>
<td>$z + z^2/5$</td>
</tr>
<tr>
<td>3</td>
<td>$(7 + 4z)/(7 + z)$</td>
<td>$z(1 + z/7)^3$</td>
</tr>
</tbody>
</table>

then (5) gives

$$(6) \quad f_0(z) = z \exp \left( \int_0^z \frac{\sinh^{-1}(t)}{t} \, dt \right) = z + z^2 + \frac{z^3}{2} + \frac{z^4}{9} - \frac{z^5}{72} - \frac{z^6}{225} + \cdots,$$

which often acts as the extremal function for the class $S^*_\rho$, yielding sharp results.

**Remark 1.1.** Note that $\sinh^{-1}(z) = \ln(z + \sqrt{1 + z^2})$. Let $w = zf'(z)/f(z)$, where $f \in S^*_\rho$. Then the class $S^*_\rho$ can be alternatively represented by $\exp(w - 1) \prec z + \sqrt{1 + z^2}$, where $z + \sqrt{1 + z^2}$ represents the Crescent shaped domain \cite{15}. Thus, there exists an exponential relation among the functions in the classes $S^*_\rho$ and $\Delta^*$.

In the present investigation, the geometrical properties of the function $1 + \sinh^{-1}(z)$ are studied and certain inclusion properties as well as radius problems are established for the class $S^*_\rho$.

2. Properties of the function $1 + \sinh^{-1}(z)$

The current section deals with the study of some geometric properties of the function $1 + \sinh^{-1}(z)$.

**Theorem 2.1.** The function $\rho(z) = 1 + \sinh^{-1}(z)$ is a convex univalent function.

**Proof.** Let $h(z) = \sinh^{-1}(z)$. Clearly, $h(0) = 0$. Since $h'(z) = 1/\sqrt{1 + z^2}$ and $\sqrt{1 + z^2} \prec \sqrt{1 + z} \in \mathcal{P}$, where $\mathcal{P}$ is the Carathéodory class. Therefore, $1/\sqrt{1 + z^2} \in \mathcal{P}$ which implies that $\Re h'(z) > 0$. Hence $\rho$ is univalent. Now a calculation yields

$$1 + \frac{zh''(z)}{h'(z)} = \frac{1}{1 + z^2}.$$

Since

$$1 + z^2 \prec \frac{1}{1 + z} \in \mathcal{P},$$

Therefore, $\Re(1 + zh''(z)/h'(z)) > 0$ which implies that $h$ (and thus $\rho$) is a convex univalent function. \qed

**Remark 2.2.** Note that $\rho'(0) > 0$ and the function $\varphi(z) = z + \sqrt{1 + z^2}$ satisfies $\varphi(z) = \overline{\varphi(z)}$. Therefore, $\rho(z) = \overline{\rho(z)}$ and hence, the domain $\Omega_\rho = \rho(\mathbb{D})$ is symmetric about the real axis.
Theorem 2.3. The domain $\Omega_\rho$ is symmetric about the line $\text{Re}(w) = 1$.

Proof. Since $\Omega_\rho$ is symmetric about the real axis, the condition $0 \leq \theta \leq \pi/2$ is sufficient to prove our result. As we know that symmetry along imaginary axis for $f \in A$ holds if $\text{Re}(f(\theta)) = -\text{Re}(f(\pi - \theta))$ and $\text{Im}(f(\theta)) = \text{Im}(f(\pi - \theta))$. Now let $h(z) = \sinh^{-1}(z) = \ln(z + \sqrt{1 + z^2})$. Then $\text{Im}(h(z)) = \arg(z + \sqrt{1 + z^2})$. For $z = re^{it}, t \in [0, \pi]$ and fixed $r \in (0, 1)$, we have the following expressions for $t \to \theta$

$$I_1 = \arg \left( r(\cos \theta + i \sin \theta) + \sqrt{1 + r^2(\cos(2\theta) + i \sin(2\theta))} \right)$$
$$= \arg \left( z + \sqrt{1 + z^2} \right),$$

and for $t \to \pi - \theta$

$$I_2 = \arg \left( r(\cos(\pi - \theta) + i \sin(\pi - \theta)) + \sqrt{1 + r^2(\cos(2(\pi - \theta)) + i \sin(2(\pi - \theta)))} \right)$$
$$= \arg \left( r(-\cos \theta + i \sin \theta) + \sqrt{1 + r^2(\cos(2\theta) - i \sin(2\theta))} \right)$$
$$= \arg \left( -z + \sqrt{1 + z^2} \right).$$

Now let us consider $(z + \sqrt{1 + z^2})/(-z + \sqrt{1 + z^2})$. On rationalising the denominator, we get

$$\frac{z + \sqrt{1 + z^2}}{-z + \sqrt{1 + z^2}} = \frac{(z + \sqrt{1 + z^2})(-z + \sqrt{1 + z^2})}{(-z + \sqrt{1 + z^2})(-z + \sqrt{1 + z^2})} = \frac{1}{|z + \sqrt{1 + z^2}|^2} = k,$$

where $k > 0$ is some real positive constant. Thus,

$$\arg \left( \frac{z + \sqrt{1 + z^2}}{-z + \sqrt{1 + z^2}} \right) = \arg(k) = 0$$

$$\Rightarrow \arg \left( z + \sqrt{1 + z^2} \right) = \arg \left( -z + \sqrt{1 + z^2} \right)$$

$$\Rightarrow I_1 = I_2.$$

Similarly, $\text{Re}(h(\theta)) = -\text{Re}(h(\pi - \theta))$ for $0 \leq \theta \leq \pi/2$. Hence, $h(z)$ is symmetric about the imaginary axis and thus, by translation property, $\rho(z)$ is symmetric about the line $\text{Re}(w) = 1$. \hfill $\square$

Now using Theorem 2.3, we obtain the next result:

Corollary 2.4. The disk $\{w : |w - 1| \leq \sinh^{-1}(r)\}$ is contained in $\rho(|z| \leq r)$ and is maximal.

Proof. Since $\min_{|z|=r} |\sinh^{-1}(z)| = |\sinh^{-1}(-r)| = \sinh^{-1}(r)$ and hence the conclusion can be drawn at once. \hfill $\square$

Theorem 2.5. We find that the following properties hold for $\rho(z) = 1 + \sinh^{-1}(z)$:
Figure 1. $\rho(D)$ lies in the annular region bounded between the circles $C_1$ and $C_2$.

(i) $\rho(-r) \leq \text{Re} \rho(z) \leq \rho(r)$ ($|z| \leq r < 1$);
(ii) $|\text{Im} \rho(z)| \leq \pi/2$ ($|z| \leq 1$);
(iii) $\rho(-r) \leq |\rho(z)| \leq \rho(r)$ ($|z| \leq r < 1$);
(iv) $|\arg \rho(z)| \leq \tan^{-1}(1/t)$, where $t = \frac{1}{8} \sqrt{\sinh^{-1}(1)(1 - \sinh^{-1}(1))}$.

Proof. (i) Since $\rho(z)$ is convex and typically real, the value of $\text{Re} \rho(z)$ falls between $\lim_{\theta \to 0} \rho(re^{i\theta})$ and $\lim_{\theta \to \pi} \rho(re^{i\theta})$, thus the result follows.

(ii) Using Theorem 2.3, it suffices to take $\theta \in [0, \pi/2]$. Then the inequality follows by letting $r$ tending to $1^-$ and observing that the function

$$\text{Im} \rho(z) = \arg \left( r \cos(\theta) + \sqrt{1 + r^2(\cos(2\theta) + i \sin(2\theta))} + ir \sin(\theta) \right)$$

is strictly increasing in the interval $[0, \pi/2]$ and hence the result follows at once.

(iii) The radially farthest and nearest points in $\rho(D)$ from origin are respectively $B$ and $A$ (see Figure 1) and therefore the result obviously holds. Moreover we observe that these points $A$ and $B$ lie on the real line and hence the bounds of $|\rho(z)|$ and $\text{Re} \rho(z)$ coincide.

The proof of (iv) is evident from Theorem 3.1(iii) so skipped here. □

Next we have the following important result:

**Lemma 2.6.** For $1 - \sinh^{-1}(1) < a < 1 + \sinh^{-1}(1)$, let $r_a$ be given by

$$r_a = \begin{cases} a - (1 - \sinh^{-1}(1)), & 1 - \sinh^{-1}(1) \leq a \leq 1; \\ 1 + \sinh^{-1}(1) - a, & 1 \leq a < 1 + \sinh^{-1}(1). \end{cases}$$

Then $\{w : |w - a| < r_a\} \subset \Omega_{\rho}$.

We omit the proof of Lemma 2.6 as it directly follows from Theorem 2.3 and Corollary 2.4.
Remark 2.7. Evidently the domain $\Omega_\rho$ is contained inside the disk $\{w : |w - 1| < \pi/2\}$.

3. Inclusion relations

This section establishes some inclusion results involving the class $S^*_\rho$ with some well-known classes.

We consider the class $M(\beta)$, first studied by Uralegaddi et al. [22], given by

$$M(\beta) := \left\{ f \in A : \Re \left( \frac{zf'(z)}{f(z)} \right) < \beta, \quad z \in \mathbb{D}, \quad \beta > 1 \right\},$$

and another interesting class introduced by Kanas and Wiśniowska [9] of $k$-starlike functions, denoted by $k - ST$ and defined by

$$k - ST := \left\{ f \in A : \Re \left( \frac{zf'(z)}{f(z)} \right) > k \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad z \in \mathbb{D}, \quad k \geq 0 \right\}.$$  

Note that $S^* = 0 - ST$ and $S^*_p = 1 - ST$, where $S^*_p$ is the class of parabolic starlike functions [17].

We establish the following inclusion relations for the class $S^*_\rho$.

**Theorem 3.1.** The class $S^*_\rho$ satisfies the following relationships:

(i) $S^*_\rho \subset S^*_0 \subset S^*$ for $0 \leq \alpha \leq 1 - \sinh^{-1}(1)$;

(ii) $S^*_\rho \subset M(\beta)$ for $\beta \geq 1 + \sinh^{-1}(1)$;

(iii) $S^*_\rho \subset SS^*(\gamma)$ for $(2/\pi) \tan^{-1}(1/t) \leq \gamma \leq 1$,

where $t = \frac{4}{\pi} \sqrt{\sinh^{-1}(1)(1 - \sinh^{-1}(1))}$;

(iv) $k - ST \subset S^*_\rho$ for $k \geq 1 + 1/\sinh^{-1}(1)$.

**Proof.** Consider $f \in S^*_\rho$ which implies $zf'(z)/f(z) < 1 + \sinh^{-1}(z)$. By Theorem 2.5, it is evident that for $z \in \mathbb{D}$,

$$1 - \sinh^{-1}(1) = \min_{|z|=1} \Re(1 + \sinh^{-1}(z)) \leq \Re \frac{zf'(z)}{f(z)}$$

and

$$\Re \frac{zf'(z)}{f(z)} \leq \max_{|z|=1} \Re(1 + \sinh^{-1}(z)) = 1 + \sinh^{-1}(1).$$

This proves (i) and (ii).

For (iii), let $w \in \mathbb{C}$, $X = \Re(w)$, $Y = \Im(w)$, and $b = 1 - \sinh^{-1}(1)$. Now consider the parabolic domain $\Gamma_p$ with the boundary curve $\partial \Gamma_p = \gamma_p : Y^2 = 4a(X - b)$. Then the focus $a$ of the smallest parabola $\gamma_p$ which contains $\Omega_\rho$ will touch the peak points $1 \pm i\pi/2$ of $S^*_\rho$ is $\pi^2/(16 \sinh^{-1}(1))$. Let $P$ be any point on the parabola $\gamma_p$ with parametric coordinates $(b + at^2, 2at)$ such that the tangent $OE$ at $P$ passes through origin for some parameter $t$. Let the equation
of the tangent $OE$ be $y = mx$, where $m = dy/dx = (dy/dt)/(dx/dt) = 1/t$. Therefore at $P$, we have

$$m = \frac{y}{x} \Rightarrow \frac{1}{t} = \frac{2at}{b + at^2},$$

which yields

$$t = \sqrt{\frac{b}{a}} = \frac{4}{\pi} \sqrt{\sinh^{-1}(1)(1 - \sinh^{-1}(1))}$$

and the argument of the tangent at $P$ of $\gamma_p$ is $\tan^{-1}(1/t)$. Since $\Omega_p \subset \Gamma_p$, it gives

$$\arg \left| \frac{z f'(z)}{f(z)} \right| = \max_{z = 1} \arg(\rho(z)) = \max_{z = 1} \arg(\gamma_p) = \tan^{-1}(1/t),$$

which demonstrates $f \in SS^*(\pi \tan^{-1}(1/t))$, where $t$ is given by (7).

To show (iv), consider $f \in k - ST$ along with the conic domain $\Gamma_k = \{w \in \mathbb{C} : \Re w > k[w - 1]\}$. For $k > 1$, let $\partial \Gamma_k$ represent the horizontal ellipse $\gamma_k : x^2 = k^2(x - 1)^2 + k^2y^2$ which may be rewritten as

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1,$$

where $x_0 = k^2/(k^2 - 1)$, $y_0 = 0$, $a = k/(k^2 - 1)$ and $b = 1/\sqrt{k^2 - 1}$. For $\gamma_k \subset \Omega_p$, the condition $x_0 + a \leq 1 + \sinh^{-1}(1)$ must hold, or equivalently $k \geq 1 + 1/\sinh^{-1}(1)$. Since $\Gamma_{k_1} \subseteq \Gamma_{k_2}$ for $k_1 \geq k_2$, it follows that for $k \geq 1 + 1/\sinh^{-1}(1)$, $k - ST \subset S_p$. Figure 2 clearly depicts these relations. \(\square\)

For our next result, we consider $\mathcal{P}_n[C,D]$, the class of functions $p(z)$ of the form $1 + \sum_{k=1}^{\infty} c_k z^k$, satisfying $p(z) < (1 + Cz)/(1 + Dz)$, where $-1 \leq D < C \leq 1$. Denote by $\mathcal{P}_n(\alpha) := \mathcal{P}_n[1 - 2\alpha, -1]$ and $\mathcal{P}_n := \mathcal{P}_n(0)$. For $n = 1$, $\mathcal{P} = \mathcal{P}_1$ is the Carathéodory class. We need the following lemmas:

**Lemma 3.2** ([18]). For $p \in \mathcal{P}_n(\alpha)$, we have

$$\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2(1 - \alpha)n r^n}{(1 - r^n)(1 + (1 - 2\alpha)r^n)}, \quad (|z| = r).$$

**Lemma 3.3** ([16]). For $p \in \mathcal{P}_n[C,D]$, we have

$$\left| \frac{p(z) - \frac{1 - CDr^{2n}}{1 - D^2 r^{2n}}}{\frac{1 - D^2 r^{2n}}{1 - D^2 r^{2n}}} \right| \leq \frac{(C - D)r^n}{1 - D^2 r^{2n}}, \quad (|z| = r).$$

Especially, for $p \in \mathcal{P}_n(\alpha)$, we have

$$\left| \frac{p(z) - \frac{1 + (1 - 2\alpha)r^{2n}}{1 - r^{2n}}}{\frac{1 + r^{2n}}{1 - r^{2n}}} \right| \leq \frac{2(1 - \alpha)r^n}{1 - r^{2n}}, \quad (|z| = r).$$

**Theorem 3.4.** Let $-1 < D < C \leq 1$. If either of the following two conditions holds:

(i) $(1 - \sinh^{-1}(1))(1 - D^2) < 1 - CD \leq 1 - D^2$ and $C - D \leq (1 - D) \sinh^{-1}(1)$;
(ii) $1 - D^2 \leq 1 - CD < (1 + \sinh^{-1}(1))(1 - D^2)$ and $C - D \leq (1 + D) \sinh^{-1}(1)$.
Let \( a_1 + D \) and dividing by \((1 - D^2)\) on either side of the inequality \((C - D) \leq (1 - D^2)\) gives \((C - D)/(1 - D^2) \leq a - (1 - \sinh^{-1}(1))\) on simplification. Also, the inequality \((1 - \sinh^{-1}(1))(1 - D^2) < 1 - CD \leq 1 - D^2\) is equivalent to \(1 - \sinh^{-1}(1) < (1 - CD)/(1 - D^2) \leq 1\). Therefore, from (8) we find \(w = z f'(z)/f(z)\) is contained inside the disk \(|w - a| < r_a\), where \(r_a = a - (1 - \sinh^{-1}(1))\) and \(0 \leq a \leq 1 - \sinh^{-1}(1)\) for \(a \leq 1\). Hence \(f \in S^*_p\) by Lemma 2.6. A similar proof can be shown when (ii) holds.

4. Radius problems

In this section, radius results for various subclasses of \(\mathcal{A}\) are established. We begin by determining sharp \(S^*_p\) \((0 \leq \alpha < 1)\), \(M(\beta)\) \((\beta > 1)\) and \(k - ST\)-radii \((k \geq 0)\) for the class \(S^*_p\). Using Theorem 3.1, we can establish that \(R_{S^*_p}(S^*_p) = R_M(\beta)(S^*_p) = 1\) for \(0 \leq \alpha \leq 1 - \sinh^{-1}(1)\) and \(\beta > 1 + \sinh^{-1}(1)\).

Theorem 4.1. If \(f \in S^*_p\), then the following results hold:
(i) For $1 - \sinh^{-1}(1) \leq \alpha < 1$, we have $f \in S^*_\alpha$ in $|z| \leq \sinh(1 - \alpha)$.

(ii) For $1 < \beta \leq 1 + \sinh^{-1}(1)$, we have $f \in \mathcal{M}(\beta)$ in $|z| \leq \sinh(\beta - 1)$.

(iii) For $k > 0$, we have $f \in k - \mathcal{ST}$ in $|z| \leq \sinh(1/(k + 1))$.

The results are sharp.

Proof. Since $f \in S^*_\rho$, $zf'(z)/f(z) \prec 1 + \sinh^{-1}(z)$ and hence for $|z| = r < 1$
Theorem 2.5 gives

$$1 - \sinh^{-1}(r) \leq \text{Re} \left[ \frac{zf'(z)}{f(z)} \right] \leq 1 + \sinh^{-1}(r),$$

thereby validating the first two parts. Also, the constants $\sinh(1 - \alpha)$ and $\sinh(\beta - 1)$ are optimal for the function $f_0$ given by (6). Now to prove (iii), note that $f \in k - \mathcal{ST}$ in $|z| < r$, if

$$\text{Re}(1 + \sinh^{-1}(w(z))) \geq k|1 + \sinh^{-1}(w(z)) - 1| = k|\sinh^{-1}(w(z))|.$$ 

Here $w$ denotes the Schwarz function. Since $\text{Re}(1 + \sinh^{-1}(w(z))) \geq 1 - \sinh^{-1}(r)$ and $|\sinh^{-1}(w(z))| \leq \sinh^{-1}(r)$, the inequality $\text{Re}(1 + \sinh^{-1}(w(z))) \geq k|\sinh^{-1}(w(z))|$ holds whenever $1 - \sinh^{-1}(r) \geq k \sinh^{-1}(r)$, which implies $r \leq \sinh(1/(1 + k))$. For the function $f_0$ given by (6) and for $z_0 = -\sinh(1/(1 + k))$, we have

$$\text{Re} \left[ \frac{z_0f'(z_0)}{f(z_0)} \right] = \text{Re}(1 + \sinh^{-1}(z_0)) = \frac{k}{k + 1} = k \sinh^{-1}(z_0) = k \left| \frac{z_0f'(z_0)}{f(z_0)} - 1 \right|.$$ 

This concludes the proof. \qed

Corollary 4.2. Substituting $k = 1$ in part (iii) above, we find that $f \in S^*_\rho$ is parabolic starlike [17] in $|z| \leq \sinh(1/2)$.

In the next result, we find the $K_\alpha$-radius for the class $S^*_\rho$.

Theorem 4.3. Let $f \in S^*_\rho$. Then $f \in K_\alpha$ in $|z| < r_\alpha$, where $r_\alpha$ is the least positive root of

$$1 - r^2 \sqrt{1 + r^2} \left( 1 - \sinh^{-1}(r) \right) \left( 1 - \alpha - \sinh^{-1}(r) \right) = 0 \quad (0 \leq \alpha < 1).$$

Proof. Let $f \in S^*_\rho$ and $w$ be a Schwarz function. Then $zf'(z)/f(z) = 1 + \sinh^{-1}(w(z))$ such that

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \sinh^{-1}(w(z)) + \frac{zw'(z)}{(1 + \sinh^{-1}(w(z))) \sqrt{1 + w^2(z)}}$$

which yields

$$\text{Re} \left[ 1 + \frac{zf''(z)}{f'(z)} \right] \geq \text{Re} \left( 1 + \sinh^{-1}(w(z)) \right) - \left| \frac{zw'(z)}{(1 + \sinh^{-1}(w(z))) \sqrt{1 + w^2(z)}} \right|.$$
We know for the Schwarz function \( w \), the inequality \( |w'(z)| \leq (1 - |w(z)|^2)/(1 - |z|^2) \) holds. Thus we observe that

\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \geq 1 - \sinh^{-1}(|z|) - \frac{|z|(1 - |w(z)|^2)}{(1 - \sinh^{-1}(|z|))(1 - |z|^2)\sqrt{1 + |z|^2}}
\]

\[
\geq 1 - \sinh^{-1}(|z|) - \frac{|z|}{(1 - \sinh^{-1}(|z|))(1 - |z|^2)\sqrt{1 + |z|^2}}.
\]

Let \( q(r) := 1 - \sinh^{-1}(r) - r/(1 - \sinh^{-1}(r))(1 - r^2)\sqrt{1 + r^2} \). We find \( q(r) \) is a decreasing function in \([0, 1]\) with \( q(0) = 1 \). Therefore \( \text{Re}(1 + zf''(z)/f'(z)) > \alpha \) in \(|z| < r_{\alpha} < 1\), where \( r_{\alpha} \) is given as the least positive root of the equation \( q(r) = \alpha \), which is same as (9) and hence the result. \( \square \)

**Remark 4.4.** Note for \( \alpha = 0 \), \( r_0 \approx 0.37198 \) which is not sharp, so the result can be further improved. The sharp \( K_0 \)-radius for the class \( S^*_p \) is \( r_0 \approx 0.400435 \), which we can guess graphically but a mathematical proof is yet to derive.

For our next Theorems 4.5–4.8, the following subclasses are required:

Let \( S^*_n[C, D] := \{ f \in A_n : zf'(z)/f(z) \in \mathcal{P}_n[C, D] \} \). Also, let \( S^*_n[1, 2\alpha, -1] = A_n \cap S^*_n \) and \( S^*_{p,n} := A_n \cap S^*_n \). Further, Ali et al. [2] studied the three classes \( S_n := \{ f \in A_n : f(z)/z \in \mathcal{P}_n \}, S^*_n[C, D] \) and \( \mathcal{C}S_n(\alpha) := \left\{ f \in A_n : \frac{f(z)}{g(z)} \in \mathcal{P}_n, g \in S^*_n(\alpha) \right\} \).

Now we obtain the \( S^*_{p,n} \)-radii for the classes defined above.

**Theorem 4.5.** For the class \( S_n \), the sharp \( S^*_{p,n} \)-radius is given by:

\[
R_{S^*_{p,n}}(S_n) = \left( \frac{\sinh^{-1}(1)}{n + \sqrt{n^2 + (\sinh^{-1}(1))^2}} \right)^{1/n}.
\]

**Proof.** Let \( f \in S_n \). Define \( s : \mathbb{D} \to \mathbb{C} \) by \( s(z) = f(z)/z \). Then \( s \in \mathcal{P}_n \) and we can obtain \( zf'(z)/f(z) - 1 = zs'(z)/s(z) \) from the above definition of \( s \). Using Lemma 2.6 and Lemma 3.2, the following holds

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{zs'(z)}{s(z)} \right| \leq \frac{2nr^n}{1 - r^{2n}} \leq \sinh^{-1}(1),
\]

or equivalently \( (\sinh^{-1}(1)r^{2n} + 2nr^n - \sinh^{-1}(1)) \leq 0 \). Therefore, the \( S^*_{p,n} \)-radius of \( S_n \) is the least positive root of \( (\sinh^{-1}(1))r^{2n} + 2nr^n - \sinh^{-1}(1) = 0 \) for \( r \in (0, 1) \). We can verify \( \text{Re}(f_0(z)/z) > 0 \) holds in \( \mathbb{D} \), where \( f_0(z) = z(1 + z^n)/(1 - z^n) \). Thus \( f_0 \in S_n \) and \( zf_0'(z)/f_0(z) = 1 + 2nz^n/(1 - z^{2n}) \).

Moreover, the result is sharp since at \( z = R_{S^*_{p,n}}(S_n) \), we obtain

\[
\frac{zf_0'(z)}{f_0(z)} - 1 = \frac{2nz^n}{1 - z^{2n}} = \sinh^{-1}(1).
\]

The proof is complete. \( \square \)
Let $F$ define the class of functions $f \in A$ satisfying $f(z)/z \in P$. The radius of univalence and starlikeness of the class $F$ is $\sqrt{2} - 1$, as shown in [12].

**Corollary 4.6.** For the class $F$, the $S^*_\rho$-radius is stated as

$$R_{S^*_\rho}(F) = -e + \sqrt{1 + e^2} \approx 0.178105.$$

**Theorem 4.7.** For the class $CS_n(\alpha)$, the sharp $S^*_\rho,n$-radius is given by

$$R_{S^*_\rho,n}(CS_n(\alpha)) = \left(\frac{\sinh^{-1}(1)}{n - \alpha + 1 + \sqrt{(n - \alpha + 1)^2 + (\sinh^{-1}(1) + (1 - \alpha))\sinh^{-1}(1)}}\right)^{1/n}.$$

**Proof.** Let $f \in CS_n(\alpha)$ and $g \in S^*_\rho,\alpha$. Considering $s(z) = f(z)/g(z)$, clearly indicates $s \in P_n$. Also, it gives

$$\frac{z f'(z)}{f(z)} = \frac{zf'(z)}{s(z)} + \frac{zg'(z)}{g(z)}.$$

The use of Lemmas 3.2–3.3 gives us

$$|\frac{zf'(z)}{f(z)} - 1 + (1 - 2\alpha)z^{2n}| \leq \frac{2(n - \alpha + 1)r^{2n}}{1 - r^{2n}}.$$

Considering $(1 + (1 - 2\alpha)z^{2n})/(1 - r^{2n}) \geq 1$, the relation $f \in S^*_\rho,n$ follows from (10) and Lemma 2.6 if the subsequent inequality is true:

$$\frac{1 + 2(n - \alpha + 1)r^{n} + (1 - 2\alpha)r^{2n}}{1 - r^{2n}} \leq 1 + \sinh^{-1}(1)$$

or equivalently, $(2 - 2\alpha + \sinh^{-1}(1))r^{2n} + 2(n - \alpha + 1)r^{n} - \sinh^{-1}(1) \leq 0$ holds. Thus, the least positive root of

$$(2 - 2\alpha + \sinh^{-1}(1))r^{2n} + 2(n - \alpha + 1)r^{n} - \sinh^{-1}(1) = 0$$

gives the $S^*_\rho,n$-radius for the class $CS_n(\alpha)$. Next examine the following functions

$$f_0(z) = \frac{z(1 + z^n)}{(1 - z^n)(1 + (1 - 2\alpha)z^n)} \text{ and } g_0(z) = \frac{z}{(1 - z^n)(1 - \alpha)/n},$$

which implies $f_0(z)/g_0(z) = (1 + z^n)/(1 - z^n)$ and $zg'_0(z)/g_0(z) = (1 + (1 - 2\alpha)z^n)/(1 - z^n)$. Moreover, $\Re(f_0(z)/g_0(z)) > 0$ and $\Re(zg'_0(z)/g_0(z)) > \alpha$ in the unit disk $D$ is obvious. Hence $f_0 \in CS_n(\alpha)$. At $z = R_{S^*_\rho,n}(CS_n(\alpha))$, the function $f_0$ defined in (11) satisfies

$$\frac{zf'_0(z)}{f_0(z)} = \frac{1 + 2(n - \alpha + 1)z^n + (1 - 2\alpha)z^{2n}}{1 - z^{2n}} = 1 + \sinh^{-1}(1),$$

which accomplish sharpness of the result. 

**Theorem 4.8.** For the class $S^*_n[C, D]$, the $S^*_\rho,n$-radius is given by

$$R_{S^*_\rho,n}(S^*_n[C, D]) = \begin{cases} \min\{1; R_1\}, & -1 < D < 0 < C \leq 1; \\ \min\{1; R_2\}, & 0 < D < C \leq 1. \end{cases}$$
where

\[ R_1 := \left( \frac{2 \sinh^{-1}(1)}{C - D + \sqrt{(C - D)^2 + 4(D^2(1 + \sinh^{-1}(1)) - CD) \sinh^{-1}(1)}} \right)^{\frac{1}{n}} \]

and

\[ R_2 := \left( \frac{2 \sinh^{-1}(1)}{C - D + \sqrt{(C - D)^2 + 4(D^2(\sinh^{-1}(1) - 1) + CD) \sinh^{-1}(1)}} \right)^{\frac{1}{n}} . \]

**Proof.** Let \( f \in S_n^*[C,D] \). From Lemma 3.3, we have

\[ \left| \frac{zf'(z)}{f(z)} - b \right| \leq \frac{(C - D)r^n}{1 - D^2r^{2n}}, \]

where \( b = (1 - CDr^{2n})/(1 - D^2r^{2n}) \), \(|z| = r\), represents the center of the disk. We infer \( b \geq 1 \) for \(-1 \leq D < 0 < C \leq 1\). From Lemma 2.6, \( f \in S_{\rho,n}^* \) depends on whether following condition is true:

\[ \frac{1 + (C - D)r^n - CDr^{2n}}{1 - D^2r^{2n}} \leq 1 + \sinh^{-1}(1), \]

which reduces to

\[ r \leq \left( \frac{2 \sinh^{-1}(1)}{C - D + \sqrt{(C - D)^2 + 4(D^2(1 + \sinh^{-1}(1)) - CD) \sinh^{-1}(1)}} \right)^{\frac{1}{n}} = R_1. \]

Further, taking \( D = 0 \), we get \( b = 1 \). Then (12) yields

\[ \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq Cr^n, \ (0 < C \leq 1). \]

Now applying Lemma 2.6 with \( a = 1 \) gives \( f \in S_{\rho,n}^* \) if \( r \leq ((\sinh^{-1}(1))/C)^{1/n} \).

For \( 0 < D < C \leq 1 \), we have \( b < 1 \). Thus, using Lemma 2.6 and (12), we have \( f \in S_{\rho,n}^* \) if the following holds:

\[ \frac{CDr^{2n} + (C - D)r^n - 1}{1 - D^2r^{2n}} \leq \sinh^{-1}(1) - 1, \]

or equivalently, if

\[ r \leq \left( \frac{2 \sinh^{-1}(1)}{C - D + \sqrt{(C - D)^2 + 4(D^2(\sinh^{-1}(1) - 1) + CD) \sinh^{-1}(1)}} \right)^{\frac{1}{n}} = R_2. \]

This concludes the proof. \( \square \)

The next theorem establishes radius results for some well-known classes mentioned earlier.
Theorem 4.9. The sharp $S^*_ρ$-radii for the classes $S^*_L$, $S^*_RL$, $S^*_C$, $S^*_e$, $Δ^*$ and $BS^*(α)$ are:

(i) $R_{S^*_ρ}(S^*_L) = \sinh^{-1}(1)(2 - \sinh^{-1}(1)) \approx 0.985928$.

(ii) $R_{S^*_ρ}(S^*_RL) = \frac{(2+1+\sqrt{2})\sinh^{-1}(1)}{5-3\sqrt{2}+(4\sqrt{2}-1+2\sin^{-1}(1))\sinh^{-1}(1)} \approx 0.964694$.

(iii) $R_{S^*_ρ}(S^*_C) = \frac{1}{2} \sqrt{2(2+3\sin^{-1}(1)) - 2} \approx 0.523831$.

(iv) $R_{S^*_ρ}(S^*_e) = \ln(1+\sin^{-1}(1)) \approx 0.632002$.

(v) $R_{S^*_ρ}(Δ^*) = \frac{\sinh^{-1}(1)(2+\sin^{-1}(1))}{2(1+\sin^{-1}(1))} \approx 0.674924$.

(vi) $R_{S^*_ρ}(BS^*(α)) = \frac{-1+\sqrt{1+4\sin^{-1}(1)^2\alpha}}{2\sin^{-1}(1)}$, $α \in [0, 1]$.

Proof.

(i) Suppose $f \in S^*_L$. We have $zf^/(z)/f(z) \prec \sqrt{1+z}$. When $|z| = r$, we obtain

$$\frac{|zf^/(z)/f(z) - 1|}{\sqrt{1+z}} \leq 1 - \sqrt{1-r} \leq \sinh^{-1}(1),$$

such that $r \leq (2 - \sinh^{-1}(1))\sinh^{-1}(1) = R_{S^*_ρ}(S^*_L)$ holds. Next examine the function

$$f_0(z) = \frac{4z}{(1+\sqrt{1+z})^2}e^{\sqrt{1+z}z-1}.$$

Since $zf_0^/(z)/f_0(z) = \sqrt{1+z}$, it follows that $f_0 \in S^*_L$. As $zf_0^/(z)/f_0(z) - 1 = -\sinh^{-1}(1)$ is obtained at $z = -R_{S^*_ρ}(S^*_L)$, the result is sharp.

(ii) Suppose $f \in S^*_RL$, we obtain

$$\frac{zf^/(z)}{f(z)} \prec \sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1-z}{1+2(\sqrt{2} - 1)z}}.$$

For $|z| = r$, the subsequent inequality holds

$$\frac{|zf^/(z)/f(z) - 1|}{\sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1-r}{1-2(\sqrt{2} - 1)r}}} \leq \sinh^{-1}(1),$$

provided

$$r \leq \frac{(2 + (1 + \sqrt{2})\sinh^{-1}(1))\sinh^{-1}(1)}{5-3\sqrt{2}+(4\sqrt{2}-1+2\sin^{-1}(1))\sinh^{-1}(1)} = R_{S^*_ρ}(S^*_RL).$$

Next observe the following function defined as

$$f_0(z) = z \exp \left(\int_0^z \frac{g_0(t) - 1}{t} dt \right),$$

where

$$g_0(t) = \sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1-t}{1+2(\sqrt{2} - 1)t}}.$$
From the definition of $f_0$, at $z = -R_{S^*_C}(S^*_R)$, we have
\[
\frac{zf_0'(z)}{f_0(z)} = \sqrt{2} - (\sqrt{2} - 1)\sqrt{1 - z \over 1 + 2(\sqrt{2} - 1)z} = 1 - \sinh^{-1}(1),
\]
which confirms the sharpness.

(iii) Suppose $f \in S^*_C$. So $zf'(z)/f(z) < 1 + 4z/3 + 2z^2/3$. This gives
\[
\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{4r}{3} + \frac{2r^2}{3} \leq \sinh^{-1}(1), \ |z| = r,
\]
for $r \leq \frac{1}{2} \left(\sqrt{2(2 + 3\sinh^{-1}(1))} - 2\right) = R_{S^*_C}(S^*_C)$. The sharpness of the result is established using the subsequent function
\[
f_0(z) = z \exp \left(\frac{4z + z^2}{3}\right),
\]
where $zf_0'(z)/f_0(z) = 1 + (4z + 2z^2)/3$ yields $f_0 \in S^*_C$, and substituting $z = R_{S^*_C}(S^*_C)$ gives $zf_0'(z)/f_0(z) = 1 + \sinh^{-1}(1)$, thereby proving the sharpness.

(iv) Suppose $f \in S^*_C$, we have $zf'(z)/f(z) < e^z$, which yields
\[
\left|\frac{zf'(z)}{f(z)} - 1\right| \leq e^r - 1 \leq \sinh^{-1}(1) \text{ holds in } |z| = r,
\]
provided $r \leq \ln(1 + \sinh^{-1}(1)) = R_{S^*_C}(S^*_C)$. Now consider
\[
f_0(z) = z \exp \left(\int_0^z e^t - 1 \, dt\right).
\]
Since $zf_0'(z)/f_0(z) = e^z$, where $f_0 \in S^*_C$, so at $z = R_{S^*_C}(S^*_C)$, we have
\[
zf_0'(z)/f_0(z) = 1 + \sinh^{-1}(1),
\]
which shows the sharpness of the result.

(v) Suppose $f \in \Delta^*$ which gives $zf'(z)/f(z) < z + \sqrt{1 + z^2}$. Then,
\[
\left|\frac{zf'(z)}{f(z)} - 1\right| \leq r + \sqrt{1 + r^2} - 1 \leq \sinh^{-1}(1), \ |z| = r,
\]
for $r \leq \frac{\sinh^{-1}(1)(2 + \sinh^{-1}(1))}{2(1 + \sinh^{-1}(1))} = R_{S^*_C}(\Delta^*)$. For sharpness, define $f_0$ as
\[
f_0(z) = z \exp \left(\int_0^z \frac{t + \sqrt{1 + t^2} - 1}{t} \, dt\right).
\]
Since $zf_0'(z)/f_0(z) = z + \sqrt{1 + z^2}$, $f_0 \in \Delta^*$, so at $z = R_{S^*_C}(\Delta^*)$, we have
\[
zf_0'(z)/f_0(z) = 1 + \sinh^{-1}(1) \text{ which shows the sharpness of the result.}
\]

(vi) Suppose $f \in B\cal S^*(\alpha)$, $\alpha \in [0, 1]$, which gives $zf'(z)/f(z) < 1 + z/(1 - \alpha z^2)$. Then,
\[
\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{r}{1 - \alpha r^2} \leq \sinh^{-1}(1), \ |z| = r,
\]
Proof. (i) Let $f_{0}(z) = z f(z) - z g(z)$. Since $z f_{0}(z)/f_{0}(z) = 1 + z/(1 - \alpha z^{2})$, where $f_{0} \in (BS^{*}(\alpha))$, so at $z = -f_{0}((BS^{*}(\alpha)))$, we have $z f_{0}(z)/f_{0}(z) = 1 - \sinh^{-1}(1)$, which ensures sharpness of the result.

Note that for $\alpha = 0$, $R_{S_{r,n}}^{*}(BS^{*}(0)) = \sinh^{-1}(1) \approx 0.881374$ and for $\alpha = 1$, $R_{S_{r,n}}^{*}(BS^{*}(1)) = \left(1 + \sqrt{1 + (2 \sinh^{-1}(1))^{2}}\right)/(2 \sinh^{-1}(1)) \approx 0.58241$. \hfill \Box

Next we present some radius problems for certain classes of functions expressed as ratio of functions:

$$F_{1} := \left\{ f \in A_{n} : \mathrm{Re} \left(\frac{f(z)}{g(z)}\right) > 0 \text{ and } \mathrm{Re} \left(\frac{g(z)}{z}\right) > 0, \; g \in A_{n}\right\},$$

$$F_{2} := \left\{ f \in A_{n} : \mathrm{Re} \left(\frac{f(z)}{g(z)}\right) > 0 \text{ and } \mathrm{Re} \left(\frac{g(z)}{z}\right) > 1/2, \; g \in A_{n}\right\},$$

and

$$F_{3} := \left\{ f \in A_{n} : \left|\frac{f(z)}{g(z)} - 1\right| < 1 \text{ and } \mathrm{Re} \left(\frac{g(z)}{z}\right) > 0, \; g \in A_{n}\right\}.$$

Theorem 4.10. For functions in the classes $F_{1}$, $F_{2}$ and $F_{3}$, the sharp $S_{r,n}$-radii, respectively, are:

(i) $R_{S_{r,n}}^{*}(F_{1}) = \left(\frac{\sqrt{4n^{2} + (\sinh^{-1}(1))^{2} - 2n}}{\sinh^{-1}(1)}\right)^{1/n}$.

(ii) $R_{S_{r,n}}^{*}(F_{2}) = \left(\frac{\sqrt{9n^{2} + 4 \sinh^{-1}(1) + 8n - 2n}}{2(n + \sinh^{-1}(1))}\right)^{1/n}$.

(iii) $R_{S_{r,n}}^{*}(F_{3}) = R_{S_{r,n}}^{*}(F_{2})$.

Proof. (i) Let $f \in F_{1}$ and consider the functions $s, d : \mathbb{D} \to \mathbb{C}$, where $s(z) = f(z)/g(z)$ and $d(z) = g(z)/z$. Clearly, $s, d \in P_{n}$. As $f(z) = zd(z)s(z)$, applying Lemma 3.2 here gives

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{4nrn}{1 - r^{2n}} \leq \sinh^{-1}(1)$$

such that

$$r \leq \left(\frac{\sqrt{4n^{2} + (\sinh^{-1}(1))^{2} - 2n}}{\sinh^{-1}(1)}\right)^{1/n} = R_{S_{r,n}}^{*}(F_{1})$$
The sharpness can be verified as follows. Examine the functions
\[ f_0(z) = z \left( \frac{1 + z^n}{1 - z^n} \right)^2 \text{ and } g_0(z) = z \left( \frac{1 + z^n}{1 - z^n} \right). \]
Evidently, \( \text{Re}(f_0(z)/g_0(z)) > 0 \) and \( \text{Re}(g_0(z)/z) > 0 \), which implies \( f_0 \in \mathcal{F}_1 \).
Further calculation yields at \( z = R_{S_{\rho,n}}(\mathcal{F}_1)e^{i\pi/n} \)
\[ \frac{zf_0'(z)}{f_0(z)} = 1 + 4nz^n/n - 1 - \sinh^{-1}(1), \]
which validates the result is sharp.

(ii) Let \( f \in \mathcal{F}_2 \) and consider the functions \( s, d : \mathbb{D} \to \mathbb{C} \), where \( s(z) = f(z)/g(z) \) and \( d(z) = g(z)/z \). Clearly, \( s \in \mathcal{P}_n(1/2) \) and \( d \in \mathcal{P}_n \). As \( f(z) = zd(z)s(z) \), applying Lemma 3.2 here gives
\[ \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{2nr^n}{1 - r^{2n}} + \frac{nr^n}{1 - r^n} \leq \frac{3nr^n + nr^{2n}}{1 - r^{2n}} \leq \sinh^{-1}(1), \]
whenever
\[ r \leq \left( \frac{\sqrt{9n^2 + 4\sinh^{-1}(1)(n + \sinh^{-1}(1)) - 3n}}{2(n + \sinh^{-1}(1))} \right)^{1/n} = R_{S_{\rho,n}}(\mathcal{F}_2). \]
Therefore, \( f \in \mathcal{S}_{\rho,n} \) holds for \( r \leq R_{S_{\rho,n}}(\mathcal{F}_2) \). Next see that \( \text{Re}(g_0(z)/z) > 1/2 \) while \( \text{Re}(f_0(z)/g_0(z)) > 0 \) for the functions
\[ f_0(z) = z(1 + z^n)/(1 - z^n)^2 \text{ and } g_0(z) = z/(1 - z^n). \]
Therefore \( f_0 \in \mathcal{F}_2 \) which verifies the sharpness for \( z = R_{S_{\rho,n}}(\mathcal{F}_2) \) such that
\[ \frac{zf_0'(z)}{f_0(z)} - 1 = \frac{3nz^n + nz^{2n}}{1 - z^{2n}} = \sinh^{-1}(1). \]

(iii) Let \( f \in \mathcal{F}_3 \) and consider the functions \( s, d : \mathbb{D} \to \mathbb{C} \), where \( s(z) = g(z)/f(z) \) and \( d(z) = g(z)/z \). Then \( d \in \mathcal{P}_n \). We can verify that \( |1/s(z) - 1| < 1 \) holds whenever \( \text{Re}(s(z)) > 1/2 \) and therefore \( s \in \mathcal{P}_n(1/2) \). As \( f(z) = zd(z)/s(z) \), on applying Lemma 3.2, we obtain
\[ \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{3nr^n + nr^{2n}}{1 - r^{2n}} \leq \sinh^{-1}(1). \]
The rest of the proof is omitted as it is analogous to proof of Theorem 4.10(ii). The sharpness can be verified as follows. Examine the functions
\[ f_0(z) = \frac{z(1 + z^n)^2}{1 - z^n} \text{ and } g_0(z) = \frac{z(1 + z^n)}{1 - z^n}. \]
Using above definitions of \( f_0 \) and \( g_0 \), we see that
\[ \text{Re} \left( \frac{g_0(z)}{f_0(z)} \right) = \text{Re} \left( \frac{1}{1 + z^n} \right) > \frac{1}{2} \text{ and } \text{Re} \left( \frac{g_0(z)}{z} \right) = \text{Re} \left( \frac{1 + z^n}{1 - z^n} \right) > 0, \]
and therefore, $f_0 \in F_3$. Now at $z = R_{p,n} \left( F_3 \right) e^{i \pi/n}$, we obtain
\[
\frac{zf_0'(z)}{f_0(z)} - 1 = \frac{3nz^n - nz^{2n}}{1 - z^{2n}} = -\sinh^{-1}(1),
\]
which serves as validation for the sharp result.

This concludes the proof. □

References

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