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LIGHTLIKE HYPERSURFACES OF AN INDEFINITE KAEHLER MANIFOLD WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. In this paper, we study three types of lightlike hypersurfaces, which are called *recurrent*, *Lie recurrent* and *Hopf* lightlike hypersurfaces, of an indefinite Kaehler manifold with a semi-symmetric non-metric connection. We provide several new results on such three types of lightlike hypersurfaces of an indefinite Kaehler manifold or an indefinite complex space form, with a semi-symmetric non-metric connection.

1. Introduction

A linear connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called a *semi-symmetric non-metric connection* if it and its torsion tensor $\bar{T}$ satisfy

$$(\bar{\nabla}_X \bar{g})(Y, Z) = -\pi(Y)\bar{g}(X, Z) - \pi(Z)\bar{g}(X, Y), \tag{1.1}$$

$$\bar{T}(X, Y) = \pi(Y)X - \pi(X)Y, \tag{1.2}$$

for any vector fields X, Y and Z on $\bar{M}$, where π is a 1-form associated with a smooth vector field ζ, which is called the *characteristic vector field*, on $\bar{M}$ by

$$\pi(X) = \bar{g}(X, \zeta).$$

The notion of semi-symmetric non-metric connection on a Riemannian manifold was introduced by Ageshe-Chafle [1, 2] and later studied by several authors.

The theory of lightlike hypersurfaces is an important topic of research in differential geometry due to its application in mathematical physics, especially in the general relativity. The study of such notion was initiated by Duggal and Bejancu [4] and later studied by many authors [5, 6]. Recently Yasar et al. [15] and Jin [8]∼[11] studied lightlike submanifolds of a semi-Riemannian manifold with a semi-symmetric non-metric connection.

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Let $\widetilde{\nabla}$ be the Levi-Civita connection of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ with respect to $\bar{g}$. We define a linear connection $\bar{\nabla}$ on $\bar{M}$ given by

$$\bar{\nabla}_X Y = \widetilde{\nabla}_X Y + \pi(Y)X \tag{1.3}$$

for any vector fields X and Y of $\bar{M}$. Then, by directed calculations from (1.3), we see that $\bar{\nabla}$ is a semi-symmetric non-metric connection. Conversely if $\bar{\nabla}$ is a semi-symmetric non-metric connection, then we can write

$$\bar{\nabla}_X Y = \widetilde{\nabla}_X Y + \psi(X, Y). \tag{1.4}$$

Substituting (1.4) into (1.1) and using the fact that $\widetilde{\nabla}$ is metric, we have

$$\bar{g}(\psi(X, Y), Z) + \bar{g}(\psi(X, Z), Y) = \pi(Y)\bar{g}(X, Z) + \pi(Z)\bar{g}(X, Y). \tag{1.5}$$

Also, from (1.4) and the fact that $\widetilde{\nabla}$ is torsion-free, it follows that

$$\begin{aligned} \psi(X, Y) - \psi(Y, X) &= \bar{\nabla}_X Y - \bar{\nabla}_Y X - \widetilde{\nabla}_X Y + \widetilde{\nabla}_Y X \\ &= \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y] \\ &= \bar{T}(X, Y). \end{aligned}$$

Thus, by using (1.2), we obtain

$$\psi(X, Y) - \psi(Y, X) = \pi(Y)X - \pi(X)Y. \tag{1.6}$$

Exchanging X with Y and Y with X to (1.5), we have

$$\bar{g}(\psi(Y, X), Z) + \bar{g}(\psi(Y, Z), X) = \pi(X)\bar{g}(Y, Z) + \pi(Z)\bar{g}(X, Y).$$

Subtracting this equation from (1.5) and using (1.6), we obtain

$$\bar{g}(\psi(X, Z), Y) = \bar{g}(\psi(Y, Z), X). \tag{1.7}$$

Again from (1.6) we get

$$\begin{aligned} \bar{g}(\psi(X, Y), Z) - \bar{g}(\psi(Y, X), Z) &= \pi(Y)\bar{g}(X, Z) - \pi(X)\bar{g}(Y, Z), \\ \bar{g}(\psi(X, Z), Y) - \bar{g}(\psi(Z, X), Y) &= \pi(Z)\bar{g}(X, Y) - \pi(X)\bar{g}(Z, Y). \end{aligned}$$

Adding these two equations and using (1.5), we have

$$\bar{g}(\psi(Y, X), Z) + \bar{g}(\psi(Z, X), Y) = 2\pi(X)\bar{g}(Y, Z).$$

Using this equation, (1.7) and the fact that $\bar{g}$ is non-degenerate, we obtain

$$\psi(X, Y) = \pi(Y)X.$$

Thus $\bar{\nabla}$ satisfies (1.3). This result implies that a linear connection $\bar{\nabla}$ on $\bar{M}$ is semi-symmetric non-metric connection if and only if $\bar{\nabla}$ satisfies (1.3).

In this paper, we study lightlike hypersurfaces M of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric non-metric connection $\bar{\nabla}$ given by (1.3). We introduce three types of lightlike hypersurfaces, named by recurrent, Lie recurrent and Hopf lightlike hypersurfaces, of an indefinite Kaehler manifold and we provide several new results on such three types of lightlike hypersurfaces of an indefinite Kaehler manifold with a semi-symmetric non-metric connection.

2. Lightlike hypersurfaces

Let $\bar{M} = (\bar{M}, \bar{g}, J)$ be an indefinite Kaeler manifold, where $\bar{g}$ is a semi-Riemannian metric and J is an indefinite almost complex structure satisfying

$$J^2 = -I, \quad \bar{g}(JX, JY) = \bar{g}(X, Y), \quad (\widetilde{\nabla}_X J)Y = 0, \tag{2.1}$$

for any vector field X and Y of $\bar{M}$, where $\widetilde{\nabla}$ is the Levi-Civita connection with respect to the metric $\bar{g}$. Let $\bar{\nabla}$ be a semi-symmetric non-metric connection on $\bar{M}$ given by (1.3). Using (1.3) and $(2.1)_3$, we see that

$$(\bar{\nabla}_X J)Y = \pi(JY)X - \pi(Y)JX. \tag{2.2}$$

Let (M, g) be a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$. Then the normal bundle $TM^{\perp}$ is a subbundle of the tangent bundle TM, of rank 1, and coincides with the radical distribution $Rad(TM) = TM \cap TM^{\perp}$. A complementary vector bundle $S(TM)$ of $Rad(TM)$ in TM is non-degenerate distribution on M, which is called a *screen distribution* on M [4], such that

$$TM = Rad(TM) \oplus_{orth} S(TM),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M. Also denote by $(2.1)_i$ the i-th equation of the three equations in (2.1). We use same notations for any others. For any null section ξ of $Rad(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique lightlike vector bundle $tr(TM)$ in $S(TM)^{\perp}$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall\, X \in \Gamma(S(TM)).$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to the screen distribution $S(TM)$, respectively. Then the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follow:

$$T\bar{M} = TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM).$$

In the sequel, let X, Y, Z and W be the vector fields on M, unless otherwise specified. Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingarten formulas of M and $S(TM)$ are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.3}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N; \tag{2.4}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.5}$$

$$\nabla_X \xi = -A^*_\xi X - \sigma(X)\xi, \tag{2.6}$$

respectively, where ∇ and ∇^* are the induced linear connections on TM and $S(TM)$ respectively, B and C are the local second fundamental forms on TM and $S(TM)$ respectively, A_N and A^*_ξ are the shape operators on TM and $S(TM)$ respectively and τ and σ are 1-forms on TM.

The connection ∇ is a semi-symmetric non-metric connection and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y) - \pi(Y)g(X,Z) - \pi(Z)g(X,Y), \tag{2.7}$$

$$T(X,Y) = \pi(Y)X - \pi(X)Y, \tag{2.8}$$

and B is symmetric on TM, where T is the torsion tensor with respect to the induced connection ∇ on M and η is a 1-form on TM such that

$$\eta(X) = \bar{g}(X, N).$$

From the fact that $B(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$, we know that B is independent of the choice of the screen distribution $S(TM)$ and satisfies

$$B(X, \xi) = 0. \tag{2.9}$$

From (2.3), (2.6) and (2.9), we obtain

$$\bar{\nabla}_X \xi = -A^*_\xi X - \sigma(X)\xi. \tag{2.10}$$

Now we set $a = \pi(N)$ and $b = \pi(\xi)$. Then the above two local second fundamental forms B and C are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y) + bg(X,Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{2.11}$$

$$C(X,PY) = g(A_N X, PY) + ag(X,PY) + \eta(X)\pi(PY), \tag{2.12}$$

$$\bar{g}(A_N X, N) = -a\eta(X), \qquad \sigma(X) = \tau(X) - b\eta(X).$$

From (2.11), A^*_ξ is $S(TM)$-valued real self-adjoint and satisfies

$$A^*_\xi \xi = 0. \tag{2.13}$$

Denote by $\bar{R}$, R and R^* the curvature tensors of the semi-symmetric non-metric connection $\bar{\nabla}$ on $\bar{M}$, and the induced linear connections ∇ and ∇^* on M and $S(TM)$ respectively. Using the Gauss-Weingarten formulas, we obtain two Gauss equations for M and $S(TM)$ such that

$$\begin{aligned} \bar{R}(X,Y)Z = {} & R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X \\ & + \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) \\ & - \tau(Y)B(X,Z) + B(T(X,Y),Z)\}N, \end{aligned} \tag{2.14}$$

$$\begin{aligned} R(X,Y)PZ = {} & R^*(X,Y)PZ + C(X,PZ)A^*_\xi Y - C(Y,PZ)A^*_\xi X \\ & + \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) - \sigma(X)C(Y,PZ) \\ & + \sigma(Y)C(X,PZ) + C(T(X,Y),PZ)\}\xi. \end{aligned} \tag{2.15}$$

The induced Ricci type tensor $R^{(0,2)}$ of M is defined by

$$R^{(0,2)}(X,Y) = trace\{Z \to R(Z,X)Y\}.$$

In general, $R^{(0,2)}$ is not symmetric. The Ricci type tensor $R^{(0,2)}$ is called the *induced Ricci tensor* [5] of M if it is symmetric. The symmetric $R^{(0,2)}$ tensor will be denoted by Ric. It is known that $R^{(0,2)}$ is symmetric if and only if the 1-form τ is closed, *i.e.*, $d\tau = 0$ on TM [9, 13].

3. Semi-symmetric non-metric connections

For a lightlike hypersurface M of an indefinite almost Hermitian manifold $\bar{M}$, it is known ([4, Section 6.2], [7]) that $J(Rad(TM))$ and $J(tr(TM))$ are vector subbundles of $S(TM)$, of rank 1 such that $Rad(TM) \cap J(Rad(TM)) = \{0\}$ and $Rad(TM) \cap J(tr(TM)) = \{0\}$. Hence $J(Rad(TM)) \oplus J(tr(TM))$ is a vector subbundle of $S(TM)$, of rank 2. Thus there exist two non-degenerate almost complex distributions D_o and D on M with respect to J such that

$$S(TM) = J(Rad(TM)) \oplus J(tr(TM)) \oplus_{orth} D_o,$$
$$D = \{Rad(TM) \oplus_{orth} J(Rad(TM))\} \oplus_{orth} D_o.$$

In this case, the decomposition form of TM is reduced to

$$TM = D \oplus J(tr(TM)). \tag{3.1}$$

Consider two local lightlike vector fields U and V such that

$$U = -JN, \qquad V = -J\xi. \tag{3.2}$$

Denote by S the projection morphism of TM on D with respect to the decomposition (3.1). Then any vector field X on M is expressed as follow:

$$X = SX + u(X)U,$$

where u and v are 1-forms locally defined on M by

$$u(X) = g(X, V), \qquad v(X) = g(X, U). \tag{3.3}$$

Using (3.2), the action JX of any $X \in \Gamma(TM)$ by J is expressed as

$$JX = FX + u(X)N, \tag{3.4}$$

where F is a tensor field of type $(1, 1)$ globally defined on M by $F = J \circ S$. Applying J to (3.4) and using (2.1) and (3.2), we have

$$F^2X = -X + u(X)U. \tag{3.5}$$

As $u(U) = 1$ and $FU = 0$, the set $(F,\, u,\, U)$ defines an indefinite almost contact structure on M and U is called the *structure vector field* of M.

In the following, let $(\bar{M},\, \bar{g})$ be an indefinite Kaehler manifold with a semi-symmetric non-metric connection $\bar{\nabla}$ given by (1.3). Applying $\bar{\nabla}_X$ to (3.2), (3.3) and (3.4) and using (2.2)∼(2.4), (2.10) (2.12) and (3.4), we have

$$B(X, U) = u(A_N X) + au(X) = C(X, V) - \eta(X)\pi(V), \tag{3.6}$$

$$\nabla_X U = F(A_N X) + aFX + \tau(X)U + \pi(U)X, \tag{3.7}$$

$$\nabla_X V = F(A^*_\xi X) + bFX - \sigma(X)V + \pi(V)X, \tag{3.8}$$

$$(\nabla_X F)Y = u(Y)A_N X - B(X, Y)U + \pi(JY)X - \pi(Y)FX, \tag{3.9}$$

$$(\nabla_X u)Y = -u(Y)\tau(X) - \pi(Y)u(X) - B(X, FY). \tag{3.10}$$

4. Indefinite complex space forms

An indefinite complex space form, denoted by $\bar{M}(c)$, is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature c such that

$$\bar{R}(X,Y)Z = \frac{c}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y + \bar{g}(JY,Z)JX - \bar{g}(JX,Z)JY + 2\bar{g}(X,JY)JZ\} \tag{4.1}$$

for any vector fields X, Y and Z of $\bar{M}$.

Comparing the tangential and transversal components of the two equations (2.14) and (4.1), and using (2.8) and (3.4), we get

$$\begin{aligned} R(X,Y)Z = {} & B(Y,Z)A_N X - B(X,Z)A_N Y \\ & + \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + \bar{g}(JY,Z)FX \\ & - \bar{g}(JX,Z)FY + 2\bar{g}(X,JY)FZ\}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} & (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ & + \{\tau(X) - \pi(X)\}B(Y,Z) - \{\tau(Y) - \pi(Y)\}B(X,Z) \\ = {} & \frac{c}{4}\{u(X)g(FY,Z) - u(Y)g(FX,Z) + 2u(Z)\bar{g}(X,JY)\}. \end{aligned} \tag{4.3}$$

Taking the scalar product with N to (2.15) and then, substituting (4.2) into the resulting equation and using (2.8), $(2.12)_2$ and (3.4), we obtain

$$\begin{aligned} & (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ & - \{\sigma(X) + \pi(X)\}C(Y,PZ) + \{\sigma(Y) + \pi(Y)\}C(X,PZ) \\ & + a\{\eta(X)B(Y,PZ) - \eta(Y)B(X,PZ)\} \\ = {} & \frac{c}{4}\{\eta(X)g(Y,PZ) - \eta(Y)g(X,PZ) + v(X)g(FY,PZ) \\ & - v(Y)g(FX,PZ) + 2v(PZ)\bar{g}(X,JY)\}. \end{aligned} \tag{4.4}$$

Definition. A screen distribution $S(TM)$ is said to be *totally umbilical* [4] if there exists a smooth function γ on a coordinate neighborhood $\mathcal{U}$ such that

$$C(X,PY) = \gamma g(X,PY). \tag{4.5}$$

From $(2.12)_{1,2}$, we see that (4.5) is equivalent to

$$A_N X = (\gamma - a)PX - \eta(X)\zeta. \tag{4.6}$$

Theorem 4.1. *Let M be a lightlike hypersurface of an indefinite complex space form $\bar{M}(c)$ with a semi-symmetric non-metric connection. If $S(TM)$ is totally umbilical, then $c = 0$ and the function γ satisfies the equations*

$$\gamma(\gamma - a) = 0, \qquad \gamma b = 0, \qquad X\gamma - \gamma\tau(X) = 0.$$

Proof. From (3.6) and (4.5), we have

$$B(X,U) = \gamma u(X) - \eta(X)\pi(V). \tag{4.7}$$

Replacing X by ξ, V and U to this equation by turns, we obtain

$$\pi(V) = 0, \qquad B(V, U) = 0, \qquad B(U, U) = \gamma. \tag{4.8}$$

Applying ∇_X to $C(Y, PZ) = \gamma g(Y, PZ)$ and using (2.7), we obtain

$$\begin{aligned}(\nabla_X C)(Y, PZ) &= (X\gamma)g(Y, PZ) \\ &\quad + \gamma\{B(X, PZ)\eta(Y) - \pi(Y)g(X, PZ) - \pi(PZ)g(X, Y)\}.\end{aligned}$$

Substituting this equation and (4.5) into (4.4), we have

$$\begin{aligned}&\{X\gamma - \gamma\sigma(X) - \frac{c}{4}\eta(X)\}g(Y, PZ) - \{Y\gamma - \gamma\sigma(Y) - \frac{c}{4}\eta(Y)\}g(X, PZ) \\ &\quad + (\gamma - a)\{B(X, PZ)\eta(Y) - B(Y, PZ)\eta(X)\} \\ &= \frac{c}{4}\{v(X)g(FY, PZ) - v(Y)g(FX, PZ) + 2v(PZ)\bar{g}(X, JY)\}.\end{aligned} \tag{4.9}$$

Replacing Y by ξ this equation and using (2.9), (3.2) and (3.3), we have

$$\begin{aligned}(\gamma - a)B(X, PY) &= \{\xi\gamma - \gamma\sigma(\xi) - \frac{c}{4}\}g(X, PY) \\ &\quad - \frac{c}{4}\{v(X)u(PY) + 2u(X)v(PY)\}.\end{aligned}$$

Taking $X = U$, $PY = V$ and alternately, taking $X = V$, $PY = U$ to this equation and using $(4.8)_2$ and the fact that B is symmetric, we have

$$\xi\gamma - \gamma\sigma(\xi) - \frac{3}{4}c = 0, \qquad \xi\gamma - \gamma\sigma(\xi) - \frac{2}{4}c = 0.$$

From the last three equations, we obtain $c = 0$, $\xi\gamma - \gamma\sigma(\xi) = 0$ and

$$(\gamma - a)B(X, Y) = 0. \tag{4.10}$$

Taking $X = Y = U$ to (4.10) and using $(4.8)_3$, we have

$$\gamma(\gamma - a) = 0.$$

Using (4.10) and the fact that $c = 0$, the equation (4.9) is reduced to

$$\{X\gamma - \gamma\sigma(X)\}g(Y, PZ) = \{Y\gamma - \gamma\sigma(Y)\}g(X, PZ). \tag{4.11}$$

Taking $X = PX$ and $Y = PY$ in (4.11) and taking into account that $S(TM)$ is a non-degenerate distribution, we obtain

$$\{PX\gamma - \gamma\sigma(PX)\}PY = \{PY\gamma - \gamma\sigma(PY)\}PX.$$

Now suppose there exists a vector field $X_o \in \Gamma(TM)$ such that $PX_o\gamma - \gamma\sigma(PX_o) \neq 0$, then it follows that all vector fields from $S(TM)$ are collinear with PX_o. This is a contradiction as $\mathrm{rank}(S(TM)) = m > 1$. Thus we obtain

$$PX\gamma - \gamma\sigma(PX) = 0.$$

Replacing Y by ξ to (4.11), we obtain

$$\{\xi\gamma - \gamma\sigma(\xi)\}g(X, Z) = 0.$$

Taking $X = Z$ to this equation such that $g(X, X) \neq 0$, we obtain

$$\xi\gamma - \gamma\sigma(\xi) = 0.$$

Consequently, we see that

$$X\gamma - \gamma\sigma(X) = 0. \tag{4.12}$$

Applying ∇_Y to (4.7) and using $(2.12)_3$, (3.7), (3.10) and (4.12), we have

$$\begin{aligned}(\nabla_X B)(Y,U) = &-\gamma\{bu(Y)\eta(X) + u(Y)\tau(X) + \pi(Y)u(X) + B(X,FY)\}\\ &- B(Y,F(A_N X)) - aB(Y,FX) - \pi(U)B(X,Y).\end{aligned}$$

Substituting this into (4.3) such that $Y = U$ and using (4.9), we obtain

$$\gamma b\{u(X)\eta(Y) - u(Y)\eta(X)\} = B(Y,F(A_N X)) - B(X,F(A_N Y)).$$

Taking $X = U$ and $Y = \xi$ to this equation and using (2.9), (4.7) and the fact that $u \circ F = 0$, we have $\gamma b = 0$. From this result, $(2.12)_3$ and (4.12), we get

$$X\gamma - \gamma\tau(X) = 0.$$

This completes the proof of the theorem. □

Definition. A lightlike hypersurface M is said to be *screen conformal* [5] if there exists a non-vanishing smooth function φ on $\mathcal{U}$ such that

$$C(X,PY) = \varphi B(X,Y). \tag{4.13}$$

Theorem 4.2. *Let M be a lightlike hypersurface of an indefinite complex space form $\bar{M}(c)$ with a semi-symmetric non-metric connection. If M is screen conformal, then $c = 0$.*

Proof. Assume that M is screen conformal. Using (3.6) and (4.13), we obtain

$$B(X,U - \varphi V) = -\,\eta(X)\pi(V).$$

Replacing X by ξ to this equation and using (2.9), we have $\pi(V) = 0$. Put

$$\mu = U - \varphi V. \tag{4.14}$$

Then μ is non-null vector field on $S(TM)$ and satisfies

$$B(X,\mu) = 0. \tag{4.15}$$

Applying ∇_X to $C(Y,PZ) = \varphi B(Y,PZ)$, we have

$$(\nabla_X C)(Y,PZ) = (X\varphi)B(Y,PZ) + \varphi(\nabla_X B)(Y,PZ).$$

Substituting this equation into (4.4) and using (4.3), we have

$$\begin{aligned}&\{X\varphi - \varphi\tau(X) - \varphi\sigma(X) + a\eta(X)\}B(Y,PZ)\\ &-\ \{Y\varphi - \varphi\tau(Y) - \varphi\sigma(Y) + a\eta(Y)\}B(X,PZ)\\ =\ &\frac{c}{4}\{\eta(X)g(Y,PZ) - \eta(Y)g(X,PZ) + [v(X) - \varphi u(X)]g(FY,PZ)\\ &-\ [v(Y) - \varphi u(Y)]g(FX,PZ) + 2[v(PZ) - \varphi u(PZ)]\bar{g}(X,JY)\}.\end{aligned}$$

Taking $Y = \xi$ and $PZ = \mu$ and using (3.2), (3.4), (4.14) and (4.15), we have

$$\frac{c}{2}\{v(X) - 3\varphi u(X)\} = 0.$$

Replacing X by V to this equation, we obtain $c = 0$. □

Theorem 4.3. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric non-metric connection ∇. If one of V or U is parallel with respect to ∇, then $\tau = 0$ and $R^{(0,2)}$ is a symmetric Ricci tensor of M. Moreover, if $\bar{M} = \bar{M}(c)$, then $c = 0$ and $\bar{M}(c)$ is flat.*

Proof. (1) If V is parallel with respect ∇, then, from (3.4) and (3.8), we have

$$J(A^*_\xi X) + bJX - \{u(A^*_\xi X) + bu(X)\}N - \sigma(X)V + \pi(V)X = 0.$$

Applying J to this and using (2.1), (2.11) and (3.2), we obtain

$$A^*_\xi X + bX - B(X,V)U + \sigma(X)\xi - \pi(V)JX = 0.$$

Taking the scalar product with ξ and N by turns and using $(2.12)_3$, we get

$$\pi(V)u(X) = 0, \qquad \tau(X) = \pi(V)v(X).$$

Taking $X = U$ to the first equation of the last two equations, we get $\pi(V) = 0$. Using this result, from the second equation we obtain $\tau = 0$. As $\tau = 0$, we see that $d\tau = 0$ and $R^{(0,2)}$ is a symmetric induced Ricci tensor of M.

As $\pi(V) = 0$ and $\sigma = -b\eta$, we obtain

$$A^*_\xi X = B(X,V)U + b\eta(X)\xi - bX.$$

Taking the scalar product with U to this and using (2.11), we obtain

$$B(X,U) = 0. \tag{4.16}$$

Applying ∇_Y to this equation and using (3.7) and $\tau = 0$, we have

$$(\nabla_X B)(Y,U) = -B(Y,F(A_N X)) - aB(FX,Y) - \pi(U)B(X,Y).$$

Substituting the last two equation into (4.3) such that $Z = U$, we have

$$B(X,F(A_N Y) - B(Y,F(A_N X) - a\{B(FX,Y) - B(X,FY)\}$$
$$= \frac{c}{4}\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X,JY)\}.$$

Taking $X = U$ and $Y = \xi$ to this and using (2.9) and (4.16), we get $c = 0$.

(2) If U is parallel with respect to ∇, then, from (3.4) and (3.7), we have

$$J(A_N X) + aJX - \{u(A_N X) + au(X)\}N + \tau(X)U + \pi(U)X = 0.$$

Applying J to this equation and using (2.1) and (3.2), we obtain

$$A_N X + aX - \{u(A_N X) + au(X)\}U - \tau(X)N - \pi(U)JX = 0.$$

Taking the scalar product with N and ξ by turns and using $(2.12)_2$, we get

$$\pi(U)v(X) = 0, \qquad \tau(X) = -\pi(U)u(X).$$

Taking $X = V$ to the first equation, we get $\pi(U) = 0$. Thus, from the second equation, we obtain $\tau = 0$. Using this results and (3.6), we obtain

$$A_N X = B(X,U)U - aX.$$

Taking the scalar product with U to this and using $(2.12)_1$, we obtain

$$C(X,U) = 0.$$

Applying ∇_Y to this equation and using the fact that $\nabla_Y U = 0$, we have

$$(\nabla_X C)(Y, U) = 0.$$

Substituting the last two equations into (4.4) with $PZ = U$, we have

$$a\{\eta(X)B(Y,U) - \eta(Y)B(X,U)\} = \frac{c}{2}\{v(Y)\eta(X) - v(X)\eta(Y)\}.$$

Replacing Y by ξ to this equation and using (2.9), we get

$$2aB(X,U) = cv(X).$$

If $a = 0$, then $c = 0$. Thus we set $a \neq 0$. Then the last equation is reduced to

$$B(X,U) = \beta v(X), \qquad \beta = c/2a. \tag{4.17}$$

Applying ∇_X to $v(Y) = g(X,U)$ and using (2.7) and the facts that $\pi(U) = 0$ and U is parallel with respect to ∇, we have

$$(\nabla_X v)Y = \beta v(X)\eta(Y) - \pi(Y)v(X). \tag{4.18}$$

Applying ∇_Y to (4.17) and using (4.18), we have

$$(\nabla_X B)(Y,U) = (X\beta)v(Y) + \beta^2 v(X)\eta(Y) - \beta\pi(Y)v(X).$$

Substituting this and (4.17) into (4.3) and using $\tau = 0$, we have

$$\begin{aligned}&(X\beta)v(Y) - (Y\beta)v(X) + \beta^2\{v(X)\eta(Y) - v(Y)\eta(X)\}\\ &= \frac{c}{4}\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X, JY)\}.\end{aligned}$$

Taking $X = U$ and $Y = \xi$ to this equation, we obtain $c = 0$. □

5. Recurrent, Lie recurrent and Hopf lightlike hypersurfaces

Definition. The structure tensor field F of M is said to be *recurrent* [12] if there exists a 1-form ϖ on M such that

$$(\nabla_X F)Y = \varpi(X)FY.$$

A lightlike hypersurface M of an indefinite Kaehler manifold $\bar{M}$ is called *recurrent* if it admits a recurrent structure tensor field F.

Theorem 5.1. *Let M be a recurrent lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric non-metric connection. Then*

(1) *the characteristic vector field ζ on $\bar{M}$ is tangent to M,*
(2) *F is parallel with respect to the induced connection ∇ on M,*
(3) *D and $J(tr(TM))$ are parallel distributions on M,*
(4) *M is locally a product manifold $\mathcal{C}_U \times M^\sharp$, where $\mathcal{C}_U$ is a null curve tangent to $J(tr(TM))$ and $M^\sharp$ is a leaf of the distribution D,*
(5) *if $\bar{M} = \bar{M}(c)$, then $c = 0$, i.e., $\bar{M}(c)$ is flat,*
(6) *if $\bar{M} = \bar{M}(c)$, then $R^{(0,2)}$ is a symmetric induced Ricci tensor of M.*

Proof. (1) From the above definition and (3.9), we get

$$\varpi(X)FY = u(Y)A_N X - B(X,Y)U + \pi(JY)X - \pi(Y)FX. \tag{5.1}$$

Replacing Y by ξ and using (2.9), (3.4) and the fact that $F\xi = -V$, we get

$$\varpi(X)V = \pi(V)X + bFX.$$

Taking the scalar product with N to this equation, we obtain

$$\pi(V)\eta(X) + bv(X) = 0.$$

Taking $X = V$ and then $X = \xi$ to this equation, we have

$$b = 0, \qquad \pi(V) = 0.$$

As $b = 0$, the characteristic vector field ζ on $\bar{M}$ is tangent to M.

(2) As $b = 0$ and $\pi(V) = 0$, we see that $\varpi(X)V = 0$. Taking the scalar product with U to this result, we get $\varpi = 0$. It follows that $\nabla_X F = 0$. Therefore, F is parallel with respect to the induced connection ∇ on M.

(3) Taking the scalar product with V to (5.1) such that $\varpi = 0$, we have

$$B(X,Y) = u(Y)u(A_N X) + \pi(JY)u(X).$$

Taking $Y = V$ and $Y = FZ$, $Z \in \Gamma(D_o)$ to this equation by turns and using the facts that $b = 0$, $u(FZ) = 0$ and $FZ = JZ$, we have

$$B(X,V) = 0, \qquad B(X,FZ) = -\pi(Z)u(X). \tag{5.2}$$

In general, by using (2.1), (2.6), (2.7), (2.11), (3.4) and (3.8), we derive

$$g(\nabla_X \xi, V) = -B(X,V) + bu(X), \qquad g(\nabla_X V, V) = \pi(V)u(X),$$
$$g(\nabla_X Z, V) = \pi(Z)u(X) + B(X,FZ), \quad \forall\, X \in \Gamma(TM),\ Z \in \Gamma(D_o).$$

From these equations and (5.2), we see that

$$\nabla_X Y \in \Gamma(D), \quad \forall\, X \in \Gamma(TM), \quad \forall\, Y \in \Gamma(D).$$

It follows that D is a parallel distribution on M.

On the other hand, taking $Y = U$ to (5.1), we get

$$A_N X = B(X,U)U - aX + \pi(U)FX. \tag{5.3}$$

Replacing X by V to this and using the fact that $B(X,V) = 0$, we have

$$A_N V = -aV + \pi(U)\xi.$$

Taking the scalar product with N and using $(2.12)_2$, we have $\pi(U) = 0$. Applying F to (5.3) and using the facts that $FU = 0$ and $\pi(U) = 0$, we get

$$F(A_N X) + aFX = 0.$$

Using the last equation and the fact that $\pi(U) = 0$, (3.7) is reduced to

$$\nabla_X U = \tau(X)U. \tag{5.4}$$

It follows that $J(tr(TM))$ is also a parallel distribution on M, *i.e.*,

$$\nabla_X U \in \Gamma(J(tr(TM))), \quad \forall\, X \in \Gamma(TM).$$

(4) As D and $J(tr(TM))$ are parallel distributions satisfying (3.1), by the decomposition theorem [3], M is locally a product manifold $\mathcal{C}_U \times M^\sharp$, where $\mathcal{C}_U$ is a null curve tangent to $J(tr(TM))$ and $M^\sharp$ is a leaf of D.

(5) Taking the scalar product with U to (5.3) and using (2.12) and the fact that $\pi(U) = 0$, we obtain

$$C(Y, U) = 0.$$

Applying ∇_X to this equation and using (5.4), we have

$$(\nabla_X C)(Y, U) = 0.$$

Replacing PZ by U to (4.4) and using the last two equations, we obtain

$$a\{\eta(X)B(Y, U) - \eta(Y)B(X, U)\} = \frac{c}{2}\{v(Y)\eta(X) - v(X)\eta(Y)\}. \tag{5.5}$$

Taking $X = \xi$ and $Y = V$ and using (2.9) and $(5.2)_1$, we have $c = 0$.

(6) As $c = 0$, taking $Y = \xi$ to (5.5) and using (2.9), we obtain

$$aB(X, U) = 0. \tag{5.6}$$

By directed calculations from (5.4), we obtain

$$R(X, Y)U = 2d\tau(X, Y)U.$$

Comparing this equation with (4.2) such that $Z = U$, we have

$$2d\tau(X, Y)U = B(Y, U)A_N X - B(X, U)A_N Y.$$

Taking the scalar product with V and using (3.6) and (5.6), we get $d\tau = 0$. Therefore, $R^{(0,2)}$ is a symmetric induced Ricci tensor of M. □

Definition. The structure tensor field F of M is said to be *Lie recurrent* [12] if there exists a 1-form ϑ on M such that

$$(\mathcal{L}_X F)Y = \vartheta(X)FY,$$

where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X, that is,

$$(\mathcal{L}_X F)Y = [X, FY] - F[X, Y].$$

The structure tensor field F is called *Lie parallel* if $\mathcal{L}_X F = 0$. A lightlike hypersurface M of an indefinite Kaehler manifold $\bar{M}$ is called *Lie recurrent* if it admits a Lie recurrent structure tensor field F.

Theorem 5.2. *Let M be a Lie recurrent lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric non-metric connection. Then*

1. *F is Lie parallel,*
2. *the 1-forms τ and σ satisfy $\tau = 0$ and $\sigma = -b\eta$,*
3. *$R^{(0,2)}$ is a symmetric induced Ricci tensor of M,*
4. *if $\bar{M} = \bar{M}(c)$, then $c = 0$ and $\bar{M}(c)$ is flat.*

Proof. (1) Using the above definition, (2.8), (3.4) and (3.9), we get

$$\vartheta(X)FY = u(Y)A_{N}X - B(X,Y)U + au(Y)X - \nabla_{FY}X + F\nabla_{Y}X. \tag{5.7}$$

Taking $Y = \xi$ to (5.7) and using (2.8) and the fact that $F\xi = -V$, we have

$$-\vartheta(X)V = \nabla_{V}X + F\nabla_{\xi}X. \tag{5.8}$$

Taking the scalar product with V to (5.8) and using $g(FX, V) = 0$, we have

$$u(\nabla_{V}X) = g(\nabla_{V}X, V) = 0. \tag{5.9}$$

Replacing X by U to this equation and using $(2.12)_3$ and (3.7), we obtain

$$\tau(V) = \sigma(V) = 0. \tag{5.10}$$

Replacing Y by V to (5.7) and using the fact that $FV = \xi$, we have

$$\vartheta(X)\xi = -B(X,V)U - \nabla_{\xi}X + F\nabla_{V}X.$$

Applying F to this equation and using (3.5) and (5.9), we obtain

$$\vartheta(X)V = \nabla_{V}X + F\nabla_{\xi}X.$$

Comparing this equation with (5.8), we get $\vartheta = 0$. Thus F is Lie parallel.

(2) Taking the scalar product with N to (5.7) and using $(2.12)_2$, we have

$$-\bar{g}(\nabla_{FY}X, N) + \bar{g}(F\nabla_{Y}X, N) = 0. \tag{5.11}$$

Replacing X by ξ to (5.11) and using (2.6), (3.2) and (3.4), we have

$$g(A^{*}_{\xi}X, U) = \sigma(FX).$$

From this equation, (2.11), $(2.12)_3$ and the fact that $v(X) = \eta(FX)$, we have

$$B(X,U) = \tau(FX). \tag{5.12}$$

Replacing X by U to this and using (3.6) and the fact that $FU = 0$, we get

$$C(U,V) = B(U,U) = 0. \tag{5.13}$$

Replacing X by V to (5.11) and using $(2.11)_2$, (3.4), (3.5) and (3.8), we have

$$g(A^{*}_{\xi}FY, U) + \sigma(Y) = 0.$$

Using this equation, (2.11) and $(2.12)_3$, we obtain

$$B(FY, U) = -\tau(Y).$$

Replacing Y by U to this and using the fact that $FU = 0$, we obtain

$$\tau(U) = \sigma(U) = 0. \tag{5.14}$$

Replacing X by U to (5.7) and using (3.5), (3.6), and (3.7), we get

$$u(Y)A_{N}U - F(A_{N}FY) - A_{N}Y - \tau(FY)U = 0. \tag{5.15}$$

Taking the scalar product with V and using (2.12), (3.6) and (5.13), we get

$$B(X,U) = -\,\tau(FX).$$

Comparing this with (5.12), we obtain $\tau(FX) = 0$. Replacing X by FY to this and using (3.6) and (5.14), we have $\tau = 0$. From $(2.12)_3$, we get $\sigma = -b\eta$.

(3) As $\tau = 0$, $d\tau = 0$ and $R^{(0,2)}$ is a symmetric induced Ricci tensor of M.

(4) As $\tau = 0$, from (5.12) we obtain

$$B(Y, U) = 0. \tag{5.16}$$

Applying ∇_X to this equation and using (3.7), we have

$$(\nabla_X B)(Y, U) = -B(Y, F(A_N X)) - aB(FX, Y) - \pi(U)B(X, Y).$$

Substituting the last two equation into (4.3), we have

$$\begin{aligned} &B(X, F(A_N Y) - B(Y, F(A_N X) - a\{B(FX, Y) - B(X, FY)\} \\ &= \frac{c}{4}\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X, JY)\}. \end{aligned}$$

Taking $X = U$ and $Y = \xi$ to this and using (2.9) and (5.16), we get $c = 0$. □

Definition. The structure vector field U on a lightlike hypersurface M of an indefinite almost complex manifold $\bar{M}$ is called *principal* [12], with respect to the shape operator A^*_ξ, if there exists a smooth function α such that

$$A^*_\xi U = \alpha U. \tag{5.17}$$

A lightlike hypersurface M of an indefinite almost complex manifold $\bar{M}$ is called a *Hopf lightlike hypersurface* [12] if it admits a principal structure vector field U, with respect to the shape operator A^*_ξ.

Taking the scalar product with X to (5.17) and using (2.11), we get

$$B(X, U) = \beta v(X), \tag{5.18}$$

where we set $\beta = \alpha + b$. From this equation and (3.6), we obtain

$$u(A_N X) = \beta v(X) - au(X). \tag{5.19}$$

Theorem 5.3. *Let M be a Hopf lightlike hypersurfaces of an indefinite complex space form $\bar{M}(c)$ with a semi-symmetric non-metric connection. Then $c = 0$.*

Proof. Substituting (3.4) into $\bar{g}(JX, Y) + \bar{g}(X, JY) = 0$, we have

$$g(FX, Y) + g(X, FY) + u(X)\eta(Y) + u(Y)\eta(X) = 0.$$

Applying ∇_X to $v(Y) = g(X, U)$ and using (2.7), $(2.12)_2$, (3.4), (3.7), (5.19) and the last equation, we obtain

$$(\nabla_X v)Y = v(Y)\tau(X) - \pi(Y)v(X) - ag(X, FY) - g(A_N X, FY). \tag{5.20}$$

Applying ∇_Y to (5.18) and using (3.7) and (5.20), we have

$$\begin{aligned} (\nabla_X B)(Y, U) &= (X\beta)v(Y) - \beta\pi(Y)v(X) - \beta ag(X, FY) - \beta g(A_N X, FY) \\ &\quad - B(Y, F(A_N X)) - aB(FX, Y) - \pi(U)B(X, Y). \end{aligned}$$

Substituting this equation and (5.18) into (4.12), we have

$$\begin{aligned} &(X\beta)v(Y) - (Y\beta)v(X) + \beta\{v(Y)\tau(X) - v(X)\tau(Y)\} \\ &+ \beta a\{g(FX, Y) - g(X, FY)\} + \beta\{g(A_N Y, FX) - g(A_N X, FY)\} \\ &+ B(X, F(A_N Y)) - B(Y, F(A_N X)) + a\{B(X, FY) - B(Y, FX)\} \end{aligned}$$

$$= \frac{c}{4}\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X, JY)\}.$$

Taking $X = \xi$ and $Y = U$ to this equation and using (2.9), $(2.12)_2$, (3.4), (5.18), (5.19) and the facts that $FU = 0$ and $F\xi = -V$, we obtain $c = 0$. □

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