The Critical Point Equation on 3-dimensional $\alpha$-cosymplectic Manifolds

ADARA M. BLAGA*
Department of Mathematics, West University of Timișoara, Bld. V. Pârvan nr. 4, 300223, Timișoara, România
e-mail: adarablaga@yahoo.com

CHIRANJIB DEY
Dhamla Jr. High School, Vill-Dhamla, P.O.-Kedarpur, Dist-Hooghly, Pin-712406, West Bengal, India
e-mail: dey9chiranjib@gmail.com

Abstract. The object of the present paper is to study the critical point equation (CPE) on 3-dimensional $\alpha$-cosymplectic manifolds. We prove that if a 3-dimensional connected $\alpha$-cosymplectic manifold satisfies the Miao-Tam critical point equation, then the manifold is of constant sectional curvature $-\alpha^2$, provided $D\lambda \neq (\xi\lambda)\xi$. We also give several interesting corollaries of the main result.

1. Introduction

In [4], Miao-Tam studied the volume functional on the space of constant scalar curvature metrics with a given boundary metric. They derived a necessary and sufficient condition for a metric to be a critical point as follows:

On a compact Riemannian manifold $(M^n, g)$, $n \geq 3$, with smooth boundary, if there exists a non-zero smooth function $\lambda : M^n \to \mathbb{R}$ (called potential function) such that

\[ Hess\lambda - (\Delta\lambda)g - \lambda S = g \text{ on } M^n \tag{1.1} \]

and $\lambda = 0$ on $\partial M^n$, where $\Delta$ is the Laplacian operator, Hess is the Hessian operator and $S$ is the Ricci tensor with respect to the metric $g$, then $g$ is said to satisfy the Miao-Tam critical condition.

In particular, if the potential function $\lambda$ is a non-zero constant, then (1.1) is just

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an Einstein metric. Recently, Hwang [2] proved that the CPE conjecture is also true under certain condition on the bounds of the potential function $\lambda$. In 2017, Wang [8] proved that if the metric of a 3-dimensional $(k, \mu)'$-almost Kenmotsu manifold satisfies the Miao-Tam critical condition, then the manifold is locally isometric either to the hyperbolic space $\mathbb{H}^3(-1)$ or to the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$. In [7], Ghosh and Patra considered the CPE in the framework of $K$-contact manifolds and $(k, \mu)$-contact manifolds.

Motivated by the above studies, in the present paper we study 3-dimensional $\alpha$-cosymplectic manifolds admitting CPE, i.e. satisfying the relation (1.1). The paper is organized as follows. In section 2, we recall the definition of $\alpha$-cosymplectic manifolds and some basic formulas and section 3 is devoted to prove our main result, precisely:

**Theorem 1.1.** If a 3-dimensional connected $\alpha$-cosymplectic manifold $(M, \phi, \xi, \eta, g, \alpha)$ satisfies CPE, then the manifold is of constant sectional curvature $-\alpha^2$, provided $D\lambda \neq (\xi \lambda)\xi$, where $D$ denotes the gradient operator with respect to $g$.

2. Preliminaries

An almost contact metric structure on a $(2n+1)$-dimensional smooth manifold $M$ consists of a 1-form $\eta$, a vector field $\xi$ (called the Reeb field), a $(1, 1)$-tensor field $\phi$ and a Riemannian metric $g$ satisfying the following conditions:

\begin{equation}
\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi
\end{equation}

and

\begin{equation}
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\end{equation}

for any $X, Y \in \chi(M)$. The above relations imply

\begin{equation}
\eta \circ \phi = 0, \quad \phi \xi = 0,
\end{equation}

\begin{equation}
g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X),
\end{equation}

for any $X, Y \in \chi(M)$.

An almost contact metric structure is said to be normal if the induced almost complex structure $J$ on the product manifold $M \times \mathbb{R}$ defined by

\[ J(X, f \frac{d}{dt}) := (\phi X - f \xi, \eta(X)\frac{d}{dt}) \]

is integrable, where $X \in \chi(M)$, $t$ is the coordinate on $\mathbb{R}$ and $f$ is a smooth function on $M \times \mathbb{R}$.

An almost contact metric structure is said to be a contact metric structure if

\begin{equation}
g(X, \phi Y) = d\eta(X, Y),
\end{equation}

for all $X, Y \in \chi(M)$. 

A contact metric structure is said to be a locally conformally flat contact metric structure if

\begin{equation}
g(\phi X, \phi Y) = g(X, Y) + d\eta(X, Y),
\end{equation}

for all $X, Y \in \chi(M)$. 

An almost contact metric structure is said to be a cosymplectic metric structure if

\begin{equation}
g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),
\end{equation}

for all $X, Y \in \chi(M)$. 

A metric structure is said to be a k-cosymplectic metric structure if

\begin{equation}
g(\phi X, \phi Y) = g(X, Y) + k\eta(X)\eta(Y),
\end{equation}

for all $X, Y \in \chi(M)$. 

A metric structure is said to be a k-cosymplectic metric structure if

\begin{equation}
g(\phi X, \phi Y) = g(X, Y) + k\eta(X)\eta(Y),
\end{equation}

for all $X, Y \in \chi(M)$.
for any $X, Y \in \chi(M)$. In this case, the 1-form $\eta$ is called the contact metric form. We define a $(1,1)$-tensor field $h$ by $h := \frac{1}{2} \mathcal{L}_\xi \phi$, where $\mathcal{L}$ denotes the Lie derivative in the direction of the vector field $\xi$. It is symmetric and satisfies $h\phi = -\phi h$. Also, we have $Tr.h = Tr.\phi h = 0$, $h\xi = 0$ and

\begin{equation}
\nabla_X \xi = -\phi X - \phi hX,
\end{equation}

for any $X \in \chi(M)$, where $\nabla$ is the Levi-Civita connection of $g$.

An almost contact metric manifold is called Kenmotsu if

\begin{equation}
(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X,
\end{equation}

for any $X, Y \in \chi(M)$.

In 2005, Kim and Pak [3] introduced the notion of almost $\alpha$-cosymplectic manifold, which is an almost contact metric manifold that satisfies $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, for $\alpha$ a real number. Recently, Erken [1] and Öztürk et. al [5, 6] obtained some fundamental properties of almost $\alpha$-cosymplectic manifolds. An $\alpha$-cosymplectic manifold is a normal almost $\alpha$-cosymplectic manifold. An $\alpha$-cosymplectic manifold with $\alpha = 0$ is a cosymplectic manifold and with $\alpha = 1$, it is a Kenmotsu manifold.

On a $(2n + 1)$-dimensional $\alpha$-cosymplectic manifold $M$, for any $X, Y \in \chi(M)$, the following relations hold:

\begin{equation}
\nabla_X \xi = \alpha[X - \eta(X)\xi],
\end{equation}

\begin{equation}
R(X,Y)\xi = \alpha^2[\eta(X)Y - \eta(Y)X],
\end{equation}

\begin{equation}
S(X, \xi) = -2n\alpha^2\eta(X),
\end{equation}

\begin{equation}
Q\xi = -2n\alpha^2\xi.
\end{equation}

3. Proof of the Main Theorem

Before proving our main result, we recall the following lemma, given by Miao-Tam.

**Lemma 3.1.** ([4, Theorem 7]) If the metric of a connected Riemannian manifold satisfies the Miao-Tam critical condition, then the scalar curvature is constant.

It is known that the Riemannian curvature tensor of a 3-dimensional Riemannian manifold $(M, g)$ is given by:

\begin{equation}
R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y],
\end{equation}
for any $X, Y, Z \in \chi(M)$, where $S$ is the Ricci tensor, $Q$ is the Ricci operator and $r$ is the scalar curvature.

Assume that $(M, \phi, \xi, \eta, g, \alpha)$ is a 3-dimensional connected $\alpha$-cosymplectic manifold which satisfies the Miao-Tam critical condition, i.e. which satisfies (1.1). Putting $Y = Z = \xi$ in (3.1) and using (2.9), (2.10) and (2.11), the Ricci operator can be written as

\begin{equation}
Q X = (\alpha^2 + \frac{r}{2})X - (3\alpha^2 + \frac{r}{2})\eta(X)\xi,
\end{equation}

for any $X \in \chi(M)$.

Taking covariant derivative of the above equation with respect to $Y$ and using (2.7) and Lemma 3.1, we obtain

\begin{equation}
(\nabla_Y Q) X = -\alpha(3\alpha^2 + \frac{r}{2})[g(X,Y)\xi + \eta(X)Y - 2\eta(X)\eta(Y)\xi],
\end{equation}

for any $X, Y \in \chi(M)$.

Taking trace of the equation (1.1), we have

\begin{equation}
\Delta \lambda = -\frac{1}{2}(r\lambda + 3),
\end{equation}

Using (3.4) in (1.1), we obtain

\begin{equation}
\nabla_X D\lambda = \lambda Q X + f X, \quad \text{where} \quad f = -\frac{1}{2}(r\lambda + 1),
\end{equation}

for any $X \in \chi(M)$, where $D$ denotes the gradient operator with respect to $g$.

Taking the covariant derivative of (3.5) with respect to $Y$, we get

\begin{equation}
\nabla_Y \nabla_X D\lambda = (Y\lambda)Q X + \lambda \nabla_Y Q X + (Y f) X + f \nabla_Y X,
\end{equation}

for any $X, Y \in \chi(M)$.

Similarly, we get

\begin{equation}
\nabla_X \nabla_Y D\lambda = (X\lambda)Q Y + \lambda \nabla_X Q Y + (X f) Y + f \nabla_X Y.
\end{equation}

Also

\begin{equation}
\nabla_{[X,Y]} D\lambda = \lambda Q [X,Y] + f [X,Y],
\end{equation}

and using (3.6), (3.7) and (3.8) we have

\begin{equation}
R(X,Y) D\lambda = \nabla_X \nabla_Y D\lambda - \nabla_Y \nabla_X D\lambda - \nabla_{[X,Y]} D\lambda
= (X\lambda)Q Y - (Y\lambda)Q X + \lambda [(\nabla_X Q) Y - (\nabla_Y Q) X]
+ (X f) Y - (Y f) X.
\end{equation}
In view of (3.3) and (3.9) yields

\[ R(X,Y)D\lambda = (X\lambda)QY - (Y\lambda)QX + \lambda\alpha(3\alpha^2 + \frac{r}{2})[\eta(X)Y - \eta(Y)X] + (Xf)Y - (Yf)X, \]

(3.10)

for any \( X, Y \in \chi(M) \).

By setting \( X = \xi \) in the above equation and using (2.10) and (3.2) we get

\[ R(\xi,Y)D\lambda = (\xi\lambda)[(\alpha^2 + \frac{r}{2})Y - (3\alpha^2 + \frac{r}{2})\eta(Y)\xi] + 2\alpha^2(Y\lambda)\xi + \lambda\alpha(3\alpha^2 + \frac{r}{2})[Y - \eta(Y)\xi] + (\xi f)Y - (Yf)\xi. \]

(3.11)

Taking inner product with \( \xi \) in the above equation, we easily compute

\[ g(R(\xi,Y)\xi,D\lambda) = 2\alpha^2[(\xi\lambda)\eta(Y) - (Y\lambda)] - (\xi f)\eta(Y) + (Yf). \]

(3.12)

On the other hand, from (2.9) we have

\[ g(R(\xi,Y)\xi,D\lambda) = \alpha^2[g(Y,D\lambda) - \eta(Y)\eta(D\lambda)]. \]

(3.13)

Making use of (3.12) and (3.13) we get

\[ \alpha^2[g(Y,D\lambda) - \eta(Y)\eta(D\lambda)] = 2\alpha^2[(\xi\lambda)\eta(Y) - (Y\lambda)] - (\xi f)\eta(Y) + (Yf). \]

(3.14)

Removing \( Y \) from both sides in the above equation, we obtain

\[ 2\alpha^2(\xi\lambda)\xi - (\xi f)\xi + Df = \alpha^2[3D\lambda - \eta(D\lambda)\xi]. \]

(3.15)

From \( f = -\frac{1}{2}(r\lambda + 1) \), we get

\[ Df = -\frac{r}{2}(D\lambda) \text{ and } \xi f = -\frac{r}{2}(\xi\lambda). \]

(3.16)

Using the above relations in (3.15) we easily have

\[ (3\alpha^2 + \frac{r}{2})[(\xi\lambda)\xi - D\lambda] = 0. \]

(3.17)

If \( D\lambda = (\xi\lambda)\xi \), then taking the covariant derivative with respect to \( X \) and using (3.5) we obtain

\[ \lambda QX - \frac{1}{2}(r\lambda + 1)X = X(\xi\lambda)\xi + \alpha(\xi\lambda)[X - \eta(X)\xi]. \]

Then taking trace we get

\[ \xi(\xi\lambda) = -2\alpha(\xi\lambda) - \frac{1}{2}(r\lambda + 3). \]

(3.18)
From (1.1) we get
\[ \nabla_X D\lambda = (\Delta \lambda)X + \lambda QX + X, \]
hence, for \( X = \xi \), together with (3.4) and (2.11) imply
\[ \nabla_\xi D\lambda = -\frac{1}{2}(4\alpha^2\lambda + r\lambda + 1)\xi. \]
Since \( \nabla_\xi D\lambda = \xi (\xi \lambda)\xi \), we deduce that
\[ \xi (\xi \lambda) = -\frac{1}{2}(4\alpha^2\lambda + r\lambda + 1) \]
which together with (3.18) imply
\[ \alpha^2\lambda - \alpha (\xi \lambda) - \frac{1}{2} = 0. \]

Then we can state:

**Proposition 3.1.** If a 3-dimensional connected \( \alpha \)-cosymplectic manifold \((M, \phi, \xi, \eta, g, \alpha)\) satisfies CPE and \( r \neq -6\alpha^2 \), then the gradient of \( \lambda \) is collinear with \( \xi \). Moreover
\[ (\xi \lambda) = \alpha \lambda - \frac{1}{2\alpha}. \]

**Corollary 3.1.** If a 3-dimensional connected \( \alpha \)-cosymplectic manifold \((M, \phi, \xi, \eta, g, \alpha)\) satisfies CPE and \( r \neq -6\alpha^2 \), then it can not be a cosymplectic manifold.

If \( 3\alpha^2 + \frac{r}{2} = 0 \), then \( r = -6\alpha^2 \). Putting the value of \( r = -6\alpha^2 \) in (3.1) and in view of (3.2), we find that manifold is of constant sectional curvature \(-\alpha^2\).

Hence we can state the following:

**Theorem 3.1.** If a 3-dimensional connected \( \alpha \)-cosymplectic manifold \((M, \phi, \xi, \eta, g, \alpha)\) satisfies CPE, then the manifold is of constant sectional curvature \(-\alpha^2\), provided \( D\lambda \neq (\xi \lambda)\xi \), where \( D \) denotes the gradient operator with respect to \( g \).

If \( \alpha = 0 \), then the manifold is a cosymplectic manifold and we have the following:

**Corollary 3.2.** If a 3-dimensional connected cosymplectic manifold \((M, \phi, \xi, \eta, g)\) satisfies CPE, then the manifold is flat, provided \( D\lambda \neq (\xi \lambda)\xi \).

If \( \alpha = 1 \), then the manifold is a Kenmotsu manifold and we have the following:

**Corollary 3.3.** If a 3-dimensional connected Kenmotsu manifold \((M, \phi, \xi, \eta, g)\) satisfies CPE, then the manifold is locally isometric to the hyperbolic space \( H^3(-1) \), provided \( D\lambda \neq (\xi \lambda)\xi \).
References


