A study on the approximation function for pairs of primes with difference 10 between consecutive primes

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연속하는 두 소수의 차가 10인 소수 쌍에 대한 근사 함수에 대한 연구

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Abstract In this paper, I provided an approximation function $Li^{*}_{2,10}(x)$ using logarithm integral for the counting function $\pi^{*}_{2,10}(x)$ of consecutive deca primes. Several personal computers and Mathematica were used to validate the approximation function $Li^{*}_{2,10}(x)$. I found the real value of $\pi^{*}_{2,10}(x)$ and approximate value of $Li^{*}_{2,10}(x)$ for various $x \le 10^{11}$. By the result of theses calculations, most of the error rates are margins of error of 0.005%. Also, I proved that the sum $C_{2,10}(\infty)$ of reciprocals of all primes with difference 10 between primes is finite. To find $C_{2,10}(\infty)$, I computed the sum $C_{2,10}(x)$ of reciprocals of all consecutive deca primes for various $x \le 10^{11}$ and I estimate that $C_{2,10}(\infty)$ probably lies in the range $C_{2,10}(\infty) = 0.4176 \pm 2.1 \times 10^{-3}$.

Key Words : approximation function, pair of primes, distribution of pair of primes, sum of reciprocals of primes, Mathematica

요 약 본 논문은 연속하는 두 소수의 차가 10인 소수의 쌍의 수에 대한 계산 함수 $\pi^{*}_{2,10}(x)$의 근사함수 $Li^{*}_{2,10}(x)$를 로그적분을 이용하여 유도하였다. $Li^{*}_{2,10}(x)$가 $\pi^{*}_{2,10}(x)$의 근사함수로 적절한지 알아보기 위하여 컴퓨터와 Mathematica 프로그램을 이용하여 $\pi^{*}_{2,10}(x)$와 $Li^{*}_{2,10}(x)$의 값을 $x \le 10^{11}$까지 구한 후 두 값의 오차율을 계산하였다. 오차율을 계산한 결과 대부분의 구간에서 오차율이 0.005% 이하로 나타났다. 또한, 두 소수의 차가 10인 소수들의 역수들의 합 $C_{2,10}(\infty)$이 유한임을 보였다. $C_{2,10}(\infty)$의 수렴값을 구하기 위하여 $C_{2,10}(10^{11})$을 구한 후, 이를 이용하여 $C_{2,10}(\infty)$의 대략적인 수렴값을 계산하였다. 그 결과 $C_{2,10}(\infty) = 0.4176 \pm 2.1 \times 10^{-3}$로 수렴함을 알 수 있었다.

주제어 : 근사함수, 소수들의 쌍, 소수들의 쌍의 분포, 소수들의 역수들의 합, Mathematica

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접수일 2020년 9월 21일 수정일 2020년 11월 15일 심사완료일 2020년 11월 18일

1. Introduction

Polignac(1849) conjectured whether there are infinitely many pairs of consecutive primes which differ by $2k$. This is the famous Polignac's conjecture. The most of study on the cluster of primes is focused on the twin prime. And several study is focused on the primes with difference 4, 6 or 8 between primes p_n and p_{n+1}[1-3]. But I extend the study of the primes with difference 10 between primes p_n and p_{n+1}. I define pairs of primes with difference $2k$ between primes as follows.

A pair of primes (p_n, p_n+2k) called twin(or cousin, sexy, octy, deca) primes for an integer $k=1$(or 2, 3, 4, 5). If $p_{n+1}=p_n+2k$ then a pair of primes (p_n, p_{n+1}) called consecutive twin(or cousin, sexy, octy, deca) primes for an integer $k=1$(or 2, 3, 4, 5). The triple of primes $(p, p+2, p+6)$ is called triple primes if a prime triple is three consecutive primes, such that the first and the last differ by 6. And there exists the other form of triple primes that is $(p, p+4, p+6)$.

Let $P_{2,2}=\{(p,p+2)|p, p+2\in P\}$ be the set of all pairs of twin primes, where P is the set of all primes. Whether $P_{2,2}$ is finite or not, Brun proved that the sum of the reciprocals of all twin primes is finite[4]. If prime number p belongs to $P_{2,2}$ then

$$B=\sum_{p\in P_{2,2}}\left(\frac{1}{p}+\frac{1}{p+2}\right)<\infty.$$

The interesting issue of twin primes is the numerical calculation Brun's constant B. Based on heuristic methods about the distribution of twin primes, many mathematicians tried to estimate the approximate value of B[5-9]. Recently Nicely calculated $B=1.9021605824\pm 3\times 10^{-9}$[10].

The encryption method for data security includes public key encryption and secret key encryption. RSA is the typical public key encryption system and DES is the typical secret key encryption system. Recently, the DES was replaced by the AES. NIST promulgated AES as the Federal Information Processing Standard(FIPS -197). AES is included in the ISO/IEC 18033-3 standard and is used in multiple encryption packages[11]. The RSA creates and uses public keys based on two large prime numbers. Since large prime numbers are used as key to RSA, a lot of research is being done on the properties related to prime numbers.

In this paper, I extend the study of primes with difference 10 between primes p_n and p_{n+1}. I will study the counting function of the primes with difference 10 between primes p_n and p_{n+1} and derive its approximation function from logarithm integral. Also, I will study the sum of reciprocals of all primes with difference 10 between primes p_n and p_{n+1}.

2. Theoretical Background

Let $P_{2,2k}$ be the set of all pairs of primes with difference $2k$ between primes, i.e.,
$P_{2,2k}=\{(p, p+2k)\mid p \text{ and } p+2k \textit{ are primes}\}$.

And, let $\pi_{2,2k}(x)$ be the counting function of primes with difference $2k$ between primes by

$$\pi_{2,2k}(x)=\#\{p\le x \mid (p, p+2k)\in P_{2,2k}\}.$$

The function $Li_{2,2k}(x)$ introduced by Hardy and Littlewood can be an approximation function to $\pi_{2,2k}(x)$ as $x\to\infty$ by the following asymptotic formula[12].

$$\pi_{2,2k}(x)\sim Li_{2,2k}(x) = 2c_2\int_2^x\frac{dt}{(\ln t)^2}\prod_{p>2,p|k}\frac{p-1}{p-2} \quad (2.1)$$

where c_2 is the twin prime constant:

$$c_2=\prod_{2<p\in P}\frac{(p-2)/(p-1)}{(p-1)/p}=\prod_{2<p\in P}\left(1-\frac{1}{(p-1)^2}\right)$$

This constant c_2 was computed to 105D by Sloane [13].

Also, let $P^{*}_{2,2k}$ be the set of all pairs of primes with difference $2k$ between p_n and p_{n+1}, i.e.,

$$P^{*}_{2,2k} = \{(\mathrm{p_n},\ \mathrm{p_{n+1}}) \mid \mathrm{p_{n+1}} = \mathrm{p_n} + 2\mathrm{k}\}$$

and let $\pi^{*}_{2,2k}(x)$ be the counting function of primes with difference $2k$ between p_n and p_{n+1}. The case $k=1$ is the famous twin prime conjecture.

Let $\pi_3(x)$ be the number of triple prime $(p, p+2, p+6)$ with $p \le x$. Then $Li_3(x)$, which was introduced by Hardy and Littlewood[8], can approximate to $\pi_3(x)$ as $x \to \infty$ by the following asymptotic formula.

$$\pi_{3(x)} \sim Li_{3(x)} = \frac{9}{2} c_3 \int_2^x \frac{dt}{(\ln t)^3} \qquad (2.2)$$

where c_3 is the triple prime constant:

$$c_3 = \prod_{5 \le p} \frac{p^2(p-3)}{(p-1)^3}.$$

This triple primes constant c_3 was computed to 45D by Harley[14].

A pair of primes $(p_{n,}\ p_{n+1})$ is called consecutive sexy primes if $p_{n+1} = p_n + 6$. Let $\pi^{*}_{2,6}(x)$ be the counting function of consecutive sexy primes. The Hardy and Littlewood type conjecture on consecutive sexy prime was formulated as follows[1]:

$$\begin{aligned}\pi^{*}_{2,6}(x) &\sim Li^{*}_{2,6}(x) \sim Li_{2,6}(x) - 2Li_3(x) \\ &= 2Li_{2,2}(x) - 2Li_3(x) \\ &= 2\left(2c_2 \int_2^x \frac{dt}{(\ln t)^2} - \frac{9}{2} c_3 \int_2^x \frac{dt}{(\ln t)^3}\right)\end{aligned}$$

where the function $Li_{2,6}(x) = 4c_2 \int_2^x \frac{dt}{(\ln t)^2}$ is an approximation of $\pi_{2,6}(x)$. The real and approximate values of the number of all consecutive sexy primes up to 5×10^{10} were $\pi^{*}_{2,6}(5 \times 10^{10}) = 215868063$ and $Li^{*}_{2,6}(5 \times 10^{10}) = 215883669$[1].

Refer to (2.1) as Hardy and Littlewood approximation function for prime quadruplets;

$$\pi_4(x) \sim Li_4(x) = \frac{27}{2} c_4 \int_2^x \frac{dt}{(\ln t)^4} \qquad (2.3)$$

where $\pi_4(x)$ represents the count of prime quadruplets $(p, p+2, p+6, p+8)$ such that $p \le x$, and c_4 is the constant

$$c_4 = \prod_{5 \le p < \infty} \frac{p^3(p-4)}{(p-1)^4} = 0.3074948787\cdots.$$

The constant c_4 has been computed to 45D by Harley[13].

A pair of primes $(p_{n,}\ p_{n+1})$ is called consecutive octy primes if $p_{n+1} = p_n + 8$. Let $\pi^{*}_{2,8}(x)$ be the counting function of consecutive octy primes. The Hardy and Littlewood type conjecture on consecutive octy prime was formulated in [2].

$$\begin{aligned}\pi^{*}_{2,8}(x) &\sim Li^{*}_{2,8}(x) \sim Li_{2,8}(x) - 2Li_3(x) + Li_4(x) \\ &= 2c_2 \int_2^x \frac{dt}{(\ln t)^2} - 9c_3 \int_2^x \frac{dt}{(\ln t)^3} + \frac{27}{2} c_4 \int_2^x \frac{dt}{(\ln t)^4}.\end{aligned}$$

And the result counting the number of all consecutive octy primes was $\pi^{*}_{2,8}(7 \times 10^{10}) = 133295081$ and approximate value calculated by logarithm integral was $Li^{*}_{2,8}(7 \times 10^{10}) = 133284729$[2].

Now, consider the sum of reciprocals of all primes with difference $2k$ between primes. Whether $P_{2,2k}$, which is the set of all pairs of primes with difference $2k$ between primes, is infinite or not, Brun proved that the sum of the reciprocals of all twin primes($k=1$) is finite[4],

$$\begin{aligned}B &= \left(\frac{1}{3} + \frac{1}{5}\right) + \left(\frac{1}{5} + \frac{1}{7}\right) + \left(\frac{1}{11} + \frac{1}{13}\right) + \cdots \\ &= \sum_{p \in P_{2,2}} \left(\frac{1}{p} + \frac{1}{p+2}\right) < \infty\end{aligned}$$

The sum B of the reciprocals of all twin primes is know as the Brun's constant.

Park and Lee proved the sum of the reciprocals of all consecutive cousin primes is finite and estimated approximately the sum of their reciprocals[1]. Let $C^{*}_{2,4}(x)$ be the sum of the reciprocals of the consecutive cousin primes which are equal or less then x, then $C^{*}_{2,4}(5 \times 10^{10})$ is 1.197054792988 and $C_{2,4}(\infty)$ probably lies in the range $1.197054 \pm 5.6 \times 10^{-6}$[1]. Also, Park and Lee proved the sum of the reciprocals of all consecutive sexy primes is finite and the sum $C^*_{2,6}(x)$ estimated approximately the sum of their reciprocals[1]. Let $C^*_{2,6}(x)$ be the sum of the reciprocals of all consecutive sexy primes. Lee and Park computed $C^*_{2,6}(x)$ for several intervals up to 5×10^{10} and estimated $C_{2,6}(\infty)=1.135835\pm1.2\times10^{-6}$ from the computation of $C^*_{2,6}(5\times10^{10})$[1].

The sum of the reciprocals of all consecutive octy primes is finite[2]. Let $C^*_{2,8}(x)$ be the sum of the reciprocals of the consecutive octy primes which are equal or less then x. Lee and Park computed $C^*_{2,8}(x)$ for several $x\le7\times10^{10}$ and estimated

$$C_{2,8}(\infty)=0.3374\pm2.6\times10^{-4}$$

from the computation of $C^*_{2,8}(7\times10^{10})$[2].

Let $\pi^*_{2,10}(x)$ be the counting function of all pairs of primes with difference 10 between p_n and $p_{n+1}(=p_n+10)$, i.e., $\pi^*_{2,10}(x)$ is the counting function of all consecutive deca primes. And let $C^*_{2,10}(\infty)$ be the sum of the reciprocals of all consecutive deca primes. In this paper, I will present the approximation function $Li^*_{2,10}(x)$ for consecutive deca primes and will calculate the values of $\pi^*_{2,10}(x)$ and $Li^{*_{2,10}}(x)$ for several intervals up to 10^{11}. Also, I will compute the sum of the reciprocals of all consecutive deca primes up to 10^{11} and estimate $C^*_{2,10}(\infty)$ from the computation of $C^*_{2,10}(x)$.

3. Materials and Methods

3.1 Computational Technique

In this study, I presented the approximation function $Li^*_{2,10}(x)$ for consecutive deca primes and calculated the values of $\pi^*_{2,10}(x)$ and $Li^{*_{2,10}}(x)$ for several intervals up to 10^{11}. Also, I computed the sum of the reciprocals of all consecutive deca primes up to 10^{11} and estimate $C_{2,10}(\infty)$ from the computation of $C^*_{2,10}(x)$ for various $x\le10^{11}$ using personal computers and Mathematica 10, which is computer algebraic program.

The calculations of $\pi^*_{2,10}(x)$, $Li^{*_{2,10}}(x)$, $C^*_{2,10}(x)$ and $C_{2,10}(\infty)$ were distributed asynchronously across several dozen personal computers, and code written by Mathematica[15]. To avoid errors, all calculations were performed in separate systems; furthermore, the count $\pi(x)$ of primes and the count $\pi_{2,2}(x)$ of twin primes were maintained and checked periodically against known values, such as those published by Riesel[16] and by Nicely([10]).

I counted the number of all consecutive deca primes up to 10^{11} using code written by Mathematica program as follows;

```
s=0;
Do[
  If[SameQ[Prime[k]+10, Prime[k+1]], N[s=1+s]]
  k++, {k, x, y} ]
Print[s]
```

Also, I calculated the sum of reciprocals of all consecutive deca primes up to 10^{11} using Mathematica program as follows;

```
s=0;
Do[
 If[SameQ[Prime[k]+10, Prime[k+1]],
 N[s=1/{N[Prime[k],40]}+1/{N[Prime[k+1],40]}+s, 40]]
   k++, {k, x, y} ]
Print[N[s, 40]]
```

3.2 Data Analysis

Let $\Delta_{2,10}(x)$ be the difference between the approximate value $Li^*_{2,10}(x)$ and the real value $\pi^*_{2,10}(x)$ and let $\epsilon_{2,10}(x)$ be the error (in %) between the approximate value $Li^*_{2,10}(x)$ and the real value $\pi^*_{2,10}(x)$. To measure the accurate of approximation function $Li^*_{2,10}(x)$ for real function $\pi^*_{2,10}(x)$, I calculate

$$\Delta_{2,10}(x) = Li^*_{2,10}(x) - \pi^*_{2,10}(x)$$

and

$$\epsilon_{2,10}(x) = \frac{\Delta_{2,10}(x)}{\pi^*_{2,10}(x)} \times 100$$

in intervals with distance 5×10^9. I find the approximation value of $C_{2,10}(\infty)$ from $C^*_{2,10}(x)$ for several intervals up to 10^{11}. To find the approximation value of $C_{2,10}(\infty)$, I calculate down to four places of decimals using variance $\sigma^2(x)$ and standard deviation $\sigma(x)$ for each interval(with distance 5×10^9) up to 10^{11} and the limit of error for $C_{2,10}(\infty)$ is estimated ± 0.0021.

4. Results

4.1 The distribution of primes with difference 10 between primes p_n and $p_{n+1} = p_n + 10$.

I consider pairs of primes which differ by 10. This is the case $k=5$ in (2.1). Let $\pi_{2,10}(x)$ be a counting function of deca primes. Then I have an approximation to $\pi_{2,10}(x)$ by (2.1):

$$\pi_{2,10}(x) \sim Li_{2,10}(x) = \frac{8}{3}c_2\int_2^x \frac{dt}{(\ln t)^2}. \qquad (4.1)$$

Before a discussion for consecutive deca primes, I observe facts for consecutive triple primes and prime quadruplets. Let $\pi^*_{2,10}(x)$ be the counting function of consecutive deca primes. Since there are no triple primes of the form (p, $p+2$, $p+4$) except (3, 5, 7), quadruple primes are of the form (p, $p+2$, $p+8$, $p+10$) and (p, $p+4$, $p+6$, $p+10$). There is no consecutive quintuple primes of the form (p, $p+2$, $p+4$, $p+6$, $p+10$). And there is no quintuple primes of the form (p, $p+2$, $p+6$, $p+8$, $p+10$). Using (2.1), (2.2), (2.3) and the above facts, I have following approximation to $\pi^*_{2,10}(x)$.

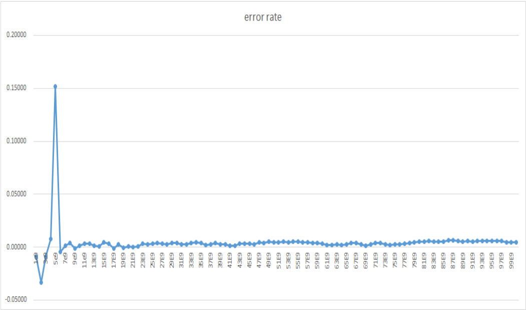
Conjecture.

$$\pi^*_{2,10}(x) \sim Li^*_{2,10}(x) = Li_{2,10}(x) - 3Li_3(x) + 2Li_4(x) \qquad (4.2)$$
$$= \frac{8}{3}c_2\int_2^x \frac{dt}{(\ln t)^2} - 3\frac{9}{2}c_3\int_2^x \frac{dt}{(\ln t)^3} + 27c_4\int_2^x \frac{dt}{(\ln t)^4}$$

〈Table 1〉 The distributions of $\pi^*_{2,10}$ up to x

x	$\pi^*_{2,10}(x)$	$Li^*_{2,10}(x)$	$\Delta_{2,10}(x)$	$\epsilon_{2,10}(x)$
1e9	3484767	3484463	-304	-0.00872
5e9	15186363	15209430	23067	0.15189
1e10	28764495	28764829	334	0.00116
1.5e10	41785314	41787293	1979	0.00474
2e10	54480401	54480649	248	0.00046
2.5e10	66935215	66937319	2104	0.00314
3e10	79207216	79210061	2845	0.00359
3.5e10	91329541	91332912	3371	0.00369
4e10	103327142	103329599	2457	0.00238
4.5e10	115213482	115217574	4091	0.00355
5e10	127004291	127010190	5899	0.00464
5.5e10	138710526	138717986	7460	0.00538
6e10	150344472	150349478	5006	0.00333
6.5e10	161907749	161911686	3937	0.00243
7e10	173405449	173410492	5043	0.00291
7.5e10	184846408	184850895	4487	0.00243
8e10	196227530	196237191	9661	0.00492
8.5e10	207561990	207573108	11118	0.00536
9e10	218849083	218861917	12834	0.00586
9.5e10	230092848	230106503	13655	0.00593
10e10	241298621	241309432	10811	0.00448

Table 1 contains a brief summary of the results of the calculation including the count $\pi^*_{2,10}(x)$ of consecutive deca primes; the values($\Delta_{2,10}(x)$) of the discrepancy and error rate($\epsilon_{2,10}(x)$) between the approximation value $Li^*_{2,10}(x)$ and the real value $\pi^*_{2,10}(x)$ for various $x \le 10^{11}$. It is not known whether there are infinitely many pairs of consecutive deca primes, and much less whether $\pi^*_{2,10}(x) \sim Li^*_{2,10}(x)$ as $x\to\infty$. However, empirical evidence suggests that the above conjecture is true.



[Fig. 1] Error rate $\epsilon_{2,10}(x)$ between $Li^*_{2,10}(x)$ and $\pi^*_{2,10}(x)$ for various $x \le 10^{11}$

4.2 The sum of reciprocals of all deca primes

We consider the sum of reciprocals of all consecutive deca prime. We will show an upper bound on deca primes using the Brun's method on the twin primes. I take the sequence to be $A = \{n(n+10) \mid n \le x\}$. Let $p > 2$, then

$$A_p = \{n(n+10) \mid n \le x,\ n(n+10) \equiv 0 \pmod{n+10}\}$$

Now $n(n+10) \equiv 0 \pmod p$ and if $n \equiv 0 \pmod p$ or $(n+10) \equiv 0 \pmod p$ since p is an odd prime. Clearly 0 and $p-10$ are two solutions in the interval $0, 1, \cdots, p-1$. So we can take $\omega(p) = 2,\ p > 2$. By the Chinese Remainder Theorem we have $|R_d(x)| \le \omega(d)$. We take the sifting primes to be $P = \{p \mid p > 2\}$. Since $S(A;P^z, x)$ counts all the deca prime pairs above z, $S(A;P^z, x) + 2z$ is an upper bound on the number of deca prime below x. Then, we have

$$W(z) = \prod_{2<p<z}\left(1-\frac{2}{p}\right) \le \prod_{2<p<z}\left(1-\frac{1}{p}\right)^2 = O\left(\frac{1}{(\ln z)^2}\right)$$

By the substitution $\ln z$ with $\ln z = \dfrac{\ln x}{\gamma \ln\ln x}$, we get the following $\pi_{2,10}(x) = O\left(x\left(\dfrac{\ln\ln x}{\ln x}\right)^2\right)$. From the equality in the above fact, we have that $\displaystyle\sum_{p \in P_{2,10}} \frac{1}{p}$ converges by following.

〈Table 2〉 The sum of reciprocals of all consecutive deca primes up to 10^{11}

x	$C_{2,10}(x)$	$C_{2,10}{'}(x)$	$C^*_{2,10}(x)$
1e9	0.273429564788614	0.141192934886911	0.414622499675525
5e9	0.283016389573410	0.132860140355393	0.415876529928803
1e10	0.286795424654724	0.129556439724620	0.416351864379341
1.5e10	0.288909428624529	0.127696501173246	0.416605929797775
2e10	0.290371052746339	0.126407901228393	0.416778953974730
2.5e10	0.291483076779116	0.125425610377285	0.416908687156401
3e10	0.292378238330522	0.124633943132541	0.417012181463062
3.5e10	0.293125812566530	0.123972124661716	0.417097937228246
4e10	0.293766707103559	0.123404322397493	0.417171029501053
4.5e10	0.294326758900009	0.122907661871760	0.417234420771769
5e10	0.294823706536723	0.122466663642146	0.417290370178868
5.5e10	0.295270020536443	0.122070373425344	0.417340393961787
6e10	0.295674953240665	0.121710759647695	0.417385712888358
6.5e10	0.296045193400774	0.121381761662161	0.417426955062935
7e10	0.296363277995482	0.121078695642913	0.417441973638392
7.5e10	0.296679023763922	0.120797868275870	0.417476892039792
8e10	0.296972840203061	0.120536316816477	0.417509157019535
8.5e10	0.297247704563115	0.120291629009059	0.417539333572174
9e10	0.297505769457101	0.120061815162496	0.417567584619593
9.5e10	0.297748941163625	0.119845215269233	0.417594156432858
10e10	0.297995941040652	0.119640430260751	0.417636371301402

$$\sum_{p\in P_{2,10}}\frac{1}{p}=\sum_{x}\frac{\pi_{2,10}(x)-\pi_{2,10}(x-1)}{x}$$
$$=\sum_{x}\pi_{2,10}(x)\left(\frac{1}{x}-\frac{1}{x+1}\right)$$
$$\le c\sum_{x}\frac{1}{x(x+1)}\frac{x(\ln\ln x)^2}{(\ln x)^2}$$
$$\le c\sum_{x}\frac{1}{x}\left(\frac{\ln\ln x}{\ln x}\right)^2=O(1)$$

This result shows that there are not very many deca primes compared with the total number of primes, since $\sum_{p\in P_{2,10}}\frac{1}{p}$ taken over only the deca primes converges, while $\sum_{p\in P}\frac{1}{p}$ extended over all primes diverges.

Let $C_{2,10}(\infty)$ be the sum of reciprocals of all primes in $P^*_{2,10}$. Since the sum of the reciprocals of all primes in $P_{2,10}$ is finite, $C_{2,10}(\infty)$ is also finite, that is

$$C_{2,10}(\infty)=\sum_{p\in P^*_{2,10}}\left(\frac{1}{p}+\frac{1}{p+10}\right)<\infty.$$

By assuming the validity of the approximation (4.2), a more rapidly converging first order extrapolation may be derived as follows. Define $C_{2,10}(x)$ as the partial sum of the reciprocals of the consecutive deca primes.

$$C_{2,10}(x)=\sum_{p\le x,p\in P^*_{2,10}}\left(\frac{1}{p}+\frac{1}{p+10}\right).$$

Then the remainder term in the series defining $C_{2,10}(\infty)$ may be approximated by

$$C_{2,10}(\infty)-C_{2,10}(x)=\sum_{x<p}\left(\frac{1}{p}+\frac{1}{p+10}\right)$$
$$\approx\int_x^\infty\left(\frac{2}{t}\frac{\frac{8}{3}c_2}{(\ln t)^2}-\frac{3}{t}\frac{\frac{27}{2}c_3}{(\ln t)^3}+\frac{4}{t}\frac{27c_4}{(\ln t)^4}\right)dt$$
$$=\frac{16}{3}c_2\int_{\ln x}^\infty\frac{du}{u^2}-\frac{81}{2}c_3\int_{\ln x}^\infty\frac{du}{u^3}+108c_4\int_{\ln x}^\infty\frac{du}{u^4}$$
$$=\frac{16c_2}{3\ln x}-\frac{81c_3}{4(\ln x)^2}+\frac{36c_4}{(\ln x)^3}$$

where I have employed the density

$$Li^{*\prime}_{2,10}(x)=\frac{8c_2}{3(\ln x)^2}-\frac{27c_3}{2(\ln x)^3}+\frac{27c_4}{(\ln x)^4}$$

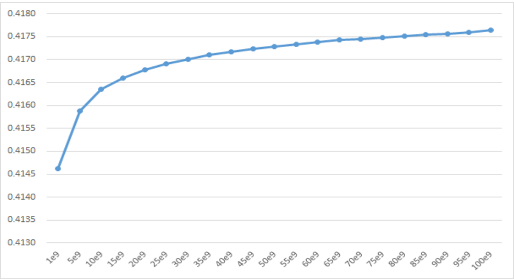
of consecutive deca primes implied by the approximation (4.2), approximated the sum of the reciprocals of twin, triple, and quadruple primes by $\frac{2}{t}$, $\frac{3}{t}$ and $\frac{4}{t}$, and used the substitution $u=\ln t$. This produce a first order extrapolation of $C_{2,10}(x)$ to $C_{2,10}(\infty)$, which I denoted by $C^*_{2,10}(x)$,

$$C^*_{2,10}(x)=C_{2,10}(x)+\frac{16c_2}{3\ln x}-\frac{81c_3}{4(\ln x)^2}+\frac{36c_4}{(\ln x)^3},$$

where $c_2=0.660161815846870$, $c_3=0.635166354604271$ and $c_4=0.307494878758327$. No effective second order extrapolation is known; however, I will present evidence that the error term is $O(1/\sqrt{x}\,\ln x)$, so that

$$C_{2,10}(\infty)=C^*_{2,10}(x)+O(1/\sqrt{x}\,\ln x)\quad ,$$
$$=C_{2,10}(x)+\frac{16c_2}{3\ln x}-\frac{81c_3}{4(\ln x)^2}+\frac{36c_4}{(\ln x)^3}+O(1/\sqrt{x}\,\ln x)$$

implying that $C^*_{2,10}(x)$ converges to $C_{2,10}(\infty)$ faster than the partial sums $C_{2,10}(x)$. I computed $C_{2,10}(x)$ and $C^*_{2,10}(x)$ for various $x\le\times10^{11}$.



[Fig. 2] Approximate value of $C_{2,10}(\infty)$

Some values are given in Table 2, where

$$C_{2,10}{}'(x)=\frac{16c_2}{3\ln x}-\frac{81c_3}{4(\ln x)^2}+\frac{36c_4}{(\ln x)^3}.$$

From my computation of $C^*_{2,10}(10^{11})$, I estimate that $C_{2,10}(\infty)$ probably lies in the range

$$C_{2,10}(\infty)=0.4176\pm2.1\times10^{-3}.$$

5. Conclusions

In this paper, I provided an approximation function $Li^*_{2,10}(x)$ for the counting function $\pi^*_{2,10}(x)$ of consecutive deca primes, which was induced by Hardy and Littlewood approximation for twin, triple and quadruplet primes. The induced approximation function of $\pi^*_{2,10}(x)$ is as follows:

$$Li^*_{2,10}(x) = \frac{8}{3}c_2\int_2^x \frac{dt}{(\ln t)^2} - 3\frac{9}{2}c_3\int_2^x \frac{dt}{(\ln t)^3} + 27c_4\int_2^x \frac{dt}{(\ln t)^4}$$

Several personal computers and Mathematica were used to validate the approximation function $Li^*_{2,10}(x)$. I found the real value of $\pi^*_{2,10}(x)$ and approximate value of $Li^*_{2,10}(x)$ for various $x \le 10^{11}$ and then calculated the gap $\Delta_{2,10}(x)$ and the error rate $\epsilon_{2,10}(x)$ between these values in the short intervals. According to the result of theses calculations, most of the error rates are margins of error of 0.005 percent. Therefore $Li^*_{2,10}(x)$ is suitable for use as the approximation function of the counting function $\pi^*_{2,10}(x)$ of consecutive deca prime for any x.

Also, I proved that the sum of reciprocals of all consecutive deca primes is finite. To obtain the approximation of the convergence value, I computed $C_{2,10}(x)$, $C_{2,10}{}'(x)$ and $C^*_{2,10}(x)$ for various $x \le 10^{11}$ and I estimate that $C_{2,10}(\infty)$ probably lies in the range

$$C_{2,10}(\infty) = 0.4176 \pm 2.1 \times 10^{-3}$$

from the computation of $C^*_{2,10}(10^{11})$.

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