REFINED ENUMERATION OF VERTICES AMONG ALL ROOTED ORDERED $d$-TREES

SANGWOOK KIM, SEUNGHYUN SEO, AND HEESUNG SHIN

Abstract. In this paper, we enumerate the cardinalities for the set of all vertices of outdegree $\geq k$ at level $\geq \ell$ among all rooted ordered $d$-trees with $n$ edges. Our results unite and generalize several previous works in the literature.

1. Introduction

For a positive integer $d$, the $n$th $d$-Fuss-Catalan number is given by

$$\text{Cat}^{(d)}_n = \frac{1}{dn+1} \binom{(d+1)n}{n} \text{ for } n \geq 0.$$  

It is a generalization of the well-known $n$th Catalan number. Like Catalan numbers, there are several combinatorial objects which are enumerated by Fuss-Catalan numbers. The most well-known object is the Fuss-Catalan path. A $d$-Fuss-Catalan path of length $(d+1)n$ is a lattice path from $(0,0)$ to $((d+1)n,0)$ using up steps $(1,d)$ and down steps $(1,-1)$ such that it stays weakly above the $x$-axis. Denote by $\mathcal{FC}^{(d)}_n$ the set of $d$-Fuss-Catalan paths of length $(d+1)n$. Another example is dissections of a $(dn+2)$-gon into $(d+2)$-gons by diagonals. There are three more combinatorial objects which are enumerated by $d$-Fuss-Catalan numbers.

Rooted ordered $d$-trees

A rooted tree can be considered as a process of successively gluing an edge (1-simplex) to a vertex (0-simplex) from the root in a half-plane, where the root is fixed in the line (1-dimensional hyperplane) as the boundary of the given half-plane. In the same way, we can define a rooted $d$-tree in $(d+1)$-dimensional lower Euclidean half-space $\mathbb{R}^{d+1}_-$ as follows: The root $r$ is a $(d-1)$-simplex fixed in the boundary of $\mathbb{R}^{d+1}_-$. From the root $(d-1)$-simplex $r$, we glue $d$-simplices (as edges) successively to one of previous $(d-1)$-simplices (as vertices) in $\mathbb{R}^{d+1}_-$. 

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By definition, if $d = 1$, a rooted $d$-tree is a rooted tree.

In a rooted tree, we can consider a linear order among all edges having one common vertex by their positions and such a tree is called a rooted ordered tree. Similarly, in higher dimensional cases, we can also give a linear order among $d$-simplices having one common $(d - 1)$-simplices naturally by their positions and such a tree is also called a rooted ordered $d$-tree. Jani, Rieper, and Zeleke [5] enumerated ordered $K$-trees, which was obtained in a similar way using $d$-simplices with $d \in K$.

**Rooted $d$-ary cacti**

A cactus is a connected simple graph in which each edge is contained in exactly one elementary cycle which is just a polygon. These graphs are also known as ‘Husimi trees’. They are introduced by Harary and Uhlenbeck [4]. If each elementary cycle has exactly $d$ edges, a cactus is called a $d$-ary cactus. Bóna et al. [2] provided enumerations of various combinatorial objects of $d$-ary cacti.

**Rooted $d$-tuplet trees**

Instead of $d$-simplices used in rooted ordered $d$-trees, we may use $(d + 1)$-gons. A root is a vertex fixed in the bounding hyperplane of a half-plane. One can glue $(d + 1)$-gons to a vertex from the root. A tree obtained in this way is called a rooted $d$-tuplet tree, and the $(d + 1)$-gons are called $d$-tuplets. As there is a linear order on the vertices in a tuplet, one can show that there is a one-to-one correspondence between rooted ordered $d$-trees with $n$ edges and rooted $d$-tuplet trees with $n$ tuplets. Thus rooted ordered $d$-trees and rooted $d$-tuplet trees are essentially the same. Note that the underlying graph of a $d$-tuplet tree is a $(d + 1)$-ary cactus.

Let $T_n^{(d)}$ be the set of rooted $d$-tuplet trees with $n$ tuplets. It is easy to see that the cardinality of $T_n^{(d)}$ is the $n$th $d$-Fuss-Catalan number $\text{Cat}_n^{(d)}$. For example, there are 22 rooted 3-tuplet trees with 3 tuplets, see Figure 1. Clearly the number of vertices among rooted $d$-tuplet tree with $n$ tuplets in $T_n^{(d)}$ is

$$ (dn + 1) \text{Cat}_n^{(d)} = \binom{(d + 1)n}{n}. \tag{1} $$

In a rooted $d$-tuplet tree, the degree of a vertex is the number of tuplets it connects. We can have the notion of the outdegree of a vertex $v$, which is the number of tuplets starting at $v$ and pointing away from the root. The level of a vertex $v$ in a rooted $d$-tuplet tree is the distance (number of tuplets) from the root to $v$. Table 1 shows the number of all vertices of outdegree $k$ at level $\ell$ among all rooted 3-tuplet trees in $T_3^{(3)}$. For example, there are 9 vertices of outdegree 1 at level 2 in $T_3^{(3)}$, see Figure 1.
In a rooted $d$-tuplet tree, there exists the unique vertex $u$ in each tuplet such that its level is less than levels of the other vertices $v_1, \ldots, v_d$. Here, $u$ is called the parent of $v_i$'s and each $v_i$ is called a child of $u$. For each vertex $v$ (except the root), there exists the unique tuplet containing $v$ toward the root, called the tuplet of $v$. Vertices with the same parent are called siblings. For two siblings $v$ and $w$, if $v$ is on the left of $w$, $v$ is called an elder sibling of $w$; meanwhile, $w$ is called a younger sibling of $v$.

Table 1. The number of vertices of outdegree $k$ at level $\ell$ among all rooted 3-tuplet trees in $T_3^{(3)}$

<table>
<thead>
<tr>
<th>$\ell \setminus k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$\sum$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>15</td>
<td>6</td>
<td>1</td>
<td>22</td>
</tr>
<tr>
<td>1</td>
<td>66</td>
<td>21</td>
<td>3</td>
<td>0</td>
<td>90</td>
</tr>
<tr>
<td>2</td>
<td>72</td>
<td>9</td>
<td>0</td>
<td>0</td>
<td>81</td>
</tr>
<tr>
<td>3</td>
<td>27</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>27</td>
</tr>
<tr>
<td>$\sum$</td>
<td>165</td>
<td>45</td>
<td>9</td>
<td>1</td>
<td>220</td>
</tr>
</tbody>
</table>

the number of vertices among all trees in the set of rooted ordered trees under some conditions.

**Theorem 1** (Eu, Seo, and Shin, 2017). Given $n \geq 1$, for any nonnegative integers $k$ and $\ell$, the number of all vertices of outdegree $\geq k$ at level $\geq \ell$ among all rooted ordered trees with $n$ edges is

$$\binom{2n-k}{n+\ell}.$$  

We give a generalization of the formula (2) for $T_n^{(d)}$ by generalizing their bijection.

**Theorem 2** (Main Result). Given $n \geq 1$, for any nonnegative integers $k$ and $\ell$, the number of all vertices of outdegree $\geq k$ at level $\geq \ell$ among all rooted $d$-tuplet trees with $n$ tuplets is

$$d^\ell \binom{(d+1)n-k}{dn+\ell}.$$  

We also find a refinement of the formula (3).

**Theorem 3**. Given $n \geq 1$, for any two nonnegative integers $i$, $j$, one nonnegative integer $k$ which is a multiple of $d$, and one positive integer $\ell$, the number of all vertices among all rooted $d$-tuplet trees with $n$ tuplets such that

- having at least $i$ elder siblings,
- having at least $j$ younger siblings,
- having at least $k$ children,
- at level $\geq \ell$

is

$$d^\ell \left(1 - \frac{\beta}{d} \frac{dn+\ell}{(d+1)n-\alpha}\right) \binom{(d+1)n-\alpha}{dn+\ell},$$  

where $\alpha$ and $\beta$ are nonnegative integers satisfying $i + j + k = \alpha d + \beta$ and $0 \leq \beta < d$.

The rest of the paper is organized as follows. In Section 2, we show the Theorem 2 bijectively. In Section 3, we give a combinatorial proof of the
Theorem 3. In Section 4, we present corollaries induced from Theorems 2 and 3.

2. A bijective proof of Theorem 2

Henceforth, a tree is assumed to be a rooted $d$-tuplet tree. Let $V$ be the set of pairs $(T,v)$ such that $v$ is a vertex of outdegree $\geq k$ at level $\geq \ell$ in $T \in \mathcal{T}^{(d)}_n$. Let $P$ be the set of sequences in $\{0,\ldots,d-1\}$ of length $\ell$. Let $L$ be the set of lattice paths of length $((d+1)n-k)$ from $(k,dk)$ to $((d+1)n, -(d+1)\ell)$, consisting of $(n-k-\ell)$ up-steps along the vector $(1,d)$ and $(dn+\ell)$ down-steps along the vector $(1,-1)$. To show Theorem 2, it is enough to construct a bijection $\Phi$ between $V$ and $P \times L$, due to

$$\#P = d^{\ell}, \quad \#L = \binom{(d+1)n-k}{n-k-\ell, dn+\ell} = \binom{(d+1)n-k}{dn+\ell}.$$ 

Three bijections $\varphi$, $\varphi$, and $\psi$

Let a reverse $d$-Fuss-Catalan path of length $(d+1)n$ be a lattice path from $(0,0)$ to $((d+1)n,0)$ using up steps $(1,d)$ and down steps $(1,-1)$ such that it stays weakly below the $x$-axis. Denote by $\mathcal{FC}^{(d)}_n$ the set of reverse $d$-Fuss-Catalan paths of length $(d+1)n$.

Before constructing the bijection $\Phi$, we introduce three bijections

$$\varphi : \mathcal{T}^{(d)}_n \rightarrow \mathcal{FC}^{(d)}_n, \quad \overline{\varphi} : \mathcal{T}^{(d)}_n \rightarrow \overline{\mathcal{FC}}^{(d)}_n, \quad \psi : \mathcal{T}^{(d)}_n \rightarrow \mathcal{FC}^{(d)}_n.$$ 

The bijection $\varphi$ corresponds a tree to a lattice path weakly above the $x$-axis by recording the steps when the tree is traversed in preorder: whenever we go down a side of a tuplet, record an up-step along the vector $(1,d)$ and whenever we go right or up a side of a tuplet, record a down-step along the vector $(1,-1)$.

Similarly, the bijection $\overline{\varphi}$ corresponds a tree to a lattice path weakly below the $x$-axis by recording the steps when the tree is traversed in preorder: whenever we go down or right a side, record a down-step along the vector $(1,-1)$ and whenever we go up a side, record an up-step along the vector $(1,d)$. An example of two bijections $\varphi$ and $\overline{\varphi}$ is shown in Figure 2.
The bijection $\psi$ corresponds a tree to a lattice path weakly above the $x$-axis by recording the steps when the tree is traversed in preorder: whenever we meet a vertex of outdegree $m$, except the last leaf, record $m$ up-steps followed by one down-step. An example of the bijection $\psi$ is shown in Figure 3.

**Step 1**

Given $(T, v) \in V$, let $D_v$ be the subtree consisting of $v$ and all its descendants in $T$, say the descendant subtree of $v$. Letting $\ell' (\geq \ell)$ be the level of $v$, consider the path from $v$ to the root $r$ of $T$

$$v(=v_0) \rightarrow v_1 \rightarrow \cdots \rightarrow v_{\ell} \rightarrow \cdots \rightarrow v_{\ell'-1} \rightarrow r(=v_{\ell'}).$$

Record the number $p_i$ of elder siblings of $v_i$ in the tuplet of $v_i$ for all $0 \leq i \leq \ell - 1$.

For all $0 \leq i \leq \ell - 1$, let $w_i$ be the youngest sibling of $v_i$ in the tuplet of $v_i$. By exchanging two subtrees $D_{v_i}$ and $D_{w_i}$, we obtain the tree $T'$.

**Step 2**

For all $1 \leq i \leq \ell - 1$ and $i = \ell'$, let $R_i$ be the subtree consisting of $v_i$ and all its descendants on the right of the tuplet of $v_{i-1}$ in $T'$. We obtain the tree $L$ by cutting the $\ell + 1$ subtrees $D_v, R_1, \ldots, R_{\ell-1}, R_{\ell}', L$ from the tree $T'$, see Figure 4.

**A construction of the bijection $\Phi$**

We will construct the bijection $\Phi$ between $V$ to $P \times L$. Given $(T, v) \in V$, let $k'(\geq k)$ be the outdegree of $v$ in $T$ and let $\ell'(\geq \ell)$ be the level of $v$ in $T$. We separate two cases:

**Case I.** If $v$ is not the root of $T$, i.e., $\ell' > 0$. We obtain the sequence $p = (p_0, \ldots, p_{\ell'-1}) \in P$ in Step 1 and $(\ell + 2)$ trees $D_v, R_1, \ldots, R_{\ell-1}, R_{\ell}', L$ after Step 2 as Figure 4.

Let $\rho$ be the mapping on the set of lattice paths defined by

$$\rho(s_1s_2 \cdots s_n) = s_2 \cdots s_n s_1,$$

where each $s_i$ is a step. Note that $\rho^m$ means to apply $\rho$ recursively $m$ times.

Clearly, the outdegree of the root of $D_v$ is $k'$. In the tree $L$, there are no younger siblings of $v$ in the tuplet of $v$ and the outdegree of vertex $v$ is 0.
Thus the lattice path \( p^{a+\ell}(\varphi(L)) \) ends with one down-step and \( \ell \) consecutive up-steps, where \( a \) is the number of vertices of \( L \) which precede \( v \) in preorder.

We define a lattice path \( P \) from \((0,0)\) to \(((d+1)n+\ell+1),-(\ell+1))\) by

\[
P = \psi(D_v) \varphi(R_1) \varphi(R_2) \cdots \varphi(R_{\ell-1}) \varphi(R_\ell') \varphi(L) \rho^{a+\ell}(\varphi(L)),
\]

where \( \varphi \) means a down-step.

**Case II.** If \( v \) is the root of \( T \), i.e., \( \ell' = 0 \). We define a sequence \( p = () \in \mathcal{P} \) and a lattice path

\[
P = \psi(T) \varphi.
\]

In all cases, the lattice path \( P \) always starts with at least \( k \) (precisely \( k' \)) consecutive up-steps and ends with one down-step and \( \ell \) consecutive up-steps as red segments in Figure 5.

By removing the first \( k \) steps and the last \( (\ell+1) \) steps from \( P \), we obtain the lattice path \( \hat{P} \) of length \(((d+1)n-k)\) from \((k,dk)\) to \(((d+1)n,-(d+1)\ell)\).
consisting of \((n-k-\ell)\) up-steps along the vector \((1,d)\) and \((dn+\ell)\) down-steps along the vector \((1,-1)\), so \(\hat{P}\) belongs to \(\mathcal{L}\).

Hence the map \(\Phi : V \to \mathcal{P} \times \mathcal{L}\) is defined by

\[
\Phi(T,v) = (p, \hat{P}).
\]

A description of the bijection \(\Phi^{-1}\)

In the Case I of the construction of the bijection \(\Phi\), given a lattice path \(P\) from \((0,0)\) to \(((d+1)n + (\ell + 1), -({\ell}+1))\), we decompose \(P\) into \((\ell + 2)\) paths \(P_D, P_1, \ldots, P_{\ell-1}, P_\ell, P_L\) by removing the leftmost down-steps from height \(-i\) to height \((-i+1)\) for \(0 \leq i \leq \ell\). Some of those paths may be empty.

Clearly all the paths \(P_D, P_1, \ldots, P_{\ell-1}, P_\ell\) are \(d\)-Fuss-Catalan path. By moving all the steps after the leftmost highest vertex in the lattice path \(P_L\) to the beginning, we obtain a reverse \(d\)-Fuss-Catalan path \(\overline{P}_L\) from \(P_L\). Since \(\varphi\), \(\overline{\varphi}\), and \(\psi\) are bijections, we can restore trees \(D_v, R_1, \ldots, R_{\ell-1}, R_\ell, L\) from \(P_D, P_1, \ldots, P_{\ell-1}, P_\ell, \overline{P}_L\).

Therefore, \(\Phi\) is a bijection between \(V\) and \(\mathcal{P} \times \mathcal{L}\) since all the remaining processes are also reversible.
3. Proof of Theorem 3

For any three nonnegative integers \(i, j, k\) and one positive integer \(\ell\), denote by \(\mathcal{V}_n^{(d)}(i, j, k; \ell)\) the set of pairs \((T, v)\) whose tree \(T\) in \(\mathcal{T}_n^{(d)}\) and vertex \(v\) in \(T\) such that
- \(v\) has at least \(i\) elder siblings in \(T\),
- \(v\) has at least \(j\) younger siblings in \(T\),
- \(v\) has at least \(k\) children in \(T\),
- \(v\) is at level \(\geq \ell\) in \(T\).

We show the following lemma, which is a particular case of Theorem 3, that is, \(i\) and \(j\) are multiples of \(d\).

**Lemma 4.** Given \(n \geq 1\), for any three nonnegative integers \(i, j, k\), all of which are multiples of \(d\), and one positive integer \(\ell\), the cardinality of \(\mathcal{V}_n^{(d)}(i, j, k; \ell)\) is

\[
\frac{d^\ell}{d} \left( \left( \frac{d+1}{d} \right) n - \alpha \right),
\]

where \(\alpha\) is the nonnegative integer satisfying \(i + j + k = ad\).

**Proof.** That a vertex \(v\) has at least \(i\) elder (or younger resp.) siblings means that there exists at least \(i/d\) (or \(j/d\) resp.) \(d\)-tuplets directly connected from the parent of \(v\) on its left (or right resp.).

A pair \((T, v)\) in \(\mathcal{V}_n^{(d)}(i, j, k, \ell)\) corresponds to a pair \((T', v)\) in \(\mathcal{V}_n^{(d)}(0, 0, i + j + k, \ell)\) under a cut-and-paste bijection \(\gamma_{i,j} : (T, v) \mapsto (T', v)\) which cuts the leftmost \(i/d\) tuplets connected from the parent \(p\) of \(v\) and pastes them at \(v\) on the left and does again the rightmost \(j/d\) tuplets connected from the parent \(p\) of \(v\) on the right, as Figure 6.
Since that \( v \) has at least \( i + j + k \) children means that the outdegree of \( v \) greater than or equal to \( \alpha = \frac{i+j+k}{d} \), this case corresponds to \( k \leftarrow \alpha \) of Theorem 2.

In Theorem 3, what to find is the cardinality of \( V_n^{(d)}(i, j, k; \ell) \) for any two nonnegative integers \( i, j, \) one nonnegative integer \( k \) which is a multiple of \( d \), and one positive integer \( \ell \).

Given \( (T, v) \in V_n^{(d)}(i, j, k; \ell) \), let \( w \) be the \( j \)th younger sibling of \( v \). By exchanging two subtrees \( D_v \) and \( D_w \), we obtain \( (T', v) \) in \( V_n^{(d)}(i, j, 0, k; \ell) \) from \( (T, v) \) in \( V_n^{(d)}(i, j, k; \ell) \). Let \( \alpha \) and \( \beta \) be the quotient and the remainder when \( i + j + k \) is divided by \( d \), that is,

\[
i + j + k = \alpha d + \beta.
\]

By applying the cut-and-paste bijection \( \gamma_{i+j-\beta, 0} \), we obtain \( (T'', v) \) in \( V_n^{(d)}(\beta, 0, 0; \alpha d; \ell) \) from \( (T', v) \) in \( V_n^{(d)}(i, j, 0, k; \ell) \). One can show that the values

\[
\#V_n^{(d)}(i, 0, \alpha d; \ell) - \#V_n^{(d)}(i + 1, 0, \alpha d; \ell)
\]

are the same for all \( 0 \leq i \leq d - 1 \) under exchanging two descendant subtrees of two siblings in the same tuplet. By telescoping, we get the formula

\[
\frac{\beta}{d} \left[ \#V_n^{(d)}(0, 0, \alpha d; \ell) - \#V_n^{(d)}(d, 0, \alpha d; \ell) \right].
\]

By Lemma 4, we have

\[
\#V_n^{(d)}(0, 0, \alpha d; \ell) = d^\ell \left( \frac{(d+1)n - \alpha}{dn + \ell} \right),
\]

\[
\#V_n^{(d)}(d, 0, \alpha d; \ell) = d^\ell \left( \frac{(d+1)n - \alpha - 1}{dn + \ell} \right).
\]

Thus we get the cardinality of \( V_n^{(d)}(\beta, 0, 0; \alpha d; \ell) \) and the desired formula (4).

4. Further results

From Theorem 2, we can obtain the following result.

**Corollary 5.** Given \( n \geq 1 \), for any two nonnegative integers \( k \) and \( \ell \), the number of all vertices of outdegree \( k \) at level \( \ell \) among \( d \)-trees in \( T_n^{(d)} \) is

\[
d^\ell dk + (d+1)\ell \left( \frac{(d+1)n - k}{dn + \ell} \right).
\]

**Proof.** By the sieve method with (3), we obtain the formula (5) from

\[
d^\ell \left( \frac{(d+1)n - k}{dn + \ell} \right) - d^\ell \left( \frac{(d+1)n - k - 1}{dn + \ell} \right)
\]

\[
- d^{\ell+1} \left( \frac{(d+1)n - k}{dn + \ell + 1} \right) + d^{\ell+1} \left( \frac{(d+1)n - k - 1}{dn + \ell + 1} \right).
\]

\( \square \)
The next result follows from Theorem 3 for $d = 1$.

**Corollary 6.** Given $n \geq 1$, for any three nonnegative integers $i$, $j$, $k$, and one positive integer $\ell$, the number of all vertices among trees in $T_n$ such that
- having at least $i$ elder siblings,
- having at least $j$ younger siblings,
- having at least $k$ children,
- at level $\geq \ell$

is
$$\binom{2n - i - j - k}{n + \ell}.$$

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