A NON-ITERATIVE RECONSTRUCTION METHOD FOR AN INVERSE PROBLEM ModeLED BY A STOKES-BRINKMANN EQUATIONS

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Abstract. This work is concerned with a geometric inverse problem in fluid mechanics. The aim is to reconstruct an unknown obstacle immersed in a Newtonian and incompressible fluid flow from internal data. We assume that the fluid motion is governed by the Stokes-Brinkmann equations in the two dimensional case. We propose a simple and efficient reconstruction method based on the topological sensitivity concept. The geometric inverse problem is reformulated as a topology optimization one minimizing a least-square functional. The existence and stability of the optimization problem solution are discussed. A topological sensitivity analysis is derived with the help of a straightforward approach based on a penalization technique without using the classical truncation method. The theoretical results are exploited for building a non-iterative reconstruction algorithm. The unknown obstacle is reconstructed using a level-set curve of the topological gradient. The accuracy and the robustness of the proposed method are justified by some numerical examples.

1. Introduction

This paper is concerned with a geometric inverse problem related to the Stokes-Brinkmann equations. It consists in reconstructing an obstacle immersed in a porous media with the help of collecting measurements of the velocity of the fluid motion. Such an inverse problem has several applications such as modeling of liquids or gas through the ground [21, 29] and microfluidics [20].

Geometric inverse problems in fluid mechanics have been the subject of various theoretical and numerical investigations. The majority of the considered applications are modeled by the Stokes equations, see for instance [3, 7, 9, 11, 12, 17] and reference therein. However, till now, there is few research works concerning inverse problems governed by the Stokes-Brinkmann system. In this context, one can cite the developed approaches in [22] and

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In the first reference, Lechleiter and Riemmüller identified the shape of a penetrable inclusion from boundary measurements using the factorization method. In [33], Yan et al., solved the considered inverse problem and proposed a method based on the minimization of a tracking cost functional using the shape gradient method. They derived the shape gradient for the tracking functional based on the continuous adjoint method and the function space parametrization technique.

In this work, to solve the considered geometric inverse problem, we propose an alternative approach based on the topological sensitivity analysis. The main idea of the proposed method consists in reformulating the inverse problem as a topology optimization one, where the obstacle is the unknown variable. The topology optimization problem consists in minimizing the so-called least squares functional with the total variation regularization. This cost functional is minimized with respect to a small topological perturbation of the obstacle by using the concept of topological sensitivity. The main advantage of this detection method is that, it provides fast and accurate numerical reconstruction algorithm.

The topological sensitivity analysis consists of studying the variation of a given cost functional with respect to the presence of a small geometric perturbation, such as the insertion of inclusions, cavities, cracks or source-terms. The main idea of this method was originally introduced by Schumacher [31] in the context of compliance minimization in linear elasticity. The first mathematical justification of this approach has been presented by Sokolowski and Zochowski [32] for Laplace operator and circular geometric perturbations. Then Masmoudi worked out a more general topological sensitivity analysis approach based on the generalized adjoint method and the so-called truncation technique [25]. By using this framework the topological sensitivity has been developed for several equations [13, 26, 28, 30]. For other investigations on the topological sensitivity concept, we refer to the book by Novotny and Sokolowski [27].

In order to introduce this concept, let us consider a shape functional $j(\Omega) = J(\psi_{\Omega})$ to be minimized where $\psi_{\Omega}$ is the solution to a given partial differential equation defined in an open and bounded domain $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial \Omega$. For $\varepsilon > 0$, let $\Omega \setminus S_{z,\varepsilon}$ be the perturbed domain obtained by removing a small topological perturbation $S_{z,\varepsilon} = z + \varepsilon S$ from the initial one (unperturbed) $\Omega$, where $z \in \Omega$ and $S \subset \mathbb{R}^2$ is a given fixed and bounded domain containing the origin. Then, the topological sensitivity analysis leads to an asymptotic expansion of the shape function $j$ on the form

$$j(\Omega \setminus S_{z,\varepsilon}) = j(\Omega) + f(\varepsilon)\delta j(z) + o(f(\varepsilon)),$$

where:

- $f(\varepsilon)$ is a positive function depending on the size $\varepsilon$ of the geometric perturbation and going to zero when $\varepsilon$ goes to zero.
the function \( z \mapsto \delta_j(z) \) is called the “topological gradient” or “topological sensitivity” of \( j \) at \( z \). Mathematically, one can express it as

\[
\delta_j(z) := \lim_{\varepsilon \to 0} \frac{j(\Omega \setminus S_{z,\varepsilon}) - j(\Omega)}{f(\varepsilon)}.
\]

Hence, if we want to minimize the cost function \( j \), the best location to insert a small perturbation in \( \Omega \) is where \( \delta j \) is most negative. In fact if \( \delta j(z) < 0 \), we have \( j(\Omega \setminus S_{z,\varepsilon}) \leq j(\Omega) \) for small \( \varepsilon \).

The topological sensitivity analysis has been developed and exploited for solving various geometric inverse problems in fluid mechanics. One can cite the works investigated by Guillaume and Idris in [15] for the Stokes system, by Hassine and Masmoud in [16] for the quasi-Stokes problem and by Amstutz in [5] for the Navier-Stokes equations. The main idea of the mathematical frameworks presented in [15] and [16] is based on the truncation technique. However, the presented approach in [5] is based on the calculus of variation technique and some preliminary estimates of the perturbed solution.

In this paper, we derive a topological sensitivity analysis for the Stokes-Brinkmann operator. The theoretical framework is based on a simplified and rigorous mathematical tools without using the complicated truncation technique. Our main idea is inspired from the well known penalization method, was generally used for imposing Dirichlet boundary conditions type.

Based on the obtained theoretical results, we propose a fast and accurate detection algorithm for solving the considered geometric inverse problem. The shape and the location of the unknown obstacle is reconstructed with the help of the topological gradient. The efficiency and accuracy of the proposed algorithm are illustrated by some numerical examples. The limits of the proposed algorithm is discussed, such as the effect of the obstacle size and the noisy measured data.

The rest of this paper is organized as follows. In Section 2, we introduce the notation for function spaces and we present the forward and inverse problem. Section 3 proves the unique existence and the stability of the considered optimization problem. While in Section 4, we derive the asymptotic expansion of the proposed cost functional. In Section 5, some numerical experiments are presented in order to show the effectiveness of the proposed method. Finally, the paper ends with some concluding remarks in Section 6.

2. The problem setting

In this section, we present the considered geometric inverse problem. Firstly, we introduce some notations and we recall the definitions of some functional spaces.

2.1. Notations

We denote by \( L^q(\Omega) \) and \( H^s(\Omega) \) the usual Lebesgue and Sobolev spaces. We note in bold the vectorial functions and spaces: \( \mathbb{L}^q(\Omega) \) and \( \mathbb{H}^s(\Omega) \). We define
an inner product for matrices by \( M : N = \sum_{i,j=1}^{2} M_{ij} N_{ij} \) for \( M, N \in \mathbb{R}^{2 \times 2} \); the associated norm is \( |M| = \sqrt{M : M} \). The corresponding inner product on \( L^2(\Omega) \) is

\[
\langle M, N \rangle_{L^2(\Omega)} = \int_{\Omega} M : N \quad \text{for} \quad M, N \in L^2(\Omega).
\]

In a Banach space \( \mathcal{Y} \), we denote the weak convergence of a sequence \( \{\zeta_n\}_n \) to \( \zeta \) by

\[
\zeta_n \rightharpoonup \zeta \quad \text{in} \quad \mathcal{Y} \quad \text{as} \quad n \to \infty.
\]

Finally, for the sake of completeness we briefly introduce the space of functions with bounded total variation. Standard properties of bounded variation functions can be found in [4]. A function \( u \) belonging to \( L^1(\Omega) \) is said to be of bounded total variation if

\[
TV(u) := \int_{\Omega} |Du| := \sup \left\{ \int_{\Omega} u \, \text{div} \, \varphi \ \bigg| \ \varphi \in C^1_c(\Omega), \ |\varphi|_{L^\infty(\Omega)} \leq 1 \right\} < \infty,
\]

where \( C^1_c(\Omega) \) is the space of continuously differentiable functions with compact support in \( \Omega \) and \( \|\cdot\|_{L^\infty(\Omega)} \) is the essential supremum norm.

The space of functions of bounded variation on \( \Omega \) is denoted by \( BV(\Omega) \), defined by

\[
BV(\Omega) = \left\{ u \in L^1(\Omega) \bigg| \int_{\Omega} |Du| < \infty \right\}.
\]

It is well known that, the set \( BV(\Omega) \) equipped with the usual norm

\[
\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + \int_{\Omega} |Du|.
\]

defines a Banach space.

2.2. The studied problem

We consider the domain \( \Omega \) as a bounded region containing a Newtonian and incompressible fluid with coefficient of kinematic viscosity \( \nu > 0 \) and has an inverse permeability \( \alpha > 0 \). Inside the fluid flow domain, we assume the existence of a rigid body (obstacle) \( \omega \subset \Omega \) (with Lipschitz boundary) immersed in the fluid. Then, the Brinkmann system describing the motion of the fluid in \( \Omega \) in the presence of the obstacle \( \omega \) is given by (see, for example [10,24])

\[
\begin{aligned}
-\nu \Delta \psi + \alpha \psi + \nabla p &= 0 \quad \text{in} \quad \Omega \setminus \overline{\Sigma}, \\
\text{div} \, \psi &= 0 \quad \text{in} \quad \Omega \setminus \overline{\Sigma}, \\
\psi &= 0 \quad \text{on} \quad \Gamma, \\
\sigma(\psi, p) n &= g \quad \text{on} \quad \Sigma, \\
\psi &= 0 \quad \text{on} \quad \partial \omega,
\end{aligned}
\]

where \( g \in H^{-1/2}(\Sigma) \) is a given function, \( \psi \) represents the velocity of the fluid, \( p \) is the pressure field and \( \sigma \) represents the stress tensor defined by

\[
\sigma(\psi, p) = -p I + 2\nu \varepsilon(\psi),
\]
with I is the $2 \times 2$ identity matrix and $e(\psi)$ is the linear strain tensor defined as

$$e(\psi) = \frac{1}{2} \left( \nabla \psi + (\nabla \psi)^{\top} \right).$$

In addition, $n$ denotes the outward normal to the boundary $\partial \Omega = \Sigma \cup \Gamma$ where $\Sigma$ and $\Gamma$ have both a nonnegative Lebesgue measure and $\Sigma \cap \Gamma = \emptyset$.

In this work, we deal with a geometric inverse problem in fluid mechanics. Our aim is to reconstruct the location and the shape of an unknown obstacle $\omega \subset \Omega$ from a measurement $\psi_d \in H^1(\Omega)$ of the velocity field.

To solve this geometric inverse problem, we introduce the following optimization problem:

$$\begin{cases}
\text{Minimize } K(\omega, \psi) := J(\omega) = \int_{\Omega \setminus \omega} \|\psi - \psi_d\|^2 \, dx + \rho P_\Omega(\omega), \\
\text{subject to } \omega \in U_{ad} \text{ and } \psi \text{ is the solution to } (1),
\end{cases}$$

where $\rho$ is a regularization parameter and $U_{ad}$ is the set of admissible sub-domains

$$U_{ad} = \left\{ \omega \subset \Omega : \omega \text{ is a subdomain in } \Omega \text{ such that } P_\Omega(\omega) < +\infty \right\},$$

with $P_\Omega(\omega)$ denotes the relative perimeter of $\omega$ in $\Omega$, defined by

$$P_\Omega(\omega) = \sup \left\{ \int_\omega \text{div } \varphi : \varphi \in C^1_c(\Omega), \|\varphi\|_{L^\infty(\Omega)} \leq 1 \right\}.$$

Remark 2.1. If $P_\Omega(\omega) < +\infty$, we say that $\omega$ has finite perimeter in $\Omega$. In this case the relative perimeter $P_\Omega(\omega)$ of $\omega$ coincides with the total variation of the distributional gradient of the characteristic function of $\omega$:

$$P_\Omega(\omega) = |D\chi_\omega|(\Omega),$$

with $\chi_\omega$ is the characteristic function of the sub-domain $\omega$.

3. Analysis of the minimization problem

In this section, we will establish some theoretical results for the minimization problem (2). Existence and uniqueness result is proved in Subsection 3.1. Then, the stability notion is discussed in Subsection 3.2. We start our analysis by the following preliminary result.

Lemma 3.1 ([14], see also [22]). The boundary value problem (1) admits a unique solution $(\psi(\omega), p(\omega))$ and there exists a constant $c > 0$ such that

$$\|\psi(\omega)\|_{H^1(\Omega)} \leq c \|g\|_{H^{-1/2}(\Sigma)}.$$ 

Remark 3.2. In the above inequality, the solution $\psi(\omega)$ is extended by zero inside the domain $\omega$, still denoted by $\psi(\omega)$. 

3.1. Existence of a minimizer

The penalization of the cost function ($L^2$-norm) by the relative perimeter is relevant for the existence and uniqueness of the optimal solution of (2), which will be proved in the following theorem.

**Theorem 3.3.** For any $\psi^d \in H^1(\Omega)$, there exists a unique minimizer $\omega^* \in U_{ad}$ to the minimization problem (2).

**Proof.** Since $J(\omega)$ is non-negative, it follows that $\inf_{\omega \in U_{ad}} J(\omega)$ is finite. Therefore, there exists a minimizing sequence $\{\omega_n\}_{n \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} J(\omega_n) = \inf_{\omega \in U_{ad}} J(\omega).$$

From the definition of the admissible set $U_{ad}$, we have $P_{\Omega}(\omega_n) < +\infty$ then $\{\chi_{\omega_n}\}_{n \in \mathbb{N}}$ is bounded in $BV(\Omega)$. Thus, $\{\chi_{\omega_n}\}_{n \in \mathbb{N}}$ is relatively compact in $L^1(\Omega)$. Therefore, there exist $\omega^* \in U_{ad}$ and a subsequence of $\{\chi_{\omega_n}\}_{n \in \mathbb{N}}$, still denoted by $\{\chi_{\omega_n}\}_{n \in \mathbb{N}}$, such that

$$\chi_{\omega_n} \rightharpoonup \chi_{\omega^*} \text{ in } L^1(\Omega) \text{ as } n \to \infty.$$

Now we prove that $\omega^*$ is indeed the unique minimizer to the problem (2).

Since each $\omega_n$ corresponds with a solution $\psi(\omega_n)$ to (1) with $\omega = \omega_n$, it follows immediately from Lemma 3.1 that the sequence $\{\psi(\omega_n)\}_{n \in \mathbb{N}}$ is also bounded in $H^1(\Omega)$. This indicates the existence of some $\psi^* \in H^1(\Omega)$ and a subsequence of $\{\psi(\omega_n)\}_{n \in \mathbb{N}}$, again still denoted by $\{\psi(\omega_n)\}_{n \in \mathbb{N}}$, such that

$$\psi(\omega_n) \rightharpoonup \psi^* \text{ in } H^1(\Omega) \text{ as } n \to \infty.$$

We claim $\psi^* = \psi(\omega^*)$. Actually, using Green’s formula on (1), we have

$$\int_{\Omega \setminus \overline{\omega}} \left( \rho \nabla \psi : \nabla \vartheta + \alpha \psi \cdot \vartheta \right) \, dx = \int_{\Sigma} g \cdot \vartheta \, ds \text{ for all } \vartheta \in V(\omega),$$

where the functional space $V(\omega)$ is defined by

$$V(\omega) = \left\{ v \in H^1(\Omega \setminus \overline{\omega}); \text{ div } v = 0 \text{ in } \Omega \text{ and } v = 0 \text{ on } \Gamma \cup \partial \omega \right\}.$$ 

By taking $\omega = \omega_n$ and $\psi = \psi(\omega_n)$ we have

$$\int_{\Omega \setminus \overline{\omega}} \left( \rho \nabla \psi(\omega_n) : \nabla \vartheta + \alpha \psi(\omega_n) \cdot \vartheta \right) \, dx = \int_{\Sigma} g \cdot \vartheta \, ds \text{ for all } \vartheta \in V(\omega_n).$$

Since (3) implies

$$\nabla \psi(\omega_n) \rightharpoonup \nabla \psi^* \text{ in } L^2(\Omega) \text{ as } n \to \infty,$$

we pass $n \to \infty$ in (4) to obtain

$$\int_{\Omega \setminus \overline{\omega}} \left( \rho \nabla \psi^* : \nabla \vartheta + \alpha \psi^* \cdot \vartheta \right) \, dx = \int_{\Sigma} g \cdot \vartheta \, ds \text{ for all } \vartheta \in V(\omega^n).$$

Because of the uniqueness of the limit, we conclude that $\psi^*$ coincides with the unique solution to (1) with $\omega = \omega^*$, that is, $\psi^* = \psi(\omega^*)$. 

Finally, using the fact that $\chi_{\omega_n} \to \chi_{\omega^*}$ in $L^1(\Omega)$ and (3), by exploiting the lower semi-continuity of the $L^2$-norm and the lower semi-continuity of the perimeter one can conclude
\[
J(\omega^*) = \int_{\Omega \setminus \omega^*} |\psi^* - \psi|^2 \, dx + \rho P_{\Omega}(\omega^*) \\
\leq \liminf_{n \to \infty} \int_{\Omega \setminus \omega_n} |\psi(\omega_n) - \psi|^2 \, dx + \rho \liminf_{n \to \infty} P_{\Omega}(\omega_n) \\
\leq \liminf_{n \to \infty} J(\omega_n) = \inf_{\omega \in \mathcal{U}_{ad}} J(\omega).
\]

3.2. Stability

Here, we establish stability result for the minimization problem (2) with respect to a small perturbation of the internal data $\psi^d$.

**Theorem 3.4.** Let $\psi^d_n \subset H^1(\Omega)$ be a sequence such that
\[
\psi^d_n \to \psi^d \text{ in } H^1(\Omega) \text{ as } n \to \infty,
\]
and $\{\omega_n\}_n$ be a sequence of minimizer of problems

\[
\text{Minimize } J_n(\omega) \text{ with } J_n(\omega) := \int_{\Omega \setminus \omega} |\psi - \psi^d_n|^2 \, dx + \rho P_{\Omega}(\omega), \quad n = 1, 2, \ldots.
\]

Then $\{\omega_n\}_n$ converges to the minimizer of (2).

**Proof.** The unique existence of each $\omega_n$ is guaranteed by Theorem 3.3. Since $P_{\Omega}(\omega_n) < +\infty$, then $\chi_{\omega_n}$ are bounded in $BV(\Omega)$. Thus $\chi_{\omega_n}$ are relatively compact in $L^1(\Omega)$. Hence, there exist $\omega^* \in \mathcal{U}_{ad}$ and a subsequence of $\{\chi_{\omega_n}\}_n$, still denoted by $\{\chi_{\omega_n}\}_n$, such that
\[
\chi_{\omega_n} \to \chi_{\omega^*} \text{ in } L^1(\Omega) \text{ as } n \to \infty.
\]

Now it suffices to show that $\omega^*$ is indeed the unique minimizer of (2). Actually, repeating the same argument as that in the proof of Theorem 3.3, we can derive
\[
\psi(\omega_n) \to \psi(\omega^*) \text{ in } H^1(\Omega) \text{ as } n \to \infty,
\]
up to taking a further subsequence. By using (5) and (6), we obtain
\[
\psi(\omega_n) - \psi^d_n \to \psi(\omega^*) - \psi^d \text{ in } H^1(\Omega) \text{ as } n \to \infty.
\]
Consequently, for any $\omega \in \mathcal{U}_{ad}$, again we exploit the lower semi-continuity of the $L^2$-norm and the lower semi-continuity of the perimeter to deduce
\[
J(\omega^*) = \int_{\Omega \setminus \omega^*} |\psi^* - \psi^d|^2 \, dx + \rho P_{\Omega}(\omega^*) \\
\leq \liminf_{n \to \infty} \int_{\Omega \setminus \omega_n} |\psi(\omega_n) - \psi^d|^2 \, dx + \rho \liminf_{n \to \infty} P_{\Omega}(\omega_n) \\
\leq \liminf_{n \to \infty} J(\omega_n) = \inf_{\omega \in \mathcal{U}_{ad}} J(\omega). \quad \square
\[
\leq \lim_{n \to \infty} \left[ \frac{1}{\Omega} \int_{\Omega} \left| \psi(n) - \psi_n \right|^2 dx + \rho P_\Omega(n) \right]
= \int_{\Omega} \left| \psi(n) - \psi_n \right|^2 dx + \rho P_\Omega(n) = J(n), \quad \forall n \in U_{ad},
\]
which implies that \( n^* \) is the minimizer of (2).

To solve the minimization problem (2) and built a numerical reconstruction algorithm, we introduce the topological sensitivity analysis method.

4. Topological sensitivity analysis

In this section, we derive a topological asymptotic expansion for the Stokes-Brinkmann operator. It consist in studying the variation of the cost functional \( K \) with respect to the presence of a small obstacle \( S_{z,\varepsilon} = z + \varepsilon S \) in the fluid flows domains \( \Omega \).

In order to compute the asymptotic expansion of \( K \), we consider the following penalized Brinkmann problem

\[
\begin{cases}
-\nu \Delta \psi_{\varepsilon} + \alpha \psi_{\varepsilon} + \delta c_{\varepsilon} \psi_{\varepsilon} + \nabla p_{\varepsilon} = 0 & \text{in } \Omega, \\
\text{div} \, \psi_{\varepsilon} = 0 & \text{in } \Omega, \\
\psi_{\varepsilon} = 0 & \text{on } \Gamma, \\
\sigma(\psi_{\varepsilon}, p_{\varepsilon}) n = g & \text{on } \Sigma,
\end{cases}
\]

where \( \delta c_{\varepsilon} \) is a piecewise constant function defined by

\[
\delta c_{\varepsilon}(x) = \begin{cases}
k & \text{if } x \in S_{z,\varepsilon}, \\
0 & \text{if } x \in \Omega \setminus S_{z,\varepsilon},
\end{cases}
\]
with \( k \) is large enough.

Remark 4.1. The penalization method is a well known technique, commonly used in the implementation of the finite element approximation method for imposing Dirichlet boundary conditions. In our case, if \( k \) goes to infinity in \( S_{z,\varepsilon} \), for a given \( \varepsilon \), the corresponding solution \( \psi_{\varepsilon} \) converges to the solution of the following perturbed Stokes-Brinkmann system

\[
\begin{cases}
-\nu \Delta \psi_{\varepsilon} + \alpha \psi_{\varepsilon} + \nabla p_{\varepsilon} = 0 & \text{in } \Omega \setminus S_{z,\varepsilon}, \\
\text{div} \, \psi_{\varepsilon} = 0 & \text{in } \Omega \setminus S_{z,\varepsilon}, \\
\psi_{\varepsilon} = 0 & \text{on } \Gamma, \\
\sigma(\psi_{\varepsilon}, p_{\varepsilon}) n = g & \text{on } \Sigma, \\
\psi_{\varepsilon} = 0 & \text{on } \partial S_{z,\varepsilon}.
\end{cases}
\]

The weak form associated with (7) reads:

\[
\begin{cases}
\text{Find } \psi_{\varepsilon} \in X_{\Gamma} \text{ such that,} \\
\mathcal{A}_\varepsilon(\psi_{\varepsilon}, v) = l_\varepsilon(v), \quad \forall v \in X_{\Gamma},
\end{cases}
\]

where the functional space \( X_{\Gamma} \), the bilinear form \( \mathcal{A}_\varepsilon \), and the linear form \( l_\varepsilon \) are defined by

\[
X_{\Gamma} = \left\{ v \in H^1(\Omega) \text{ such that } \text{div} \, v = 0 \text{ and } v = 0 \text{ on } \Gamma \right\},
\]
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\[ A_\varepsilon(\psi_\varepsilon, v) = \int_\Omega \nu \nabla \psi_\varepsilon : \nabla v \, dx + \int_\Omega (\alpha + \delta \varepsilon) \psi_\varepsilon \cdot v \, dx, \]

\[ l_\varepsilon(v) = \int_\Sigma g \cdot v \, ds. \]

According to the above statements, the cost functional \( K \) is then defined in the perturbed domain as

\[ K(S_{z,\varepsilon}, \psi_\varepsilon) = J(\varepsilon) = \int_\Omega \| \psi_\varepsilon - \psi_0 \|^2 \, dx + \rho P_\Omega(S_{z,\varepsilon}). \]

In the particular case \( S_{z,\varepsilon} = \emptyset \) (i.e., \( \varepsilon = 0 \)), the cost functional \( K \) is defined by \( L^2 \)-norm without the regularization term:

\[ K(\emptyset, \psi_0) = J(0) = \int_\Omega \| \psi_0 - \psi_0 \|^2 \, dx, \]

where \( \psi_0 \) is the solution to

\[
\begin{cases}
-\nu \Delta \psi_0 + \alpha \psi_0 + \nabla p_0 = 0 & \text{in } \Omega, \\
\text{div } \psi_0 = 0 & \text{in } \Omega, \\
\psi_0 = 0 & \text{on } \Gamma, \\
\sigma(\psi_0, p_0) n = g & \text{on } \Sigma.
\end{cases}
\]

In order to establish the expected asymptotic expansion, we start our analysis by the following lemma. It provides an estimate of the perturbed velocity field.

**Lemma 4.2.** Let \( \psi_\varepsilon \) and \( \psi_0 \) be the solutions to the problems (7) and (15), respectively. Then, there exists a positive constant \( c \) independent of \( \varepsilon \) such that

\[ \| \psi_\varepsilon - \psi_0 \|_{H^1(\Omega)} \leq c \varepsilon^{1+\tau} \]

for any \( 0 < \tau < 1 \).

**Proof.** From (7) and (15) and using Green’s formula, we obtain

\[
\begin{align*}
\int_\Omega \nu |\nabla (\psi_\varepsilon - \psi_0)|^2 & \, dx + \int_\Omega (\alpha + \delta \varepsilon) |\psi_\varepsilon - \psi_0|^2 \, dx \\
& \quad + \int_\Omega \delta \varepsilon \psi_0 \cdot v \, dx = 0 \ \forall v \in X_T.
\end{align*}
\]

By taking \( v = \psi_\varepsilon - \psi_0 \) in (16) as a test function, we get

\[
\begin{align*}
\int_\Omega \nu |\nabla (\psi_\varepsilon - \psi_0)|^2 & \, dx + \int_\Omega (\alpha + \delta \varepsilon) |\psi_\varepsilon - \psi_0|^2 \, dx = -\int_{S_{z,\varepsilon}} \psi_0 \cdot (\psi_\varepsilon - \psi_0) \, dx.
\end{align*}
\]

From the Cauchy-Schwarz inequality and the smoothness of \( \psi_0 \) in \( S_{z,\varepsilon} \), there exists a positive constant \( c_1 \) independent of \( \varepsilon \) such that

\[
\begin{align*}
\int_\Omega \nu |\nabla (\psi_\varepsilon - \psi_0)|^2 & \, dx + \int_\Omega (\alpha + \delta \varepsilon) |\psi_\varepsilon - \psi_0|^2 \, dx \\
& \leq \| \psi_0 \|_{L^2(S_{z,\varepsilon})} \| \psi_\varepsilon - \psi_0 \|_{L^2(S_{z,\varepsilon})} \leq c_1 \varepsilon \| \psi_\varepsilon - \psi_0 \|_{L^2(S_{z,\varepsilon})}.
\end{align*}
\]
Notice that, with the help of Hölder inequality and the Sobolev embedding theorem, one can derive

\[ \| \psi_\varepsilon - \psi_0 \|_{L^2(S_{z,\varepsilon})} \leq c_2 \varepsilon^{1/q} \| \psi_\varepsilon - \psi_0 \|_{L^{2p}(S_{z,\varepsilon})} \leq c_3 \varepsilon^\tau \| \psi_\varepsilon - \psi_0 \|_{H^1(\Omega)} \]

for any \( 1 < q < \infty \) with \( 1/p + 1/q = 1 \). Let us denote \( \tau = 1/q \) which implies \( 0 < \tau < 1 \).

Inserting (18) into (17), we obtain

\[ \int_\Omega \nu \left| \nabla \left( \psi_\varepsilon - \psi_0 \right) \right|^2 dx + \int_\Omega (\alpha + \delta c_\varepsilon) \left| \psi_\varepsilon - \psi_0 \right|^2 dx (19) \]

\[ \leq c_4 \varepsilon^{\tau + 1} \| \psi_\varepsilon - \psi_0 \|_{H^1(\Omega)}. \]

On the other hand, we have

\[ \min \{ \nu, \alpha \} \left\| \psi_\varepsilon - \psi_0 \right\|_{H^1(\Omega)}^2 \leq \int_\Omega \nu \left| \nabla \left( \psi_\varepsilon - \psi_0 \right) \right|^2 dx + \int_\Omega (\alpha + \delta c_\varepsilon) \left| \psi_\varepsilon - \psi_0 \right|^2 dx. \]

Inserting (19) into the above inequality, we get

\[ \min \{ \nu, \alpha \} \left\| \psi_\varepsilon - \psi_0 \right\|_{H^1(\Omega)}^2 \leq c_4 \varepsilon^{\tau + 1} \| \psi_\varepsilon - \psi_0 \|_{H^1(\Omega)}. \]

Consequently,

\[ \| \psi_\varepsilon - \psi_0 \|_{H^1(\Omega)} \leq c \varepsilon^{\tau + 1} \text{ with } c = \frac{c_4}{\min \{ \nu, \alpha \}}. \]

The following lemma gives a preliminary estimate of the variation \( \mathcal{K}(S_{z,\varepsilon}, \psi_\varepsilon) - \mathcal{K}(\emptyset, \psi_0) \) with respect to \( \varepsilon \).

**Lemma 4.3.** The cost functional \( \mathcal{K} \) is differential with respect to \( \psi_0 \), such that

\[ DK(\emptyset, \psi_0)w = 2 \int_\Omega \left( \psi_0 - \psi_0^d \right) \cdot w \ dx \ \forall w \in H^1(\Omega) \]

and we have

\[ \mathcal{K}(S_{z,\varepsilon}, \psi_\varepsilon) - \mathcal{K}(\emptyset, \psi_0) = DK(\emptyset, \psi_0)(\psi_\varepsilon - \psi_0) + o(\varepsilon^2). \]

**Proof.** The verification of the differentiability of \( \mathcal{K} \) with respect to \( \psi_0 \) such that

\[ DK(\emptyset, \psi_0)w = 2 \int_\Omega \left( \psi_0 - \psi_0^d \right) \cdot w \ dx \ \forall w \in H^1(\Omega) \]

is trivial. By subtracting (14) from (13), one can obtain

\[ \mathcal{K}(S_{z,\varepsilon}, \psi_\varepsilon) - \mathcal{K}(\emptyset, \psi_0) \]

\[ = \int_\Omega \left| \psi_\varepsilon - \psi_\varepsilon^d \right|^2 dx - \int_\Omega \left| \psi_0 - \psi_0^d \right|^2 dx + \rho P_\Omega(S_{z,\varepsilon}) \]

\[ = \int_\Omega \left( \left| \psi_\varepsilon - \psi_0 \right|^2 + \left| \psi_0 - \psi_0^d \right|^2 \right) dx - \int_\Omega \left| \psi_0 - \psi_\varepsilon^d \right|^2 dx + \rho P_\Omega(S_{z,\varepsilon}) \]

\[ = 2 \int_\Omega \left( \psi_\varepsilon - \psi_0 \right) \cdot \left( \psi_0 - \psi_0^d \right) dx + \int_\Omega \left| \psi_\varepsilon - \psi_0 \right|^2 dx + \rho P_\Omega(S_{z,\varepsilon}) \]
\[ D_K(\emptyset, \psi_0)(\psi_\varepsilon - \psi_0) = 2 \int_\Omega (\psi_\varepsilon - \psi_0) \cdot (\psi_0 - \psi_0^\varepsilon) \, dx. \]

Using Lemma 4.2, the second term on the right-hand-side of the equality (22) admits the following approximation
\[ \int_\Omega \left| \psi_\varepsilon - \psi_0 \right|^2 \, dx = o(\varepsilon^2). \]

To estimate the last term in the right-hand-side of (22), we need to take \( \rho = \varepsilon^3 \) then,
\[ \rho P_\Omega(S_{z,\varepsilon}) = o(\varepsilon^2), \]
since \( P_\Omega(S_{z,\varepsilon}) < +\infty \). Therefore;
\[ K(S_{z,\varepsilon}, \psi_\varepsilon) - K(\emptyset, \psi_0) = D_K(\emptyset, \psi_0)(\psi_\varepsilon - \psi_0) + o(\varepsilon^2). \]

\[ \text{□} \]

Now, we are ready to state the main result of this section. It concerns the expected topological asymptotic expansion of the cost function \( J \).

**Theorem 4.4.** Let \( S_{z,\varepsilon} = z + \varepsilon S \) be a small obstacle in the fluid flow domain \( \Omega \) and let \( J \) be a cost function of the form
\[ J(\varepsilon) = \int_\Omega \left| \psi_\varepsilon - \psi_0^\varepsilon \right|^2 \, dx + \rho P_\Omega(S_{z,\varepsilon}). \]

Then the cost function \( J \) has the following asymptotic expansion:
\[ J(\varepsilon) - J(0) = k|S|\varepsilon^2 \mathcal{G}(z) + o(\varepsilon^2), \]
where \( |S| \) is the Lebesgue measure (volume) of \( S \) and \( \mathcal{G} \) is the topological gradient defined in \( \Omega \) by
\[ \mathcal{G}(z) = \psi_0(z) \cdot \vartheta_0(z), \]
with \( \vartheta_0 \) is the solution to the adjoint problem: find \( \vartheta_0 \in X_\Gamma \) such that
\[ A_0(w, \vartheta_0) = -DK(\emptyset, \psi_0)w \quad \forall w \in X_\Gamma. \]

**Proof.** Let us introduce the Lagrangian \( \mathcal{L}_\varepsilon \) defined by
\[ \mathcal{L}_\varepsilon(u, v) = \mathcal{K}(S_{z,\varepsilon}, u) + \mathcal{A}_\varepsilon(u, v) - l_\varepsilon(v) \quad \forall u, v \in X_\Gamma. \]

Setting \( u = \psi_\varepsilon \) in the above equality and using the fact that \( \psi_\varepsilon \) is the weak solution to (9), we obtain
\[ \mathcal{L}_\varepsilon(\psi_\varepsilon, v) = \mathcal{K}(S_{z,\varepsilon}, \psi_\varepsilon) \quad \forall v \in X_\Gamma. \]

Hence,
\[ J(\varepsilon) - J(0) = \mathcal{L}_\varepsilon(\psi_\varepsilon, v) - \mathcal{L}_0(\psi_0, v) \]
\[ = \mathcal{K}(S_{z,\varepsilon}, \psi_\varepsilon) - \mathcal{K}(\emptyset, \psi_0) + \mathcal{A}_\varepsilon(\psi_\varepsilon, v) - \mathcal{A}_0(\psi_0, v) + l_0(v) - l_\varepsilon(v). \]

(24)
The linear form \( l \) is independent of \( \varepsilon \), then
\[
(25) \quad l_0(v) - l_\varepsilon(v) = 0, \quad \forall v \in X_0.
\]
For all \( v \in X_0 \), the variation of the bilinear form is given by
\[
\mathcal{A}_\varepsilon(\psi_\varepsilon, v) - \mathcal{A}_0(\psi_0, v) = \int_\Omega \alpha \left( \psi_\varepsilon - \psi_0 \right) \cdot v \, dx + \int_{S_{z,\varepsilon}} k \psi_\varepsilon \cdot v \, dx.
\]
Choosing \( v = \vartheta_0 \) in the above equality, where \( \vartheta_0 \) is a solution to (23), we obtain
\[
\mathcal{A}_\varepsilon(\psi_\varepsilon, \vartheta_0) - \mathcal{A}_0(\psi_0, \vartheta_0) = \mathcal{A}_0(\psi_\varepsilon - \psi_0, \vartheta_0) + \int_{S_{z,\varepsilon}} k \psi_\varepsilon \cdot \vartheta_0 \, dx.
\]
By taking \( w = \psi_\varepsilon - \psi_0 \) as a test function in (23), we deduce that
\[
\mathcal{A}_\varepsilon(\psi_\varepsilon, \vartheta_0) - \mathcal{A}_0(\psi_0, \vartheta_0) = -D K(\emptyset, \psi_0)(\psi_\varepsilon - \psi_0) + \int_{S_{z,\varepsilon}} k \psi_\varepsilon \cdot \vartheta_0 \, dx.
\]
Then, it follows from (24), (25), (26) and (21) that
\[
(27) \quad J(\varepsilon) - J(0) = \int_{S_{z,\varepsilon}} k \psi_\varepsilon \cdot \vartheta_0 \, dx + o(\varepsilon^2).
\]
To this end, we start with the following decomposition
\[
(28) \quad \int_{S_{z,\varepsilon}} k \psi_\varepsilon \cdot \vartheta_0 \, dx = k \int_{S_{z,\varepsilon}} \psi_0 \cdot \vartheta_0 \, dx + k \int_{S_{z,\varepsilon}} (\psi_\varepsilon - \psi_0) \cdot \vartheta_0 \, dx.
\]
Let us first focus on the first term in the right-hand side of (28). Using the change of variable \( x = z + \varepsilon y \), we derive
\[
(29) \quad k \int_{S_{z,\varepsilon}} \psi_0 \cdot \vartheta_0 \, dx = k \varepsilon^2 \int_S \psi_0(z) \cdot \vartheta_0(z) \, dy + k \varepsilon^2 \int_S \left[ \psi_0(z + \varepsilon y) \cdot \vartheta_0(z + \varepsilon y) - \psi_0(z) \cdot \vartheta_0(z) \right] \, dy.
\]
From the fact that \( \psi_0 \) and \( \vartheta_0 \) are regular near \( z \), we have
\[
\lim_{\varepsilon \to 0} \int_S \left[ \psi_0(z + \varepsilon y) \cdot \vartheta_0(z + \varepsilon y) - \psi_0(z) \cdot \vartheta_0(z) \right] \, dy = 0.
\]
Hence, the second term in (29) can be neglected with respect to \( \varepsilon^2 \) and one can deduce
\[
k \int_{S_{z,\varepsilon}} \psi_0 \cdot \vartheta_0 \, dx = k |S| \varepsilon^2 \psi_0(z) \cdot \vartheta_0(z) + o(\varepsilon^2).
\]
For the other term in the right-hand side of (28) using Hölder inequality, we derive
\[
\left| \int_{S_{z,\varepsilon}} (\psi_{\varepsilon} - \psi_0) \cdot \vartheta_0 \, dx \right| \leq \left\| \vartheta_0 \right\|_{L^2(S_{z,\varepsilon})} \left\| \psi_{\varepsilon} - \psi_0 \right\|_{L^2(S_{z,\varepsilon})} \\
\leq \left\| \vartheta_0 \right\|_{L^2(S_{z,\varepsilon})} \left\| \psi_{\varepsilon} - \psi_0 \right\|_{H^1(\Omega)}.
\]
We know using elliptic regularity that \( \vartheta_0 \) is uniformly bounded in \( S_{z,\varepsilon} \). Thus
\[
\left\| \vartheta_0 \right\|_{L^2(S_{z,\varepsilon})}^2 = \int_{S_{z,\varepsilon}} \left| \vartheta_0 \right|^2 \, dx \leq c \int S \varepsilon^2 \leq c \varepsilon^2.
\]
Consequently,
\[
\left\| \vartheta_0 \right\|_{L^2(S_{z,\varepsilon})} \leq c \varepsilon.
\]
Inserting (31) into (30), we get
\[
\left| \int_{S_{z,\varepsilon}} (\psi_{\varepsilon} - \psi_0) \cdot \vartheta_0 \, dx \right| \leq c \varepsilon \left\| \psi_{\varepsilon} - \psi_0 \right\|_{H^1(\Omega)}.
\]
From Lemma 4.2, we deduce that
\[
\left| \int_{S_{z,\varepsilon}} (\psi_{\varepsilon} - \psi_0) \cdot \vartheta_0 \, dx \right| \leq c \varepsilon^{2+\tau}.
\]
Since \( \tau > 0 \), the term \( c \varepsilon^{2+\tau} \) can be neglected with respect to \( \varepsilon^2 \). Therefore,
\[
\left| \int_{S_{z,\varepsilon}} (\psi_{\varepsilon} - \psi_0) \cdot \vartheta_0 \, dx \right| = o(\varepsilon^2),
\]
which achieve the proof of the theorem. \( \square \)

5. Numerical results

In this section, we present some numerical investigations showing the efficiency of the proposed method. The aim is to reconstruct the location and shape of an unknown obstacle \( \omega \) inserted inside the fluid flow domain \( \Omega \) from internal data. The proposed numerical approach is based on the topological sensitivity analysis method.

From Theorem 4.4, the functional \( K \) admits the following topological asymptotic expansion:
\[
K(S_{z,\varepsilon}, \psi_{\varepsilon}) = K(\emptyset, \psi_0) + \varepsilon^2 k[\mathcal{G}(z) + o(\varepsilon^2)),
\]
with \( \mathcal{G} \) is the topological gradient, defined in \( \Omega \) by
\[
\mathcal{G}(z) = \psi_0(z) \cdot \vartheta_0(z).
\]
where $\psi_0$ is the solution to (15) and $\vartheta_0$ is the adjoint state, solution to

$$
\begin{cases}
-\nu \Delta \vartheta_0 + \alpha \vartheta_0 + \nabla p_0 = -2(\psi_0 - \psi^d) & \text{in } \Omega, \\
\text{div } \vartheta_0 = 0 & \text{in } \Omega, \\
\vartheta_0 = 0 & \text{on } \Gamma, \\
\sigma(\vartheta_0, p_0)n = 0 & \text{on } \Sigma.
\end{cases}
$$

Since the term $o(\varepsilon^2)$ can be neglected and $k\varepsilon^2|S|$ is a positive constant, one can remark if the topological gradient $G(z) < 0$, we have $K(S_z, \varepsilon, \psi_\varepsilon) \leq K(\emptyset, \psi_0)$.

Then, in order to minimize $K$, the best location $z$ of the small obstacle $S_z$ in the fluid domain $\Omega$ is where $G$ is most negative.

Based on this remark, we propose in this section a simple and fast numerical reconstruction algorithm. To present the considered procedure, we introduce some notations. Let $\delta_{\text{min}}$ be the most negative value of the topological gradient $G$ in $\Omega$ (i.e., $\delta_{\text{min}} = \min_{x \in \Omega} G(x)$). For all $\gamma \in [0, 1]$, we denote by $\omega_\gamma$ the zone defined as

$$
\omega_\gamma = \{ x \in \Omega; G(x) \leq (1 - \gamma)\delta_{\text{min}} \}.
$$

According to the main idea of the topological sensitivity analysis method, the unknown obstacle $\omega$ is likely to be located at the zone where the topological gradient $G$ is the most negative. The main steps of our numerical procedure are summarized in the following “one-shot” algorithm.

**One-shot algorithm.**

- Solve the direct problem (15) and the associated adjoint problem (34).
- Compute the topological gradient $G(x) = \psi_0(x) \cdot \vartheta_0(x)$, $x \in \Omega$.
- Reconstruct the unknown obstacle

$$
\omega = \{ x \in \Omega; G(x) \leq (1 - \gamma^*)\delta_{\text{min}} \},
$$

where $\gamma^* \in (0, 1)$ such that

$$
J(\omega_{\gamma^*}) \leq J(\omega_\gamma) \ \forall \gamma \in (0, 1).
$$

In this one iteration algorithm:

- the location of $\omega$ is given by the point $z \in \Omega$ where the topological gradient $G$ is most negative (i.e., $z = \text{arg min}_{x \in \Omega} G(x)$).
- the optimal size of the constructed obstacle $\omega$ is approximated by a level-set curve of the topological gradient $G$. 

**Remark 5.1.** In the particular case when the exact obstacle $\omega$ is known, the best value $\gamma^*$ of the parameter $\gamma$ can be determined as the minimum of the following error functional,

$$
E(\gamma) = \left[ \text{meas}(\omega \cup \omega_\gamma) - \text{meas}(\omega \cap \omega_\gamma) \right] / \text{meas}(\omega), \ \forall \gamma \in (0, 1),
$$

where $\text{meas}(B)$ is the Lebesgue measure of the set $B \subset \mathbb{R}^2$. 

Similar topological gradient algorithms have already been illustrated in [6] for the identification of cracks from overdetermined boundary data, in [18, 19] for the reconstruction of discontinuous source functions, in [7] for the detection of small gas bubbles in Stokes flow, and for detection of multiple impedance obstacles in [23], and so on.

In this paper, we extend this approach for the reconstruction of an obstacle \( \omega \) immersed in a fluid governed by the Stokes-Brinkmann equation.

In our numerical implementation, we use synthetic data, i.e., the measurement \( \psi^d \) is generated by numerical resolution of the problem (1). The square domain \( \Omega = (0, 1) \times (0, 1) \) is used as a mould filled with a viscous and incompressible fluid. The two components \( \Sigma \) and \( \Gamma \) of the boundary \( \partial \Omega \) are described in Figure 1. The direct problem (15) and the adjoint problem (34) are solved with the help of a \( P_2/P_1 \) finite elements method. The approximated solutions are computed using uniform mesh with step size \( h = 1/100 \). The numerical procedure is implemented using the free software FreeFem ++.

Next, we present some reconstruction results showing the efficiency of the proposed one-shot algorithm.

![Figure 1. Domain \( \Omega \) with boundary \( \partial \Omega = \Sigma \cup \Gamma \)](image)

### 5.1. Reconstruction of some obstacles

This section is devoted to the reconstruction of obstacles having circular or elliptical shapes with no noise added to the simulated data.

**Example 1.** *Reconstruction of a circular-shaped obstacle.* In this example, we test our procedure to detect a circular-shaped obstacle. More precisely, we want to reconstruct an obstacle \( \omega \) described by the disc centered at \( z = (0.5, 0.5) \) with radius \( r = 0.05 \). The obtained reconstruction results are illustrated in Figure 2.
As one can observe in Figure 2, the unknown obstacle (see Figure 2(b), black line) is located at the region where the topological gradient $G$ is the most negative (see Figure 2(a), red zone). It is approximated by a level-set curve of the topological gradient (see Figure 2(b) red lines).

The optimal approximation of the exact shape of the actual obstacle, described with black line in Figure 2(b) (circle centered at $(0.5, 0.5)$ with radius $r = 0.05$), is obtained by minimizing the function $E$ and the determination of the optimal value $\gamma^*$ of the parameter $\gamma$:

$$\gamma^* = \arg \min_{\gamma \in \{\gamma_1, \ldots, \gamma_L\}} E(\gamma).$$

For more details, the variation of the function $E(\gamma)$ is plotted in Figure 3. Figure 3(b) shows the exact boundary $\partial \omega$ (represented with black line) and the obtained one (represented with red line).

(a) Negative zone (red zone) of $G$  
(b) Iso-values of the topological gradient $G$

**Figure 2.** Detection of a circular-shaped obstacle

(a) Variation of $E$ with respect to $\gamma$  
(b) Reconstruction with $\gamma^* = 0.35$

**Figure 3.** Exact (black line) and reconstructed (red lines) obstacles
As one can observe here, our numerical algorithm worked very well for this example. It provide an efficient reconstruction result for the considered circular-shaped obstacle.

**Example 2. Reconstruction of ellipse-shaped obstacle.** In this example, we aim to reconstruct an obstacle described by an ellipse centered at \((0.5, 0.5)\). The obtained results are presented in Figure 4.

![Iso-values of \(G\)](a) Iso-values of \(G\) ![Reconstruction with \(\gamma^* = 0.57\)](b) Reconstruction with \(\gamma^* = 0.57\)

**Figure 4.** Reconstruction of an ellipse-shaped obstacle

Once again, we obtain a good reconstruction of the considered obstacle. As one can see in Figure 4(a), the unknown obstacle is located in the zone where the topological gradient is negative. Its boundary (see Figure 4(a) red lines) is approximated by the level-set curve given by the optimal value \(\gamma^* = 0.15\).

In conclusion, the proposed one iteration reconstruction algorithm works well for the identification of circular and elliptical-shaped obstacles. It provides an efficient estimate of the location and shape of the unknown obstacle.

The last part of this section is concerned with the limits and the features of the proposed algorithm. The influence of the size of the obstacle on the accuracy of the reconstruction procedure is discussed in the next paragraph.

5.2. Influence of the size of the obstacle

Here, we aim to evaluate the influence of the obstacle size on the quality of the reconstruction results. In order to do that, we apply our algorithm to detect circular-shaped obstacles having variable radius \(r\). More precisely, the obstacle to be detected is defined by the disc centered at \((0.5, 0.5)\) with a variable radius \(r \in \{0.03, 0.06, 0.12, 0.18\}\). The obtained detection results are shown in Figure 5.

![Reconstruction results](a) Reconstruction results (b) Reconstruction results

From the obtained numerical results, one can conclude that:
- when the obstacle is relatively small, the quality of reconstruction is quite efficient (see Figure 5(a)-(b)),

...
– when the obstacle becomes “too big” (see Figure 5(b)-(d)), the algorithm is not able to provide an acceptable approximation.

Figure 5. Reconstruction of obstacles having variables sizes

Next, we investigate the robustness of the method with respect to noisy measurement.

5.3. Reconstruction results with noisy data

In this paragraph, we test the robustness of our one iteration reconstruction method when the measurement $\psi^d$ is corrupted with Gaussian random noise. More precisely, the measurement $\psi^d$ is replaced by

$$\psi_d^\delta = \psi^d + \delta \psi^d,$$

where $\delta \psi^d$ is a Gaussian random noise with mean zero and standard deviation $\delta \| \psi^d \|_\infty$, where $\delta$ is a parameter.

In this test, we reconstruct an ellipse centered at $0.5, 0.5). The obtained results are illustrated in Figure 6. From the reconstruction results in Figure 6, one can notice that if the noise level no more than 20%, that our algorithm is
able to detect the location and the shape of obstacle, whereas for a noise level larger than 30% the reconstruction becomes wrong.

Figure 6. Reconstruction results with noisy data
5.4. The main feature of our procedure

Compared to the classical geometric reconstruction algorithm, our proposed algorithm has several advantages as:

- Only one iteration is needed to identify the location of the unknown obstacle which significantly reduces the running times.
- The shape of the unknown obstacle is approximated by a level-set curve of a scalar function, which gives it more flexibility and a wide range of applications.
- Unlike the classical geometric reconstruction algorithms such as level-set method [2, 8] or homogenization method [1], our numerical approach is not sensitive to the initial geometry. The topological sensitivity analysis method consists in creating some geometric perturbations in the initial domain.

6. Concluding remarks

In this work, we have proposed an efficient reconstruction approach for solving a geometric inverse problem in fluid mechanics. The fluid flow is governed by the two-dimensional Stokes-Brinkmann system. The considered inverse problem is reformulated as a topology optimization one by minimizing a least-square function. The existence and stability of the optimization problem solution have been proved. The proposed reconstruction method is based on the topological sensitivity concept. A topological asymptotic expansion is derived with the help of simplified mathematical analysis tools without using the complicate truncation technique [16]. The topological gradient (the leading term of the asymptotic expansion) has been exploited for building an efficient numerical reconstruction algorithm. Our proposed method is general and can be adapted for solving various inverse problems.

In this work, focused on the topological sensitivity analysis and a non-iterative reconstruction method, several mathematical issues of high interest could not be discussed. The identifiability issue is one of them. The full identifiability problem is, however, up to our knowledge, still an open one which deserves attention. Reconstructing obstacle from partial interior measurements of the velocity is also an interesting problem to tackle, for several causes may lead to such a situation, especially when zones of the fluid flow domain are not accessible to measurements.

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