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CLASSIFICATION ON ARITHMETIC FUNCTIONS AND CORRESPONDING FREE-MOMENT L-FUNCTIONS

Ilwoo Cho

Abstract. In this paper, we provide a classification of arithmetic functions in terms of identically-free-distributedness, determined by a fixed prime. We show then such classifications are free from the choice of primes. In particular, we obtain that the algebra $\mathfrak{A}_p$ of equivalence classes under the quotient on $\mathcal{A}$ by the identically-free-distributedness is isomorphic to an algebra $\mathbb{C}^2$, having its multiplication $(\bullet)$; $(t_1, t_2) \bullet (s_1, s_2) = (t_1 s_1, t_1 s_2 + t_2 s_1)$.

1. Introduction

In [7], the author and Gillespie established the free probabilistic model $(\mathcal{A}, \varphi_x)$ on the algebra $\mathcal{A}$ consisting of all *arithmetic functions* by constructing *linear functionals* φ_x on it, for all positive real numbers x.

As a continued study of such a free probabilistic model on $\mathcal{A}$, the author considered "truncated" linear functionals $\varphi_{x<y}$, in [6]. They contain information about primes between x and y, and induce free probabilistic data for the primes, where $x < y \in \mathbb{R}^+$. In particular, we showed that the free distributional data of $f \in \mathcal{A}$ is characterized by each prime.

Arithmetic functions are main objects in modern *number theory*. They are the functions f defined from the natural numbers $\mathbb{N}$ into the complex numbers $\mathbb{C}$. Especially, they are tools for constructing (classical) *Dirichlet L-functions*,

$$L_f(s) = \sum_{k=1}^{\infty} \frac{f(k)}{k^s} \quad \text{for all } s \in \mathbb{C},$$

on $\mathbb{C}$, for $f \in \mathcal{A}$. The entire-ness of L-functions is an important and interesting topic in analysis. So, they are main objects for investigating modern number theory; *combinatorial number theory*, *L-function theory*, and *analytic number theory*, etc (e.g., [1], [2], [11], [12], [13], [14] and [17]).

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1.1. Background

We concentrate on the fact the collection $\mathcal{A}$ of all arithmetic functions forms an algebra, under the usual functional addition and convolution. Recall that if f_1, f_2 are arithmetic functions in $\mathcal{A}$, then the *convolution* $f_1 * f_2$ becomes an arithmetic function in $\mathcal{A}$, too, where

$$f_1 * f_2(n) \overset{\text{def}}{=} \sum_{d|n} f_1(d) f_2\left(\frac{n}{d}\right)$$

for all $n \in \mathbb{N}$, where "$d \mid n$" means "d divides n," or "n is divisible by d," for $d \in \mathbb{N}$.

In [7], by understanding $\mathcal{A}$ as an algebra, we defined linear functionals φ_x on $\mathcal{A}$ by

$$\varphi_x(f) \overset{\text{def}}{=} \sum_{p:\text{primes},\, p \leq x} f(p) \quad \text{for all } f \in \mathcal{A}.$$

It provides corresponding *free probability on* $\mathcal{A}$, i.e., each element f of $\mathcal{A}$ can have its distributional data in terms of φ_x. The existence of the system $\{\varphi_x\}_{x \in \mathbb{R}^+}$ guarantees how the real numbers $\mathbb{R}$ acts on $\mathcal{A}$. We have studied free-probabilistic properties of $\{(\mathcal{A}, \varphi_x)\}_{x \in \mathbb{R}^+}$, and considered the relations between free probabilistic data and well-known number-theoretic results in [7].

In [6], we defined a new type of linear functionals $\varphi_{x<y}$, for $x < y$ in $\mathbb{R}^+$;

$$\varphi_{x<y}(f) \overset{\text{def}}{=} \sum_{p:\text{prime},\, x < p \leq y} f(p) \quad \text{for all } f \in \mathcal{A},$$

and established the corresponding free probability space $(\mathcal{A}, \varphi_{x<y})$.

We study relations between $(\mathcal{A}, \varphi_{x<y})$ and $\{(\mathcal{A}, \varphi_t)\}_{t \in \mathbb{R}^+}$ in [6]. We realized that every prime p acts as a linear functional g_p on $\mathcal{A}$, i.e.,

$$g_p(f) \overset{\text{def}}{=} f(p) \quad \text{for all } f \in \mathcal{A}.$$

1.2. Overview

In this paper, based on the results of [6] and [7], we concentrate on studying (i) free moments of arithmetic functions, (ii) the identically free-distributedness conditions, and (iii) the corresponding *free-moment L-functions* in the sense of [6].

The main results are the *classification theorem* of the arithmetic algebra $\mathcal{A}$, and that of the algebra $\mathcal{L}_p$, consisting of all free-moment L-functions in $(\mathcal{A}, g_p)$, under an equivalence relation induced by identically free-distributedness.

In Section 2, we briefly introduce free probability theory providing a main tool for our study. In Section 3, free moments of arithmetic functions are computed. From the computation, we construct a classification on $\mathcal{A}$. Corresponding free-moment L-functions are studied in Section 4.

2. Free probability

Let $\mathfrak{A}$ be an arbitrary algebra, and let $\psi : \mathfrak{A} \to \mathbb{C}$ be a linear functional on $\mathfrak{A}$. Then the pair $(\mathfrak{A}, \psi)$ is called a free probability space (*over* $\mathbb{C}$). All operators $a \in (\mathfrak{A}, \psi)$ are called *free random variables* (See [16] and [18]). Remark that free probability spaces are dependent upon the choice of linear functionals.

Let $a_1, \ldots, a_s$ be a free random variable in a $(\mathfrak{A},\psi)$, for $m \in \mathbb{N}$. The *free moments of* $a_1, \ldots, a_m$ are determined by the quantities

$$\psi(a_{i_1} \cdots a_{i_n})$$

for all $(i_1, \ldots, i_n) \in \{1, \ldots, m\}^n$, for all $n \in \mathbb{N}$.

And the *free cumulants* $k_n(a_{i_1}, \ldots, a_{i_n})$ *of* $a_1, \ldots, a_m$ is determined by the *Möbius inversion*;

$$\begin{aligned} k_n(a_{i_1}, \ldots, a_{i_n}) &= \sum_{\pi \in NC(n)} \psi_\pi(a_{i_1}, \ldots, a_{i_n})\mu(\pi, 1_n) \\ &= \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \psi_V(a_{i_1}, \ldots, a_{i_n})\mu\left(0_{|V|},\ 1_{|V|}\right)\right) \end{aligned}$$

for all $(i_1, \ldots, i_n) \in \{1, \ldots, m\}^n$, for all $n \in \mathbb{N}$, where $\psi_\pi(\cdots)$ means the *partition-depending moments*, and $\psi_V(\cdots)$ means the *block-depending moment*; for example, if

$$\pi = \{(1, 5, 7),\ (2, 3, 4), (6)\} \text{ in } NC(7),$$

with three blocks $(1, 5, 7), (2, 3, 4)$, and (6), then

$$\begin{aligned} \psi_\pi\left(a_{i_1}, \ldots, a_{i_7}\right) &= \psi_{(1,5,7)}(a_{i_1}, \ldots, a_{i_7})\psi_{(2,3,4)}(a_{i_1}, \ldots, a_{i_7})\psi_{(6)}(a_{i_1}, \ldots, a_{i_7}) \\ &= \psi(a_{i_1} a_{i_5} a_{i_7})\psi(a_{i_2} a_{i_3} a_{i_4})\psi(a_{i_6}). \end{aligned}$$

Here, the set $NC(n)$ means the *noncrossing partition set* over $\{1, \ldots, n\}$, which is a *lattice* with the inclusion $\leq$, such that

$$\theta \leq \pi \overset{\text{def}}{\Longleftrightarrow} \forall V \in \theta,\ \exists B \in \pi \text{ s.t.}, V \subseteq B,$$

where $V \in \theta$ or $B \in \pi$ means that V is a *block of* θ, respectively, B is a block of π, and $\subseteq$ means the usual set inclusion. This lattice has its minimal element $0_n = \{(1), (2), \ldots, (n)\}$, the n-block partition, and its maximal element $1_n = \{(1, \ldots, n)\}$, the one-block partition.

Especially, a partition-depending free moment $\psi_\pi(a, \ldots, a)$ is determined by

$$\psi_\pi(a, \ldots, a) = \prod_{V \in \pi} \psi\left(a^{|V|}\right),$$

where $|V|$ means the cardinality of V.

Also, μ is the *Möbius functional* from $\mathcal{NC}$ into $\mathbb{C}$, where

$$\mathcal{NC} = \bigcup_{n=1}^{\infty} \left(NC(n) \times NC(n)\right),$$

i.e., it satisfies

$$\mu(\pi, \theta) = 0 \text{ for all } \pi > \theta \text{ in } NC(n),$$

and

$$\mu(0_n, 1_n) = (-1)^{n-1} c_{n-1}, \text{ and } \sum_{\pi \in NC(n)} \mu(\pi, 1_n) = 0$$

for all $n \in \mathbb{N}$, where

$$c_k = \frac{1}{k+1} \begin{pmatrix} 2k \\ k \end{pmatrix} = \frac{1}{k+1} \frac{(2k)!}{k!k!}$$

means the *k-th Catalan numbers*, for all $k \in \mathbb{N}$. Notice that since each $NC(n)$ is a well-defined lattice, if $\pi < \theta$ are given in $NC(n)$, one can decide the "interval"

$$[\pi, \theta] = \{\delta \in NC(n) : \pi \leq \delta \leq \theta\},$$

and it is always lattice-isomorphic to

$$[\pi, \theta] = NC(1)^{k_1} \times NC(2)^{k_2} \times \cdots \times NC(n)^{k_n}$$

for some $k_1, \ldots, k_n \in \mathbb{N}$, where $NC(l)^{k_t}$ means "l blocks of π generates k_t blocks of θ," for $k_j \in \{0, 1, \ldots, n\}$, for all $n \in \mathbb{N}$. By the multiplicativity of μ on $NC(n)$, for all $n \in \mathbb{N}$, if an interval $[\pi, \theta]$ in $NC(n)$ satisfies the above set-product relation, then we have

$$\mu(\pi, \theta) = \prod_{j=1}^{n} \mu(0_j, 1_j)^{k_j}.$$

(For details, see [16]. For applications, see [9] and [10].)

In fact, the free moments of free random variables and the free cumulants of them provide equivalent free distributional data. For example, if a free random variable a in $(\mathfrak{A}, \psi)$ is a self-adjoint operator in a topological $*$-algebra $\mathfrak{A}$ in the sense that: $a^* = a$, then both free moments $\{\psi(a^n)\}_{n=1}^{\infty}$ and free cumulants $\{k_n(a, \ldots, a)\}_{n=1}^{\infty}$ give its spectral distributional data.

However, their uses are different case-by-case. For instance, to study the free distribution of fixed free random variables, the computation and investigation of free moments is better; and to study (inner) free structures of free random variables in $\mathfrak{A}$, the computation and observation of free cumulants is better (See [15] and [16]).

We say two *subalgebras* A_1 *and* A_2 *of* $\mathfrak{A}$ *are free in* $(\mathfrak{A}, \psi)$, if all "mixed" free cumulants of A_1 and A_2 vanish. Two *subsets* X_1 *and* X_2 *are free in* $(\mathfrak{A}, \psi)$, if the subalgebras A_1 and A_2 generated by X_1 respectively X_2 are free in $(\mathfrak{A}, \psi)$. Similarly, two *free random variables* a_1 *and* a_2 *are free in* $(\mathfrak{A}, \psi)$, if two subsets $\{a_1\}$ and $\{a_2\}$ are free in $(\mathfrak{A}, \psi)$.

Assume that $\{A_i\}_{i \in \Lambda}$ is a family of subalgebras of $\mathfrak{A}$, generating $\mathfrak{A}$, i.e., $\mathfrak{A}$ is generated by $\{A_i\}_{i \in \Lambda}$. If A_i's are free from each other in $(\mathfrak{A}, \psi)$, then we say $\mathfrak{A}$ is a *free product algebra* with its *free blocks* $\{A_i\}_{i \in \Lambda}$, and denote this relation by

$$\mathfrak{A} = \underset{i \in \Lambda}{*_{\mathbb{C}}} A_i.$$

3. Free moments of arithmetic functions determined by primes

Define

$$\mathcal{A} \overset{\text{def}}{=} \{f : \mathbb{N} \to \mathbb{C} : f \text{ is a function}\},$$

i.e., $\mathcal{A}$ is the set of all *arithmetic functions*. Then under the usual functional addition, if $t_1, t_2 \in \mathbb{C}$, and $f_1, f_2 \in \mathcal{A}$, then

$$t_1 f_1 + t_2 f_2 \in \mathcal{A}.$$

Thus, the set $\mathcal{A}$ is a vector space over $\mathbb{C}$. Consider now the *convolution* $(*)$ on $\mathcal{A}$,

$$f_1 * f_2(n) \overset{\text{def}}{=} \sum_{d|n} f_1(d) f_2\left(\frac{n}{d}\right) \quad \text{for all } n \in \mathbb{N}.$$

Then $f_1 * f_2$ is an arithmetic function in $\mathcal{A}$, too.

Since this convolution is associative,

$$(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3),$$

and since it is distributed,

$$f_1 * (f_2 + f_3) = f_1 * f_2 + f_1 * f_3,$$
$$(f_1 + f_2) * f_3 = f_1 * f_3 + f_2 * f_3$$

for all $f_1, f_2, f_3 \in \mathcal{A}$, the vector space $\mathcal{A}$ becomes an algebra equipped with $(*)$.

Definition 3.1. We call $\mathcal{A}$ the arithmetic(-functional) algebra.

Remark that

$$f_1 * f_2 = f_2 * f_1 \quad \text{for all } f_1, f_2 \in \mathcal{A}.$$

So, the arithmetic algebra $\mathcal{A}$ is a commutative algebra.

Define a linear functional g_p on $\mathcal{A}$ for a fixed prime p by

$$g_p(f) \overset{\text{def}}{=} f(p) \quad \text{for all } f \in \mathcal{A}. \tag{3.1}$$

Definition 3.2. Let $\mathcal{A}$ be the arithmetic algebra, and let g_p be the linear functional in the sense of (3.1) for a fixed prime p. Then the free probability space $(\mathcal{A}, g_p)$ is called the arithmetic p-prime probability space of $\mathcal{A}$.

In [6], we obtained the following free moment computation needed for our study.

Proposition 3.1 (See [6]). *Let $(\mathcal{A}, g_p)$ be the arithmetic p-prime-probability space, for a fixed prime p. If f is a free random variable in $(\mathcal{A}, g_p)$, then*

$$g_p(f^{(n)}) = n f(1)^{n-1} f(p) \quad \textit{for all } n \in \mathbb{N}, \tag{3.2}$$

where $f^{(n)} = f * \cdots * f$ (n-copies).

The formula (3.2) is proven inductively, by the general relation (3.3) below,

$$g_p(f_1 * f_2) = g_p(f_1)f_2(1) + f_1(1)g_p(f_2) \tag{3.3}$$

for all $f_1, f_2 \in \mathcal{A}$, for all primes p (See [6]).

Recall that two free random variables a_j of free probability spaces $(\mathfrak{A}_j, \psi_j)$, for $j = 1, 2$, are said to be *identically (free-)distributed*, if

$$\psi_1(a_1^n) = \psi_2(a_2^n) \text{ for all } n \in \mathbb{N},$$

where $\mathfrak{A}_j$ are (pure-algebraic) algebras and ψ_j are linear functionals on A_j, for $j = 1, 2$.

By (3.2) and (3.3), one obtains the following identically distributedness conditions.

Theorem 3.2. *Let f_1 and f_2 be arithmetic functions in the arithmetic algebra $\mathcal{A}$. Suppose $f_j \in (\mathcal{A}, g_{p_j})$ for $j = 1, 2$. Then f_1 and f_2 are identically distributed if and only if*

$$f_1(1) = f_2(1), \text{ and } f_1(p_1) = f_2(p_2) \text{ in } \mathbb{C}. \tag{3.4}$$

Proof. ($\Leftarrow$) Assume that

$$f_1(1) = w = f_2(1) \text{ and } f_1(p_1) = u = f_2(p_2) \text{ in } \mathbb{C}$$

for $f_j \in (\mathcal{A}, g_{p_j})$, for $j = 1, 2$. Then

$$g_{p_1}(f_1^{(n)}) = nf_1(1)^{n-1}f_1(p_1) = nw^{n-1}u,$$

and

$$g_{p_2}(f_2^{(n)}) = nf_2(1)^{n-1}f_2(p_2) = nw^{n-1}u$$

for all $n \in \mathbb{N}$, by (3.2). Therefore, the free random variables $f_j \in (\mathcal{A}, g_{p_j})$ are identically distributed from each other, for $j = 1, 2$.

($\Rightarrow$) Suppose f_j are free random variables in the arithmetic p_j-prime probability spaces $(\mathcal{A}, g_{p_j})$, for $j = 1, 2$. And assume that they are identically distributed, i.e.,

$$\begin{aligned} g_{p_1}(f_1^{(n)}) &= nf_1(1)^{n-1}f_1(p_1) \\ &= nf_2(1)^{n-1}f_2(p_2) = g_{p_2}(f_2^{(n)}) \end{aligned}$$

for all $n \in \mathbb{N}$.

Assume now that either $f_1(1) \neq f_2(1)$, or $f_1(p) \neq f_2(p)$ in $\mathbb{C}$.

First, assume $f_1(p) \neq f_2(p)$ in $\mathbb{C}$. Then the first free moments satisfy that

$$g_{p_1}(f_1) = f_1(p_1) \neq f_2(p) = g_{p_2}(f_2),$$

and hence, f_1 and f_2 are not identically distributed. This contradicts our assumption that f_1 and f_2 are identically distributed.

Suppose that $f_1(p_1) = d = f_2(p_2)$ in $\mathbb{C}$, and assume $f_1(1) \neq f_2(1)$ in $\mathbb{C}$. Then

$$g_{p_1}(f_1^{(2)}) = 2f_1(1)f_1(p_1) = 2f_1(1)d$$

$$\neq 2f_2(1)d = 2f_2(1)f_2(p_2) = g_{p_2}(f_2^{(2)}),$$

by (3.3). Therefore, f_1 and f_2 are not identically distributed. It contradicts our assumption.

Finally, suppose that $f_1(1) \neq f_2(1)$ and $f_1(p_1) \neq f_2(p_2)$ in $\mathbb{C}$. Then

$$g_{p_1}(f_1) = f_1(p_1) \neq f_2(p_2) = g_{p_2}(f_2),$$

and hence, f_1 and f_2 are not identically distributed. It also contradicts our assumption.

Therefore, if $f_1(1) = f_2(1)$ and $f_1(p_1) = f_2(p)$, then $f_1 \in (\mathcal{A}, g_{p_1})$ and $f_2 \in (\mathcal{A}, g_{p_2})$ are identically distributed. □

The above theorem characterizes the identically free-distributedness of arithmetic functions in terms of certain quantities.

Corollary 3.3. *Let $(\mathcal{A}, g_p)$ be the arithmetic p-prime probability space and let $f_1, f_2 \in (\mathcal{A}, g_p)$. Then two free random variables f_1 and f_2 are identically distributed if and only if*

$$f_1(1) = f_2(1) \text{ and } f_1(p) = f_2(p) \text{ in } \mathbb{C}. \tag{3.5}$$

Also, the above characterization (3.5) provides an equivalence relation on a fixed arithmetic p-prime probability space $(\mathcal{A}, g_p)$.

Let $(\mathcal{A}, g_p)$ be a fixed arithmetic p-prime probability space for a prime p. Define a relation $\mathcal{R}_p$ on $(\mathcal{A}, g_p)$ by

$$f_1 \mathcal{R}_p f_2 \overset{\text{def}}{\Longleftrightarrow} f_1 \text{ and } f_2 \text{ are identically distributed} \tag{3.6}$$

in $(\mathcal{A}, g_p)$. Then the relation $\mathcal{R}_p$ is an equivalence relation. Indeed,

(i) $f \mathcal{R}_p f$ for all $f \in (\mathcal{A}, g_p)$,

(ii) $f_1 \mathcal{R}_p f_2 \Longrightarrow f_2 \mathcal{R}_p f_1$ for all $f_1, f_2 \in (\mathcal{A}, g_p)$, and

(iii) $f_1 \mathcal{R}_p f_2$ and $f_2 \mathcal{R}_p f_3 \Longrightarrow f_1 \mathcal{R}_p f_3$

for all $f_1, f_2, f_3 \in (\mathcal{A}, g_p)$, by (3.4) and (3.5).

So, one can get *equivalence classes* $[f]_p$ of $f \in \mathcal{A}$ in $(\mathcal{A}, g_p)$, by

$$\begin{aligned} [f]_p &= \{h \in (\mathcal{A},\ g_p) : h\ \mathcal{R}_p\ f\} \\ &= \{h \in (\mathcal{A},\ g_p) : g_p(h^n) = g_p(f^n), \forall n \in \mathbb{N}\}. \end{aligned} \tag{3.7}$$

By (3.4) and (3.5), one obtains the following lemma.

Lemma 3.4. *Let $(\mathcal{A}, g_p)$ be the arithmetic p-prime probability space for a prime p. Then*

$$[f]_p = \{h \in (\mathcal{A}, g_p) : h(1) = f(1) \text{ and } h(p) = f(p)\} \tag{3.8}$$

for all $f \in (\mathcal{A}, g_p)$.

Proof. The proof of (3.8) is trivial by (3.5) and (3.7). □

The above relation (3.8) provides a classification of $\mathcal{A}$, by identically distributedness on the arithmetic algebra $\mathcal{A}$, in terms of primes, i.e., without considering all free-moments $g_p\left(f^{(n)}\right)$ of $f \in \mathcal{A}$, by investigating two quantities $f(1)$ and $f(p)$, one can characterize the identically distributedness in $\mathcal{A}$, in terms of primes p.

Theorem 3.5. *Let $(\mathcal{A}, g_p)$ be the arithmetic p-prime probability space for a fixed prime p. Then*

$$\mathcal{A} = \underset{(\alpha,\ \lambda)\in\mathbb{C}^2}{\sqcup}[\alpha, \lambda]_p, \tag{3.9}$$

where

$$[\alpha, \lambda]_p \overset{def}{=} \{f \in \mathcal{A} : f(1) = \alpha,\ f(p) = \lambda\}$$

for all $(\alpha, \lambda) \in \mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$. Equivalently,

$$\mathcal{A}/\mathcal{R}_p = \mathbb{C}^2 \quad \textit{for all primes } p, \tag{3.10}$$

set-theoretically.

Proof. By (3.8), each $\mathcal{R}_p$-equivalence class has its form,

$$[f]_p = \{h \in \mathcal{A} : h(1) = f(1) \text{ and } h(p) = f(p)\}$$

for all $f \in (\mathcal{A}, g_p)$. So, one can realize that the equivalence classes $[f]_p$ are completely determined by two $\mathbb{C}$-quantities $f(1)$ and $f(p)$, i.e., every $\mathcal{R}_p$-equivalence class $[f]_p$ is determined by $\alpha = f(1)$ and $\lambda = f(p)$. So, we can re-write $[f]_p$ by

$$[f(1), f(p)]_p = [\alpha, \lambda]_p, \tag{3.11}$$

as in (3.9). Since

$$\mathcal{A} = \underset{f\in\mathcal{A}}{\cup}[f]_p = \underset{(f(1),f(p))\in\mathbb{C}^2}{\cup}[f(1), f(p)]_p$$

by (3.11)

$$\mathcal{A} = \underset{(\alpha,\lambda)\in\mathbb{C}^2}{\sqcup}[\alpha, \lambda]_p$$

by the uniqueness of $\mathbb{C}$-pairs in $\mathbb{C}^2$.

Therefore, the quotient set $\mathcal{A}/\mathcal{R}_p$ is equipotent (or bijective) to $\mathbb{C}^2$, set-theoretically. □

Observe now that the quotient set $\mathfrak{A}_p = \mathcal{A}/\ \mathcal{R}_p$ becomes an algebra, i.e., one can define

$$[f_1]_p + [f_2]_p \overset{\text{def}}{=} [f_1 + f_2]_p, \tag{3.12}$$

and

$$([f_1]_p)\,([f_2]_p) \overset{\text{def}}{=} [f_1 * f_2]_p, \tag{3.13}$$

on $\mathfrak{A}_p$, for all $f_1, f_2 \in (\mathcal{A}, g_p)$.

Indeed, one can get that, for any $t_1, t_2 \in \mathbb{C}$,

$$t_1[f_1]_p + t_2[f_2]_p = [t_1 f_1 + t_2 f_2]_p,$$

and hence, $\mathfrak{A}_p$ is a vector space over $\mathbb{C}$, furthermore,

$$(([f_1]_p)([f_2]_p))([f_3]_p) = ([f_1]_p)(([f_2]_p)([f_3]_p)),$$

in $\mathfrak{A}_p$, for all $f_1, f_2, f_3 \in (\mathcal{A}, g_p)$, because

$$(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3) \text{ on } \mathcal{A}.$$

Furthermore, the operations (3.12) and (3.13) satisfy the distributedness,

$$\begin{aligned} ([f_1]_p)([f_2]_p + [f_3]_p) &= ([f_1]_p)([f_2 + f_3]_p) \\ &= ([f_1 * (f_2 + f_3)]_p) \\ &= ([f_1 * f_2 + f_1 * f_3]_p) \\ &= [f_1 * f_2]_p + [f_1 * f_3]_p \\ &= ([f_1]_p)([f_2]_p) + ([f_1]_p)([f_3]_p), \end{aligned}$$

similarly,

$$([f_1]_p + [f_2]_p)([f_3]_p) = ([f_1]_p)([f_3]_p) + ([f_2]_p)([f_3]_p)$$

for all $f_1, f_2, f_3 \in \mathcal{A}$. Therefore, one can understand the quotient set $\mathfrak{A}_p = \mathcal{A}/\mathcal{R}_p$ as an algebra.

Lemma 3.6. *Let $\mathfrak{A}_p$ be the quotient set $\mathcal{A}/\mathcal{R}_p$, where $\mathcal{R}_p$ is the equivalence relation* (3.6) *on the arithmetic algebra $\mathcal{A}$. Then $\mathfrak{A}_p$ is an algebra over $\mathbb{C}$ with* (3.12) *and* (3.13).

Now, let $f_j \in (\mathcal{A}, g_p)$, with

$$f_j(1) = \alpha_j, \text{ and } f_j(p) = \lambda_j$$

for $j = 1, 2$. Then

$$(f_1 + f_2)(1) = \alpha_1 + \alpha_2,$$

and

$$(f_1 + f_2)(p) = \lambda_1 + \lambda_2.$$

Under the same hypothesis,

$$f_1 * f_2(1) = (f_1(1))(f_2(1)) = \alpha_1\alpha_2$$

and

$$\begin{aligned} (f_1 * f_2)(p) &= f_1(1)f_2(p) + f_1(p)f_2(1) \\ &= \alpha_1\lambda_2 + \alpha_2\lambda_1, \end{aligned}$$

by (3.3). So, if f_1 and f_2 are given as above, then, by using the equivalent notations in (3.9), one has

$$\begin{aligned} [f_1]_p + [f_2]_p &= [f_1 + f_2]_p \\ &= [\alpha_1 + \alpha_2,\ \lambda_1 + \lambda_2]_p, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} ([f_1]_p)([f_2]_p) &= [f_1 * f_2]_p \\ &= [\alpha_1\alpha_2,\ \alpha_1\lambda_2 + \alpha_2\lambda_1]_p. \end{aligned} \tag{3.15}$$

Define now binary operations $(+)$ and $(\bullet)$ on

$$\mathbb{C}^2 = \{(\alpha, \lambda) : \alpha, \lambda \in \mathbb{C}\}$$

by

$$(\alpha_1, \lambda_1) + (\alpha_2, \lambda_2) \overset{\text{def}}{=} (\alpha_1 + \alpha_2, \lambda_1 + \lambda_2), \tag{3.16}$$

respectively,

$$(\alpha_1, \lambda_1) \bullet (\alpha_2, \lambda_2) \overset{\text{def}}{=} (\alpha_1 \alpha_2, \alpha_1 \lambda_2 + \alpha_2 \lambda_1) \tag{3.17}$$

for all $\alpha_j, \lambda_j \in \mathbb{C}$, for $j = 1, 2$.

It is not difficult to check that the set $\mathbb{C}^2$, equipped with $(+)$ and $(\bullet)$, is an algebra over $\mathbb{C}$, because for any $(\alpha, \lambda) \in \mathbb{C}^2$, there always exists at least one $f \in \mathcal{A}$, such that

$$(\alpha, \lambda) = [f]_p.$$

Notation. We denote this algebra $\mathbb{C}^2$ with (3.16) and (3.17) by $\mathfrak{C}^2$.

Then, we obtain the following classification theorem.

Theorem 3.7. *Let $\mathfrak{A}_p$ be the quotient algebra $\mathcal{A}/\mathcal{R}_p$ with* (3.12) *and* (3.13), *and let $\mathfrak{C}^2$ be the algebra $\mathbb{C}^2$ equipped with* (3.16) *and* (3.17). *Then*

$$\mathfrak{A}_p \overset{algebra}{=} \mathfrak{C}^2, \tag{3.18}$$

where "$\overset{algebra}{=}$" means "being algebra-isomorphic," for all primes p.

Proof. Set-theoretically, two algebras $\mathfrak{A}_p$ and $\mathfrak{C}^2$ are equipotent, by (3.10). Define a bijective morphism

$$\varphi : \mathfrak{C}^2 \to \mathfrak{A}_p$$

by

$$\varphi\left((\alpha,\ \lambda)\right) = [f]_p,$$

where f is a free random variable of $(\mathcal{A}, g_p)$, such that

$$f(1) = \alpha, \text{ and } g_p(f) = f(p) = \lambda.$$

Again, by (3.10), φ is bijective under the quotient determined by the equivalence relation $\mathcal{R}_p$ of (3.6).

This morphism φ satisfies

$$\begin{aligned} &\varphi\left(t_1(\alpha_1,\ \lambda_1)\ +\ t_2(\alpha_2,\ \lambda_2)\right) \\ &= \varphi\left((t_1\alpha_1,\ t_1\lambda_1) + (t_2\alpha_2,\ t_2\lambda_2)\right) \\ &= \varphi\left((t_1\alpha_1 + t_2\alpha_2,\ t_1\lambda_1 + t_2\lambda_2)\right) \\ &= [t_1 f_1 + t_2 f_2]_p \text{ (with } f_j(1) = \alpha_j, \text{ and } f_j(p) = \lambda_j \text{ for } j = 1, 2,) \\ &= t_1[f_1]_p + t_2[f_2]_p \text{ (by (3.14))} \\ &= t_1\varphi\left((\alpha_1,\ \lambda_1)\right) + t_2\varphi\left((\alpha_2,\ \lambda_2)\right), \end{aligned}$$

in $\mathfrak{A}_p$, for all $t_j \in \mathbb{C}$, and $(\alpha_j, \lambda_j) \in \mathfrak{C}^2$, for $j = 1, 2$. So, the morphism φ is a bijective linear transformation (i.e., a vector-space isomorphism).

Also, one can get

$$\begin{aligned}\varphi\left((\alpha_1,\ \lambda_1)\bullet(\alpha_2,\ \lambda_2)\right) &= \varphi\left((\alpha_1\alpha_2,\ \alpha_1\lambda_2+\alpha_2\lambda_1)\right)\\ &= [f_1 * f_2]_p\end{aligned}$$

by (3.15), and by the bijectivity of φ, where

$$f_j(1)=\alpha_j,\ \text{and}\ f_j(p)=\lambda_j \ \text{ for } j=1,2,$$

and hence,

$$\varphi\left((\alpha_1,\ \lambda_1)\bullet(\alpha_2,\ \lambda_2)\right) = \left([f_1]_p\right)\left([f_2]_p\right) = \left(\varphi\left((\alpha_1,\ \lambda_1)\right)\right)\left(\varphi\left((\alpha_2,\ \lambda_2)\right)\right),$$

in $\mathfrak{A}_p$, for all $(\alpha_j,\lambda_j)\in\mathfrak{C}^2$, for $j=1,2$. Thus the bijective linear transformation φ is an algebra-homomorphism.

Therefore, φ is an algebra-isomorphism, and hence, two algebras $\mathfrak{A}_p$ and $\mathfrak{C}^2$ are algebra-isomorphic, for all primes p. □

The above classification theorem is interesting not only because it provides an isomorphism theorem of $\mathfrak{A}_p=\mathcal{A}/\mathcal{R}_p$, but also it is satisfied for all primes.

Therefore, we obtain a following isomorphism theorem, under the identically-distributedness.

Corollary 3.8. *Let* $\mathfrak{A}_p=\mathcal{A}/\mathcal{R}_p$, *and* $\mathfrak{A}_q=\mathcal{A}/\mathcal{R}_q$, *for primes* p *and* q, *where* $\mathcal{R}_p$ *and* $\mathcal{R}_q$ *are in the sense of* (3.6). *Then*

$$\mathfrak{A}_p \overset{Algebra}{=} \mathfrak{A}_q. \tag{3.19}$$

Proof. By (3.18), we have

$$\mathfrak{A}_p \overset{\text{Algebra}}{=} \mathfrak{C}^2 \overset{\text{Algebra}}{=} \mathfrak{A}_q.$$

□

4. Free-moment L-functions induced by primes

In this section, we construct certain L-functions induced by arithmetic functions and primes. As in Section 3, let $(\mathcal{A}, g_p)$ be the arithmetic p-prime probability space for a fixed prime p.

In [8], the author and Gillespie introduced *free-moment L-functions* induced by a fixed linear functional on an algebra. If (A,φ) is an arbitrary free probability space, consisting of an algebra A (or a normed algebra, or a $*$-algebra, or a C^*-algebra, or a von Neumann algebra, etc), and a linear functional φ (resp., a continuous linear functional, resp., a linear functional with the $(*)$-property: $\varphi(a^*)=\overline{\varphi(a)}$, for all $a\in A$, resp., a C^*-topology continuous linear functional with $(*)$-property, resp., a W^*-topology continuous linear functional with $(*)$-property, etc), then one can define the *free-moment L-function in* (A,φ),

$$L : A\times\mathbb{C}\to\mathbb{C}$$

by

$$L(a, s) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{\varphi(a^n)}{n^s} \tag{4.1}$$

for all $a \in A$, and $s \in \mathbb{C}$. In particular, in [8], we considered a special case where A is the von Neumann algebra $L^{\infty}(\mathbb{A}_{\mathbb{Q}})$, and

$$\varphi(f) = \int_{\mathbb{A}_{\mathbb{Q}}} f d\rho \quad \text{for all } f \in A,$$

where $\mathbb{A}_{\mathbb{Q}}$ is the *Adele ring* equipped with its unbounded measure ρ on the σ-algebra of $\mathbb{A}_{\mathbb{Q}}$.

We now are interested in such free-moment L-functions in our setting. Let's fix the arithmetic p-prime probability space $(\mathcal{A},\ g_p)$ for a prime p.

Proposition 4.1. *Let $(\mathcal{A}, g_p)$ be the arithmetic p-prime probability space. Then for any $f \in \mathcal{A}$, the free-moment L-function $L_p(f, s)$ satisfies*

$$L_p(f, s) = f(p) \left(\sum_{n=1}^{\infty} \frac{f(1)^{n-1}}{n^{s-1}} \right). \tag{4.2}$$

Proof. By the definition (4.1) of free-moment L-functions, if f is a free random variable in $(\mathcal{A}, g_p)$, then

$$L_p(f, s) = \sum_{n=1}^{\infty} \frac{g_p(f^{(n)})}{n^s} = \sum_{n=1}^{\infty} \frac{n f(1)^{n-1} f(p)}{n^s}$$

by (3.5)

$$L_p(f, s) = \sum_{n=1}^{\infty} \frac{f(1)^{n-1} f(p)}{n^{s-1}} = f(p) \left(\sum_{n=1}^{\infty} \frac{f(1)^{n-1}}{n^s} \right). \qquad \square$$

By (4.2), we obtain the following corollary.

Corollary 4.2. *Let $(\mathcal{A}, g_p)$ be the arithmetic p-prime probability space for a prime p, and let $f \in (\mathcal{A}, g_p)$. If f is unital in the sense that: $f(1) = 1$, then*

$$L_p(f, s) = f(p) \left(\zeta(s-1) \right), \tag{4.3}$$

where $\zeta(s) \overset{def}{=} \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann zeta-function.

Proof. Assume that a free random variable f is unital in the sense that: $f(1) = 1$. Then, by (4.2),

$$L_p(f, s) = f(p) \left(\sum_{n=1}^{\infty} \frac{f(1)^{n-1}}{n^{s-1}} \right) = f(p) \left(\sum_{n=1}^{\infty} \frac{1}{n^{s-1}} \right) = f(p)\zeta(s-1),$$

where ζ is the Riemann zeta-function, $\zeta(t) = \sum_{n=1}^{\infty} \frac{1}{n^t}$ for $t \in \mathbb{C}$. $\square$

Motivated by (4.3), one can get the following theorem.

Theorem 4.3. *Let $(\mathcal{A}, g_p)$ be the arithmetic p-prime probability space for a prime p, and let $f \in (\mathcal{A}, g_p)$, and $L_p(f,\ s)$, the corresponding free-moment L-function of f.*

$$\text{If } |f(1)| \leq 1, \text{ then } L_p(f,s) \text{ converges, whenever } \mathrm{Re}(s-1) > 1. \tag{4.4}$$

$$\text{If } |f(1)| > 1, \text{ then } L_p(f,s) \text{ diverges, for any } s \in \mathbb{C}. \tag{4.5}$$

Proof. First, assume that $f(1) = 0$, equivalently, $f \in K_1$ in $\mathcal{A}$. Then,

$$L_p(f,s) = f(p)\left(\sum_{n=1}^{\infty} \frac{f(1)^{n-1}}{n^{s-1}}\right) = 0,$$

and hence it is convergent in $\mathbb{C}$, so it satisfies the statement (4.4).

Assume now that $f(1) \neq 0$, equivalently, $f \in K_1^c$ in $\mathcal{A}$. Observe now that

$$\frac{f(1)^{n-1}}{n^{s-1}} = \frac{e^{\log f(1)^{n-1}}}{e^{\log n^{s-1}}} = \frac{e^{(n-1)\log f(1)}}{e^{(s-1)\log n}} = e^{(n-1)\log f(1)-(s-1)\log n} \tag{4.6}$$

for all $n \in \mathbb{N}$.

Suppose that $0 < |f(1)| < 1$. Then the modulus of the power term of (4.6) satisfies

$$\mathrm{Re}\left((n-1)\log f(1) - (s-1)\log n\right) < 0,$$

whenever $\mathrm{Re}(s-1) > 1$. Therefore, the series

$$\sum_{n=1}^{\infty} e^{(n-1)\log f(1)-(s-1)\log n} = \sum_{n=1}^{\infty} \frac{f(1)^{n-1}}{n^{s}}$$

converges whenever $\mathrm{Re}(s-1) > 1$.

Assume now that $|f(1)| = 1$. Take f as a unital element of $\mathcal{A}$, i.e., $f(1) = 1$. Then, by (4.3),

$$L_p(f,s) = f(p)\zeta(s-1),$$

and hence, it converges whenever $\mathrm{Re}(s-1) > 1$. More generally, if $|f(1)| = 1$, then

$$|L_p(f,s)| = |f(p)|\,|\zeta(s-1)|\,,$$

and hence, it converges, whenever $\mathrm{Re}(s-1) > 1$.

Therefore, if $|f(1)| \leq 1$, then the corresponding free-moment $L_p(f,s)$ is convergent, whenever $\mathrm{Re}(s-1) > 1$.

If $|f(1)| > 1$, then

$$\mathrm{Re}\left((n-1)\log f(1) - (s-1)\log n\right) > 0,$$

and, if $n \to \infty$,

$$\mathrm{Re}\left((n-1)\log f(1) - (s-1)\log n\right) \to \infty$$

for any fixed $s \in \mathbb{C}$. Therefore,

$$\left|\lim_{n\to\infty} e^{(n-1)\log f(1)-(s-1)\log n}\right| = e^{\infty} = \infty,$$

and hence, the series $\sum_{n=1}^{\infty} \frac{f(1)^{n-1}}{n^{s-1}}$ diverges in $\mathbb{C}$, whenever $|f(1)| > 1$. □

The above theorem characterizes the convergence of our free-moment L-functions, i.e., the free-moment L-functions $L_p(f, s)$ converges, if $|f(1)| \leq 1$, and $\text{Re}(s-1) > 1$.

Define now arithmetic functions a_f in $\mathcal{A}$ induced by other arithmetic functions f of $\mathcal{A}$ by

$$a_f(n) \overset{\text{def}}{=} f(1)^{n-1} \quad \text{for all } n \in \mathbb{N}.$$

By definition, one can define

$$a_{a_{a_{a_{\ddots_{a_f}}}}}, \text{ etc, for } f \in \mathcal{A},$$

inductively.

By definition, our free-moment L-function $L_p(f, s)$ on the arithmetic p-prime probability space $(\mathcal{A}, g_p)$ is determined by

$$\begin{aligned} L_p(f, s) &\overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{g_p(f^n)}{n^s} = \sum_{n=1}^{\infty} \frac{n f(1)^{n-1} f(p)}{n^s} \\ &= f(p) \left(\sum_{n=1}^{\infty} \frac{f(1)^{n-1}}{n^{s-1}} \right) \\ &= f(p) \left(\sum_{n=1}^{\infty} \frac{a_f(n)}{n^{s-1}} \right) \end{aligned}$$

for all $n \in \mathbb{N}$, $f \in \mathcal{A}$. So, we obtain the following theorem.

Theorem 4.4 (Also, see [6]). *Let $(\mathcal{A}, g_p)$ be the arithmetic p-prime probability space. Then, for a free-moment L-function $L_p(f, s)$, there exists an arithmetic function $a_f \in \mathcal{A}$, such that* (i) $a_f(n) = f(1)^{n-1}$ *for all $n \in \mathbb{N}$, and* (ii)

$$L_p(f, s) = f(p) \left(L(a_f,\ s-1) \right) \tag{4.7}$$

for all $f \in \mathcal{A}, s \in \mathbb{C}$, where $L(a_f, s-1)$ means the classical Dirichlet L-function induced by $a_f \in \mathcal{A}$, in the sense of Section 1.

The above relation (4.7) provides a characterization of free-moment L-functions in $(\mathcal{A}, g_p)$ in terms of classical L-functions: free-moment L-functions $L_p(f, s)$ of $f \in (\mathcal{A}, g_p)$ are understood as the products of the quantities $f(p) = g_p(f)$, and classical Dirichlet L-functions induced by $a_f \in \mathcal{A}$.

Let f, h be in $(\mathcal{A}, g_p)$, and let $L_p(f, s)$ and $L_p(h, s)$ be the corresponding free-moment L-functions. Then, by (4.7),

$$L_p(f, s) = f(p) \left(L(a_f,\ s-1) \right),$$

and

$$L_p(h, s) = h(p) \left(L(a_h,\ s-1) \right).$$

Thus, one can get the following lemma.

Lemma 4.5. (4.8) *Let* $L_p(f,s)$ *and* $L_p(h,s)$ *be the free-moment* L*-functions of* f, $h \in (\mathcal{A}, g_p)$. *Then*

$$(L_p(f,\ s))\,(L_p(h,\ s)) = (f(p)h(p))\,(L(a_f * a_h,\ s-1)).$$

(4.9) *Let* $L_p(f,s)$ *and* $L_q(h,s)$ *be the free-moment* L*-functions of* $f \in (\mathcal{A}, g_p)$ *and* $h \in (\mathcal{A}, g_q)$, *where* p *and* q *are primes. Then*

$$(L_p(f,\ s))\,(L_q(h,\ s)) = (f(p)h(q))\,(L(a_f * a_h,\ s-1)).$$

Proof. First, the formula (4.8) is trivial by (4.7). Note that, if h_1 and h_2 are arithmetic functions, then the corresponding classical L-functions satisfy that

$$(L(h_1,s))\,(L(h_2,\ s)) = L(h_1 * h_2, s).$$

Therefore, even though p and q are primes, the corresponding free-moment L-functions are understood as products of certain quantities and classical Dirichlet L-functions. So,

$$\begin{aligned}(L_p(f,\ s))\,(L_q(h,\ s)) &= \left(f(p)\sum_{n=1}^{\infty}\frac{a_f(n)}{n^{s-1}}\right)\left(h(p)\left(\sum_{n=1}^{\infty}\frac{a_h(n)}{n^{s-1}}\right)\right)\\ &= (f(p)h(p))\left(\sum_{n=1}^{\infty}\frac{(a_f * a_h)(n)}{n^{s-1}}\right)\\ &= (f(p)h(p))\,L\left(a_f * a_h,\ s-1\right).\end{aligned}$$

So, the formula (4.9) holds. As a special case of (4.9) (where $p = q$), the formula (4.8) holds, too. □

By (4.8) and (4.9), we obtain the following theorem.

Theorem 4.6. *Let* $p_1, \ldots, p_N$ *be primes which are not necessarily distinct, for* $N \in \mathbb{N}$. *Let* f_j *be the free random variables of the arithmetic* p_j*-prime probability space* $(\mathcal{A}, g_{p_j})$, *for* $j = 1, \ldots, N$. *Then*

$$\prod_{j=1}^{N}\left(L_{p_j}(f_j,\ s)\right) = \left(\prod_{j=1}^{N} f_j(p_j)\right)\left(L\left(\mathop{*}_{j=1}^{N} a_{f_j},\ s-1\right)\right). \tag{4.10}$$

By (4.4), (4.5), and (4.10), one obtains the following corollary.

Corollary 4.7. *Let* $p_1, \ldots, p_N$ *be primes, for* $N \in \mathbb{N}$. *Let* f_j *be the free random variables of the arithmetic* p_j*-prime probability space* $(\mathcal{A}, g_{p_j})$, *and let* $L_{p_j}(f_j, s)$ *be the corresponding free-moment* L*-functions, for* $j = 1, \ldots, N$.

(4.11) *If* $|f_j(1)| \leq 1$ *for all* $j = 1, \ldots, N$, *then* $\prod_{j=1}^{N}\left(L_{p_j}(f_j,\ s)\right)$ *is convergent whenever* $\mathrm{Re}(s-1) > 1$.

(4.12) *If* $L_{p_j}(f_j, s)$ *are convergent whenever* $\mathrm{Re}(s-1) > 1$ *for all* $j = 1, \ldots, N$, *then* $\prod_{j=1}^{N}\left(L_{p_j}(f_j,\ s)\right)$ *is convergent whenever* $\mathrm{Re}(s-1) > 1$.

Recall now the identically-distributedness on $\mathcal{A}$ (in terms of primes). Two free random variables f_1 and f_2 are identically distributed in $(\mathcal{A}, g_p)$, if and only if

$$f_1(1) = f_2(1), \text{ and } f_1(p) = f_2(p), \text{ in } \mathbb{C},$$

moreover, the quotient algebra $\mathfrak{A}_p = \mathcal{A}/\mathcal{R}_p$ is algebra-isomorphic to the algebra $\mathfrak{C}^2$.

So, under the identically distributedness on the arithmetic algebra $\mathcal{A}$, one can get the following theorem.

Theorem 4.8. *Let* $(\alpha, \lambda) \in \mathfrak{C}^2$*. Then there exist free-moment L-functions of* $f \in (\mathcal{A}, g_p)$ *"for all primes p," such that*

$$L_p(f,\ s) = \lambda \left(\sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{n^{s-1}} \right). \tag{4.13}$$

Proof. Since the algebra $\mathfrak{C}^2$ is algebra-isomorphic to $\mathfrak{A}_p = \mathcal{A}/\mathcal{R}_p$, for every prime p, by (3.18) and (3.19). So, there exists at least one arithmetic function f in $(\mathcal{A}, g_p)$, for each prime p, such that

$$f(1) = \alpha, \text{ and } g_p(f) = f(p) = \lambda,$$

such that $[f]_p \in \mathfrak{A}_p$, for any prime p. Then, by the identically-distributedness $\mathcal{R}_p$ on $(\mathcal{A}, g_p)$, for any $h \in [f]_p$,

$$L_p(h,\ s) = h(p) \left(\sum_{n=1}^{\infty} \frac{h(1)^{n-1}}{n^{s-1}} \right)$$

by (4.7)

$$L_p(h,\ s) = \lambda \left(\sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{n^{s-1}} \right).$$

□

Notice that the above theorem holds for all primes.

Define a set

$$\mathcal{L}_p \overset{\text{def}}{=} \{L_p(f, s) : f \in (\mathcal{A}, g_p)\} \tag{4.14}$$

for all primes p. Then this set $\mathcal{L}_p$ becomes an algebra under the usual functional addition $(+)$ and the usual functional multiplication $(\cdot)$.

Define a morphism ψ_p from the arithmetic algebra $\mathcal{A}$ into $\mathcal{L}_p$ by

$$\psi_p(f) \overset{\text{def}}{=} L_p(f, s), \tag{4.15}$$

then it is a well-defined algebra-epimorphism, satisfying

$$\psi_p(f_1 * f_2) = (\psi_p(f_1))\,(\psi_p(f_2))$$

for all $f_1, f_2 \in (\mathcal{A}, g_p)$, for each prime p. So, one can verify the following isomorphism theorem.

Theorem 4.9. *Let $\mathfrak{A}_p = \mathcal{A}/\mathcal{R}_p$, and assume $\Psi_p = \psi_p/\mathcal{R}_p$ is the quotient map from $\mathfrak{A}_p$ to $\mathcal{L}_p$, where $\mathcal{L}_p$ is an algebra* (4.14)*, and ψ_p is in the sense of* (4.15)*. Then Ψ_p is an algebra-isomorphism. Equivalently,*

$$\mathfrak{A}_p \overset{Algebra}{=} \mathcal{L}_p. \tag{4.16}$$

Proof. Let $\mathfrak{A}_p = \mathcal{A}/\mathcal{R}_p$, and let $\mathcal{L}_p$ be in the sense of (4.14). Since $\mathcal{A}$ is epimorphic to $\mathcal{L}_p$, the quotient map Ψ_p is an algebra-homomorphism from $\mathfrak{A}_p$ to $\mathcal{L}_p$. Since each $L_p(f, s)$ satisfies

$$L_p(f, s) = f(p) \sum_{n=1}^{\infty} \frac{f(1)^{n-1}}{n^2},$$

if $(f(1), f(p)) = (\alpha, \lambda)$ in $\mathbb{C}^2$, then

$$L_p(h, s) = \lambda \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{n^{s-1}}$$

for all $h \in [f]_p$. Thus, under quotient, f is uniquely determined up to $\mathcal{R}_p$, i.e., Ψ_p is injective.

So, $\Psi_p : \mathfrak{A}_p \to \mathcal{L}_p$ is an algebra-isomorphism, equivalently, two algebras $\mathfrak{A}_p$ and $\mathcal{L}_p$ are isomorphic. □

By (4.16), we obtain a following isomorphism theorem, too.

Corollary 4.10. *Let $\mathcal{L}_p$ be an algebra in the sense of* (4.14)*, for primes p. Then*

$$\mathcal{L}_p \overset{Algebra}{=} \mathfrak{C}^2 \quad \textit{for all primes } p, \tag{4.17}$$

where $\mathfrak{C}^2$ is an algebra in the sense of (3.18)*.*

Proof. The proof is done by (3.19). □

So, the algebra $\mathcal{L}_p$ of all g_p-free-moment L-functions induced by $\mathcal{A}$ is classified by $\mathfrak{C}^2$.

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Department of Mathematics
St. Ambrose University
421 Ambrose Hall, 518 W. Locust St.
Davenport, Iowa, 52803, USA
E-mail address: choilwoo@sau.edu