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Integral Transforms in Electromagnetic Formulation

(Invited Paper)

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Abstract

In this research, integral transform technique for electromagnetic scattering formulation is reviewed. Electromagnetic boundary-value problems are presented to demonstrate how the integral transforms are utilized in electromagnetic propagation, antennas, and electromagnetic interference/compatibility. Various canonical structures of slotted conductors are used for illustration; moreover, Fourier transform, Hankel transform, Mellin transform, Kontorovich-Lebedev transform, and Weber transform are presented. Starting from each integral transform definition, the general procedures for solving Helmholtz's equation or Laplace's equation for the potentials in the unbounded region are reviewed. The boundary conditions of field continuity are incorporated into particular formulations. Salient features of each integral transform technique are discussed.

Key Words: Boundary-Value Problems, Electromagnetic Scattering, Integral Transforms.

I. Introduction

Various integral transforms have been extensively used in the formulation of electromagnetic scattering, radiation, antennas, and electromagnetic interference-related problems. The integral transform technique [1, 2] is an indispensable tool for representing the fields in the unbounded (open) region. This technique is often combined with the mode-matching method to solve electromagnetic boundary-value problems. The purpose of the present paper is to review the integral transforms that are applied in conjunction with the mode-matching method.

In the paper, we will limit our discussion to the canonical slotted conductors, which will enable us to use the technique of variable separation. The use of Fourier transform, Hankel transform, Mellin transform, Kontorovich-Lebedev transform, and Weber transform is discussed. We will show how these integral transforms can be incorporated into the pertinent electromagnetic boundary-value problems, which are formulated in terms of Helmholtz's equation or Laplace's equation. Depending on the problem geometries, a certain type of integral transforms must be chosen to facilitate the solutions to Helmholtz's equation or Laplace's equation for the potentials. From the integral transform definitions, we will show how the potentials in the unbounded region can be obtained. In the next section, we begin with Fourier transform. The time convention $\exp(-i\omega t)$ is used throughout the analysis.

II. Fourier Transform

The Fourier transform pair is as follows:

$$\tilde{f}(\varsigma) = \int_{-\infty}^{\infty} f(z)\, e^{-i\varsigma z} dz\,, \tag{1}$$

$$f(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\varsigma)\, e^{i\varsigma z} d\varsigma\,. \tag{2}$$

The Fourier transform technique has long been used for electromagnetic scattering, diffraction, and antenna applications. We will discuss the formulation of Fourier transform by considering radiation from slotted circular waveguides. Fig. 1 shows a circular waveguide with a narrow circumferential slot array. The circular waveguide is infinitely long in the *z*-direction. An incident field is assumed to propagate within the circular wa-

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veguide ($\rho < a$) from below. In region I ($\rho < a$), the scattered electric and magnetic vector potentials in the z-direction are needed for the field description.

The *z*-component of the scattered electric vector potential satisfies Helmholtz's equation, $(\nabla^2 + k^2)F_z(\rho,\phi,z) = 0$, where k is the wave number in region I. Helmholtz's equation in the cylindrical coordinates (ρ,ϕ,z) is as follows:

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} + k^2\right]F_z(\rho,\phi,z) = 0. \tag{3}$$

We will represent the scattered electric vector potential $F_z(\rho,\phi,z)$ based on the Fourier transform and Fourier series. Since the problem geometry is open $(-\infty < z < \infty)$ in the *z*-direction and periodic in the ϕ-direction with 2π-periodicity, we let:

$$F_z(\rho,\phi,z) = \sum_{n=-\infty}^{\infty} e^{in\phi}\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{F}_1(\rho,\varsigma)e^{i\varsigma z}d\varsigma. \tag{4}$$

We substitute Eq. (4) into Eq. (3) to obtain Bessel's differential equation:

$$\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) - \frac{n^2}{\rho^2} - \varsigma^2 + k^2\right]\tilde{F}_1(\rho,\varsigma) = 0. \tag{5}$$

Since the field is finite at the origin, the Bessel function of the first kind, $J_n(\kappa\rho)$, is chosen as $\tilde{F}_1(\rho,\varsigma) = \tilde{F}_2(\varsigma)J_n(\kappa\rho)$, where $\kappa = \sqrt{k^2 - \varsigma^2}$. Hence:

$$F_z^I(\rho,\phi,z) = \sum_{n=-\infty}^{\infty} e^{in\phi}\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{F}_2(\varsigma)J_n(\kappa\rho)e^{i\varsigma z}d\varsigma. \tag{6}$$

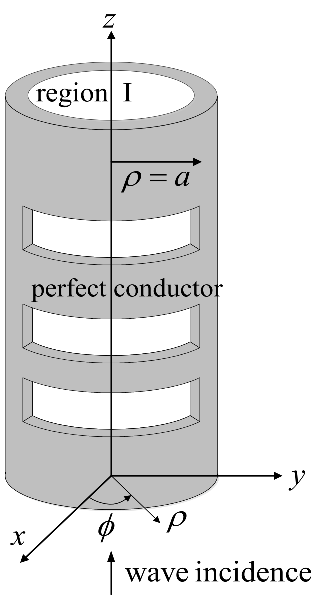

Fig. 1. A perfectly conducting circular waveguide of inner radius ($\rho = a$) with a narrow circumferential slot array.

A complete radiation analysis using the boundary conditions is available in [3].

III. Hankel Transform

The Hankel transform pair is as follows:

$$\tilde{f}(\varsigma) = \int_0^{\infty} f(\rho)J_n(\rho\varsigma)\rho d\rho, \tag{7}$$

$$f(\rho) = \int_0^{\infty}\tilde{f}(\varsigma)J_n(\varsigma\rho)\varsigma d\varsigma. \tag{8}$$

The Hankel transform is useful for the analysis of scattering from circular apertures. Consider electromagnetic scattering from a circular aperture in an infinitely extended, perfectly conducting plane, as shown in Fig. 2.

Assume that a uniform plane wave is incident on a circular aperture from below. The transmitted field in region I, which is above the slotted conducting plane, can be written in terms of the *z*-component of the transmitted electric and magnetic vector potentials. The *z*-component of the transmitted electric vector potential in region I, $F_z(\rho,\phi,z)$, satisfies Eq. (3). Since region I is open ($\rho > 0$) in the ρ-direction and periodic in the ϕ-direction with 2π-periodicity, it is expedient to use the Hankel transform and Fourier series representations simultaneously as:

$$F_z(\rho,\phi,z) = \sum_{n=-\infty}^{\infty} e^{in\phi}\int_0^{\infty}\tilde{F}_1(z,\varsigma)J_n(\varsigma\rho)\varsigma d\varsigma. \tag{9}$$

Substituting Eq. (9) into Eq. (3) yields:

$$\left(\frac{d^2}{dz^2} + \kappa^2\right)\tilde{F}_1(z,\zeta) = 0, \tag{10}$$

where $\kappa = \sqrt{k^2 - \varsigma^2}$. Since the transmitted field must vanish when z goes to infinity, we choose $\tilde{F}_1(\rho,\varsigma) = \tilde{F}_2(\varsigma)e^{i\kappa z}$. Hence:

$$F_z(\rho,\phi,z) = \sum_{n=-\infty}^{\infty} e^{in\phi}\int_0^{\infty}\tilde{F}_2(\varsigma)e^{i\kappa z}J_n(\varsigma\rho)\varsigma d\varsigma. \tag{11}$$

The field representations and some computations are available in [4].

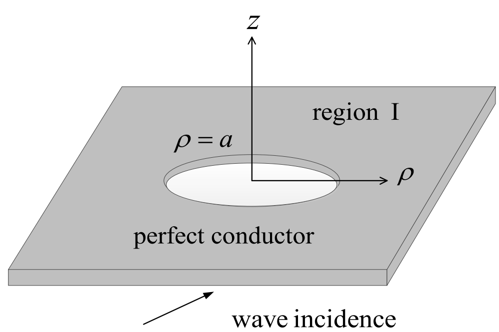

Fig. 2. A circular aperture of radius *a* in an infinitely extended, thick perfectly conducting plane.

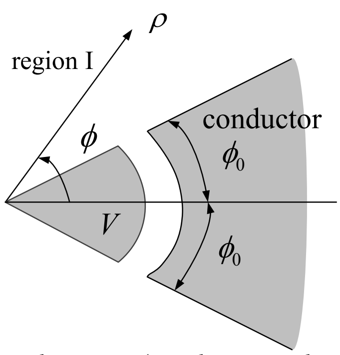

Fig. 3. A slotted two-dimensional conducting wedge, where the wedge is symmetric with respect to $\phi = 0$.

IV. Mellin Transform

The Mellin transform pair is:

$$\tilde{f}(\varsigma) = \int_0^\infty f(\rho)\rho^{\varsigma-1}d\rho, \quad (12)$$

$$f(\rho) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(\varsigma)\rho^{-\varsigma}d\varsigma. \quad (13)$$

Consider the two-dimensional (ρ,ϕ) electrostatic boundary-value problem of a slotted conducting wedge, as shown in Fig. 3. The electrostatic potential V is applied across the conducting wedge. The electrostatic potential $\Phi(\rho,\phi)$ in region I ($\phi_0 < \phi < 2\pi - \phi_0$) is given by the two-dimensional Laplace equation as:

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right]\Phi(\rho,\phi) = 0. \quad (14)$$

We will determine $\Phi(\rho,\phi)$ in region I, assuming:

$$\Phi(\rho,\phi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{\Phi}_1(\phi,\varsigma)\rho^{-\varsigma}\, d\varsigma. \quad (15)$$

Substituting Eq. (15) into Eq. (14) gives:

$$\left(\frac{d^2}{d\phi^2} + \varsigma^2\right)\tilde{\Phi}_1(\phi,\varsigma) = 0. \quad (16)$$

Since the electric field at $\phi = \pi$ exists only in the radial direction ρ, it is possible to represent the potential as $\tilde{\Phi}_1(\varsigma,\phi) = \tilde{\Phi}_2(\varsigma)\cos\varsigma(\phi-\pi)$. Thus:

$$\Phi(\rho,\phi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{\Phi}_2(\varsigma)\cos\varsigma(\phi-\pi)\rho^{-\varsigma}d\varsigma. \quad (17)$$

A complete solution to the potential problem is provided in [5].

V. Kontorovich-Lebedev Transform

The Kontorovich-Lebedev transform pair is as follows:

$$\tilde{f}(\varsigma) = \int_0^\infty f(\rho)H_\varsigma^{(1)}(k\rho)\rho^{-1}d\rho, \quad (18)$$

$$f(\rho) = \frac{1}{2}\int_{-i\infty}^{i\infty} \tilde{f}(\varsigma)\varsigma J_\varsigma(k\rho)d\varsigma, \quad (19)$$

where $H_\varsigma^{(1)}(k\rho)$ is the Hankel function of the first kind. The incident wave impinges on the structure, as shown in Fig. 4. Assume that the scattering problem is two-dimensional (ρ,ϕ) and the z-component of the scattered magnetic vector potential $A_z(\rho,\phi)$ solely describes the scattered field. Helmholtz's equation for $A_z(\rho,\phi)$ is:

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + k^2\right]A_z(\rho,\phi) = 0. \quad (20)$$

We express $A_z(\rho,\phi)$ in region I ($0 < \phi < \gamma$) as:

$$A_z(\rho,\phi) = \frac{1}{2}\int_{-i\infty}^{i\infty} \tilde{A}_1(\phi,\varsigma)\varsigma J_\varsigma(k\rho)d\varsigma. \quad (21)$$

Substituting Eq. (21) into Eq. (20) yields:

$$\left(\frac{d^2}{d\phi^2} + \varsigma^2\right)\tilde{A}_1(\phi,\varsigma) = 0. \quad (22)$$

Hence, the solution is:

$$\tilde{A}_1(\phi,\varsigma) = \tilde{A}_2(\varsigma)e^{i\varsigma\phi} + \tilde{A}_3(\varsigma)e^{-i\varsigma\phi}. \quad (23)$$

Finally, we obtain:

$$A_z(\rho,\phi) = \frac{1}{2}\int_{-i\infty}^{i\infty}\left[\tilde{A}_2(\varsigma)e^{i\varsigma\phi} + \tilde{A}_3(\varsigma)e^{-i\varsigma\phi}\right]\varsigma J_\varsigma(k\rho)d\varsigma. \quad (24)$$

A complete wedge-scattering analysis using the boundary conditions can be found in [6].

VI. Weber Transform

The Weber transform pair is [7]:

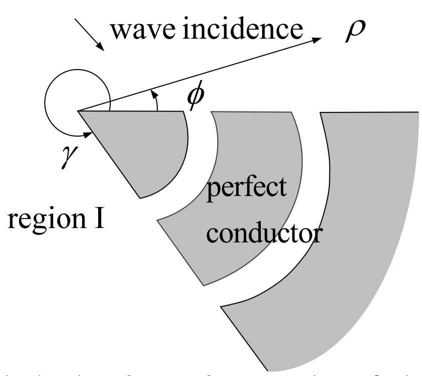

Fig. 4. A multiply slotted, two-dimensional, perfectly conducting wedge.

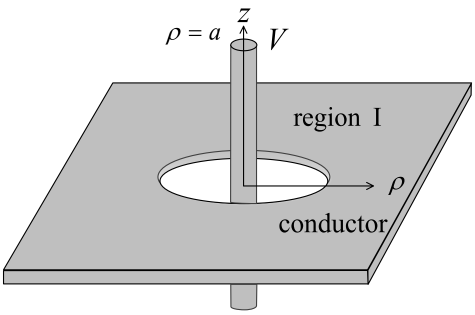

Fig. 5. An infinitely long conducting cylinder of radius *a* piercing a thick conducting plane of infinite extent.

$$\tilde{f}(\varsigma) = \int_a^\infty f(\rho) Z_\nu(\varsigma\rho)\rho \, d\rho, \tag{25}$$

$$f(\rho) = \int_0^\infty \tilde{f}(\varsigma) \frac{Z_\nu(\varsigma\rho)}{J_\nu^2(\varsigma a) + N_\nu^2(\varsigma a)} \varsigma \, d\varsigma, \tag{26}$$

where $Z_\nu(\varsigma\rho) = J_\nu(\varsigma\rho) N_\nu(\varsigma a) - N_\nu(\varsigma\rho) J_\nu(\varsigma a)$. Note that $J_\nu(\varsigma\rho)$ and $N_\nu(\varsigma\rho)$ are the Bessel functions of the first and second kinds of order ν, respectively. Consider the electrostatic boundary-value problem in Fig. 5. The electrostatic potential V is applied between the infinitely long conducting cylinder of radius *a* and a perforated thick conducting plane of infinite extent. Based on the superposition, we decompose the original problem of Fig. 5 into equivalent ones [8].

We wish to determine the electrostatic potential for the equivalent problem. The electrostatic potential $\Phi(\rho, z)$ in region I, which is above the slotted plane and exterior to the cylinder $(z > 0,\ a < \rho < \infty)$, is given by:

$$\left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{\partial^2}{\partial z^2} \right] \Phi(\rho, z) = 0. \tag{27}$$

If the boundary condition for the equivalent problem requires $\Phi(a, z) = 0$, it is possible to let:

$$\Phi(\rho, z) = \int_0^\infty \tilde{\Phi}_1(z, \varsigma) \frac{Z_0(\varsigma\rho)}{J_0^2(\varsigma a) + N_0^2(\varsigma a)} \varsigma \, d\varsigma. \tag{28}$$

Substituting Eq. (28) into Eq. (27) gives:

$$\left(\frac{d^2}{dz^2} - \varsigma^2 \right) \tilde{\Phi}_1(z, \varsigma) = 0. \tag{29}$$

If the boundary condition is such that the potential is zero when $z \to \infty$, we choose $\tilde{\Phi}_1(z, \varsigma) = \tilde{\Phi}_2(\varsigma) e^{-\varsigma z}$. Hence, the electrostatic potential is [7, 8]:

$$\Phi(\rho, z) = \int_0^\infty \tilde{\Phi}_2(\varsigma) e^{-\varsigma z} \frac{Z_0(\varsigma\rho)}{J_0^2(\varsigma a) + N_0^2(\varsigma a)} \varsigma \, d\varsigma. \tag{30}$$

A complete potential analysis is given in [8].

VII. Conclusion

In this paper, the integral transform technique in electromagnetic boundary-value problems was shown. Fourier transform, Hankel transform, Mellin transform, Kontorovich-Lebedev transform, and Weber transform were introduced. Starting from Helmholtz's equation or Laplace's equation, pertinent potential expressions for the open regions were derived. The integral transform technique can be adequately applied to electromagnetic scattering and radiation problems, in particular when the scattering geometries have canonical cylindrical shapes.

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