COMPUTATION OF $\lambda$-INARIANT

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ABSTRACT. We give an explicit formula for the computation of Iwasawa $\lambda$-invariants and an example of the computation using our method.

1. Introduction

Let $K$ be an imaginary quadratic field and $p$ be an odd prime. It is well-known (see [1] and [2]) that there exist non-negative integers $\lambda_p(K)$ and $\nu_p(K)$ such that the exact power of $p$ dividing the class number $h(K_n)$ is equal to $\lambda_p(K)n + \nu_p(K)$ for all sufficiently large $n$. Here $K_n$ is the $n$-th layer of the cyclotomic $\mathbb{Z}_p$-extension of $K$. Fukuda [3] computed $\lambda_p(K)$ using theorems of Gold and Iwasawa’s construction of $p$-adic $L$-function attached to $K$. In a paper [6], we gave another method to compute $\lambda_p(K)$ using Sinnott’s construction of $p$-adic $L$-function and Kida’s formula. Examples of computation of $\lambda_p(K)$ were given for $p = 3$ in the paper. In this paper, we compute $\lambda_p(K)$ for primes greater than 5 using our method in the paper [6].

2. Computation of $\lambda$-invariant

We briefly explain our method in the paper [6] for computing $\lambda_p(K)$. Let $\Lambda$ be the ring of $\mathbb{Z}_p$-valued measures on $\mathbb{Z}_p$. Then $\Lambda$ is isomorphic to
the ring $\mathbb{Z}_p[[T - 1]]$; explicitly, if $\alpha \in \Lambda$, then the power series associated to $\alpha$ is defined by

$$F(T) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \left( \frac{x}{n} \right) \, d\alpha(T - 1)^n,$$

where $\left( \frac{x}{n} \right) = \frac{x(x-1)\cdots(x-n+1)}{n!}$.

Let $c > 1$ be an integer prime to $p$ and the conductor of a nontrivial first kind character $\chi$ of $K$, and let $\varepsilon : \mathbb{Z} \to \mathbb{Z}_p$ be the function defined by $\varepsilon(a) = \chi(a)$, if $a$ is not divisible by $c$, and $\varepsilon(a) = \chi(a)(1 - c)$ if $a$ is divisible by $c$. Define

$$F_\varepsilon(T) = \sum_{a=1}^{f} \varepsilon(a)T^a, \frac{1}{1 - T^f},$$

where $f$ is any multiple of the minimal period of $\varepsilon$. It is known that $F_\varepsilon(T)$ lies in $\mathbb{Z}_p[[T - 1]]$. Hence it corresponds to a measure in $\Lambda$. Let $G(T)$ be the power series in $\mathbb{Z}_p[[T - 1]]$ corresponding to the measure

$$(\sum_{\eta \in \mathcal{V}} \alpha \circ \eta|_{U}) \circ \phi,$$

where $\mathcal{V}$ is the group of $p - 1$-th roots of unity in $\mathbb{Z}_p$, $U = 1 + p\mathbb{Z}_p$ and $\phi$ is the isomorphism $\phi : \mathbb{Z}_p \simeq U$ given by $\phi(y) = (1 + p)^y$.

If $F(T)$ is an element of $\mathbb{Z}_p[[T - 1]]$, write $F(T) = p^mF_0(T)$, $F_0(T) = \sum_{n \geq 0} a_n(T - 1)^n$, where $a_n \neq 0$ mod $p$ for some $n$. Then the $\lambda$-invariant of $F(T)$ is defined by

$$\lambda(F(T)) = \min\{n : a_n \neq 0 \text{ mod } p\}$$

Sinnott [7] proved that

$$\lambda_p(K) = \lambda(G(T))$$

when $p \geq 5$. Moreover we have Kida’s formula [5]:

$$p\lambda(G(T)) = \lambda(\sum_{\eta \in \mathcal{V}} \alpha \circ \eta|_{U}).$$

In the paper [6], we computed the power series $Q(T)$ corresponding to the measure $\sum_{\eta \in \mathcal{V}} \alpha \circ \eta|_{U}$.

**Theorem 1.**

$$Q(T) = \sum_{\eta \in \mathcal{V}} \frac{\sum_{a=1}^{f} \varepsilon(a)T^a\eta}{1 - T^f\eta},$$
where \( f \) is a multiple of the minimal period of \( \varepsilon \) and \( p \).

**Proof.** See the proof of Theorem 2 in [6]. \( \square \)

To compute \( \lambda(Q(T)) \) explicitly, we need to replace \( \eta \) by an integer \( i_\eta \).

**Lemma 1.** Let \( f(T) \) be in \( \mathbb{Z}_p[[T - 1]] \). Then 
\[
\lambda(f(T)) = \lambda(f(T^\beta))
\]
for \( \beta \in 1 + p\mathbb{Z}_p \).

**Proof.** Note that if \( f(T) \) is the power series associated to a measure \( \alpha \), then \( f(T^\beta) \) is the power series associated to a measure \( \alpha \circ \beta^{-1} \). So \( f(T^\beta) \) is in \( \mathbb{Z}_p[[T - 1]] \). We may write \( f(T) = \sum_{n=0}^{\infty} a_n(T - 1)^n \). By the definition of \( \lambda \) we see that \( a_n \equiv 0 \mod p \) for \( n < \lambda(f(T)) \) and \( a_{\lambda(f(T))} \not\equiv 0 \mod p \).

Since 
\[
T^\beta = \sum_{n=0}^{\infty} \binom{\beta}{n} (T - 1)^n \equiv 1 + \beta(T - 1) + \text{higher terms} \\
\equiv T + \text{higher terms} \mod p,
\]
it is easy to check that \( \lambda(f(T)) = \lambda(f(T^\beta)) \). \( \square \)

For \( \eta \in V \), let \( 1 \leq i_\eta, j_\eta \leq (p - 1) \) be integers such that \( \eta \equiv i_\eta \mod p \) and \( i_\eta j_\eta \equiv 1 \mod p \). Now we give a formula to compute \( \lambda \)-invariants for imaginary quadratic fields.

**Theorem 2.** For primes \( p \geq 5 \), we have 
\[
\lambda_p(K) = \frac{1}{p} \lambda\left(\sum_{\eta \in V} \frac{\sum_{a \equiv j_\eta} \varepsilon(a)T^{ai_\eta}}{1 - Tj_\eta}\right).
\]

**Proof.**
\[
\lambda_p(K) = \lambda(G(T)) = \frac{1}{p} \lambda\left(\sum_{\eta \in V} \alpha \circ \eta|_U\right)
\]
\[
= \frac{1}{p} \lambda(Q(T)) = \frac{1}{p} \lambda\left(\sum_{\eta \in V} \frac{\sum_{a \equiv j_\eta} \varepsilon(a)T^{ai_\eta}}{1 - Tj_\eta}\right).
\]
The last equality comes from Lemma 1 with \( \beta = \eta^{-1}i_\eta \). \( \square \)

We give an example.
Example 1. For $K = \mathbb{Q}(\sqrt{-127})$ and $p = 5$, we can choose $c = 2, f = 1270$. Moreover, $\varepsilon(a) = (\frac{a}{127})(-1)^{a+1}$, where $(\frac{a}{p})$ is the Jacobi symbol. Hence we have

$$
\lambda_5(\mathbb{Q}(\sqrt{-127})) = \frac{1}{5} \left( \sum_{a \equiv 1(5)}^{1270} \varepsilon(a) T^a \frac{1}{1 - T^{1270}} + \sum_{a \equiv 3(5)}^{1270} \varepsilon(a) T^{2a} \frac{1}{1 - T^{2 \times 1270}} 
+ \sum_{a \equiv 2(5)}^{1270} \varepsilon(a) T^{3a} \frac{1}{1 - T^{3 \times 1270}} + \sum_{a \equiv 4(5)}^{1270} \varepsilon(a) T^{4a} \frac{1}{1 - T^{4 \times 1270}} \right).
$$

$$
= \frac{1}{5} \lambda((T - 1)^{10} + (T - 1)^{11} + \text{higher terms (mod } p)) = 2,
$$

which agrees with the Table 1 of [4]. We used Maple for the second equality.

References


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