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EULER-MARUYAMA METHOD FOR SOME NONLINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH JUMP-DIFFUSION

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ABSTRACT. In this paper we discussed Euler-Maruyama method for stochastic differential equations with jump diffusion. We give a convergence result for Euler-Maruyama where the coefficients of the stochastic differential equation are locally Lipschitz and the pth moments of the exact and numerical solution are bounded for some $p > 2$.

1. INTRODUCTION

Stochastic differential equations (SDEs) arise in the modeling of many phenomena in physics, biology, climatology, economics, etc. when uncertainties or random influences (called noises) are taken into account. These random effects are not only introduced to compensate for the defects in some deterministic models, but also are often rather intrinsic phenomena. In finance, the Black-Scholes-Merton stochastic equations are used to model the option price. In climatology, the stochastic Lorenz system is used to study the flow of the atmosphere. The stochastic lattice differential equations are used to model systems such as cellular neural networks with applications to image processing, pattern recognition, and brain science.

Most SDEs arising in practice are nonlinear, and cannot be solved explicitly, so numerical methods for SDEs have recently a great deal of attention (see [2,4,5,8,11,12,13,14]). Euler-Maruyama is a technical for approximate numerical solution of a stochastic differential equation. It is a simple generalization of the Euler method for ordinary differential equations to stochastic differential equations (see [1,7,9,10]).

In this paper we study the numerical solution of the stochastic differential equations with jump-diffusion of the form

$$du(x,t) = f(x,t,u(x,t^-))dt + g(x,t,u(x,t^-))dW(t) + h(x,t,u(x,t^-))dN(t) + \sum_{|q|\le 2m} A_q(x,t)D^q u(x,t^-)dt,\ 0 \le t \le T,\ u(x,0^-) = u_0(x). \tag{1.1}$$

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Here $u(x,t^-)$ denotes $lim_{s\to t^-}u(x,s)$, $x\in R^\nu$ (R^ν is the ν-dimensional Euclidean space), $u\in R^n$, $W(t)$ is an n-dimensional Brownian motion, $N(t)$ is a scalar Poisson with intensity λ, $f:R^{n+\nu+1}\to R^n$, $g:R^{n+\nu+1}\to R^{n\times n}$, $h:R^{n+\nu+1}\to R^n$ and $A_q:R^\nu\times[0,T]\to R^{n\times n}$, where $(A_q,|q|\le 2m)$ is a family of square matrices whose elements are sufficiently smooth functions on $R^\nu\times[0,T]$ and $D^q=D_1^{q_1}.....D_\nu^{q_\nu}, D_i=\frac{\partial}{\partial x_i}$, $q=(q_1,...q_\nu)$ is ν-dimensional multi-index, (see [6]). Following Petrovsky it is assumed that

$$det\,[(-1)^m\sum_{|q|=2m}A_q(x,t)\sigma^q-\lambda I]=0$$

has roots satisfy the inequality $Re\ \lambda(x,t,\sigma)\le -\eta|\sigma|^m$ for all $x\in R^\nu,\ t\ge 0$, where η is a positive constant, $\sigma\in R^\nu$, $|\eta|=(\eta_1^2+\ ...\ +\eta_\nu^2)^{\frac{1}{2}},\sigma^q=\sigma_1^{q_1}\ ...\ \sigma_\nu^{q_\nu}$, and I is the unit matrix. Throughout this paper, $|.|$ denotes both the Euclidean vector norm and the forbenius matrix norm.

We suppose that the following conditions are satisfied:

(1) The coefficients of the operator $\sum_{|q|\le 2m}A_q(x,t)D^q$ are continuous in $t\in[0,T]$, moreover, the continuity in t of the coefficients $\sum_{|q|\le 2m}A_q(x,t)D^q$ is uniform with respect to $x\in R^\nu$.

(2) The coefficients of $\sum_{|q|\le 2m}A_q(x,t)D^q$ are bounded on $R^\nu\times[0,T]$ and satisfy the Holder condition with respect to x.

Under these conditions for the system

$$\frac{\partial v}{\partial t}=\sum_{|q|\le 2m}A_q(x,t)D^qv, \tag{1.2}$$

there exists a fundamental solution matrix $K(x,y,t,\theta)$ which satisfies the following conditions:

(i) $\frac{\partial K}{\partial t}, D^qK\in C(G_1), |\ q\ |\le 2m,$
where $G_1=R^{2\nu}\times(0,T)\times(0,T)$.

(ii) $|\ D^qK(x,y,t,\theta)\ |\le\frac{b_1}{(t-\theta)^\beta}e^z;\ t>\theta, z=\frac{-b_2|x-y|^{2m}}{t-\theta}, \beta=1/2(\nu+|\ q\ |), |\ q\ |\le 2m,$
where $|\ x\ |$ is the norm $(x_1^2+x_2^2+....+x_\nu^2)^{\frac{1}{2}}, |\ K\ |$ a suitable norm of the square matrix K, b_1 and b_2 are positive constant.

(iii) The function v defined by

$$v(x,t)=\int_{R^\nu}K(x,y,t,0)v_0(y)dy,$$

represents the unique solution of the parabolic system

$$\frac{\partial v}{\partial t}=\sum_{|q|\le 2m}A_q(x,t)D^qv, \tag{1.3}$$

with the initial conditions

$$v(x,0) = v_0(x),\ (\frac{\partial v}{\partial t}, D^q v \in C(G_2), |\ q\ | \leq 2m),\ G_2 = R^\nu \times (0,T). \tag{1.4}$$

The existence of such functions depends on the parabolicity of the system (1.3) and on the smoothness of the coefficients of such systems (see [3,6]).

We shall use the notations

$$\sup_x |v(x,t)| = \parallel v(.,t) \parallel,\ \sup_x |e(x,t)| = \parallel e(.,t) \parallel$$

where $e(x,t) = v(x,t) - u(x,t)$. We assume that f, g and h satisfy Lipschitz condition and linear growth condition, that is, for $a = f = g = h$, given any $R > 0$ there exists a constant $\gamma_R > 0$ such that

$$\parallel a(.,t,u) - a(.,t,v) \parallel^2 \leq \gamma_R \parallel u - v \parallel^2, \tag{1.5}$$

$$\parallel a(.,t,u) \parallel^2 \leq \gamma_R (1 + \parallel u \parallel^2). \tag{1.6}$$

We also assume finite moment bounds for the initial data; that is, for any $p > 0$ there is a finite M_p such that

$$E \parallel u(.,0^-) \parallel^p < M_p. \tag{1.7}$$

For a given, constant, stepsize $\Delta t > 0$, we define the split-step backward Euler (SSBE) method for (1.1) by $v(x,0) = u(x,0^-)$ and

$$v_k^*(x) = v_k(x) + f(x,t_k,v_k^*)\Delta t, \tag{1.8}$$

$$v_{k+1}^* = v_k^*(x) + g(x,t_k,v_k^*)\Delta W_k + h(x,t_k,v_k^*)\Delta N_k + \sum_{|q|\leq 2m} A_q(x,t_k) D^q v_k^*(x)\Delta t. \tag{1.9}$$

Here, v_k is the approximation to $u(x,t_k)$ for $t_k = k\Delta t$, with $\Delta W_k = W(t_{k+1}) - W(t_k)$ and $\Delta N_k = N(t_{k+1}) - N(t_k)$ representing the increments of the Brownian motion and the Poisson process, respectively. A key component in our analysis is the compensated Poisson process

$$\overline{N}(t) = N(t) - \lambda t,$$

which is a martingale. Defining

$$f_\lambda(x,t,u) = f(x,t,u) + \lambda h(x,t,u), \tag{1.10}$$

we may rewrite the jump-diffusion Ito SDE (1.1) in the form

$$du(x,t) = f_\lambda(x,t,u(x,t^-))dt + g(x,t,u(x,t^-))dW(t) + h(x,t,u(x,t^-))d\overline{N}(t) + \sum_{|q|\leq 2m} A_q(x,t) D^q u(x,t^-)dt, \tag{1.11}$$

We note that $f_\lambda(x,t,u(x,t^-))$ also satisfies Lipschitz condition and linear growth condition with larger constant.

The compensated Poisson process motivates an alternative to the SSBE method in (1.8)–(1.9). We define the compensated split-step backward Euler (CSSBE) method for (1.1) by $v_0 = u(x,0^-)$ and

$$v_k^*(x) = v_k(x) + f_\lambda(x,t_k,v_k^*)\Delta t, \tag{1.12}$$

$$v_{k+1}^* = v_k^*(x) + g(x,t_k,v_k^*)\Delta W_k + h(x,t_k,v_k^*)\Delta\overline{N}_k + \sum_{|q|\le 2m} A_q(x,t_k)D^q v_k^*(x)\Delta t, \quad (1.13)$$

where $\Delta\overline{N}_k = \overline{N}(t_{k+1}) - \overline{N}(t_k)$.

2. The Euler Method

In this section we prove the strong convergence of EM. Considering the SDE in compensated form, (1.11), motivates the explicit method

$$v_{k+1}(x) = v_k(x) + f_\lambda(x,t_k,v_k)\Delta t + g(x,t_k,v_k)\Delta W_k + h(x,t_k,v_k)\Delta\overline{N}_k + \sum_{|q|\le 2m} A_q(x,t_k)D^q v_k(x)\Delta t, \quad (2.1)$$

which we refer to as the compensated Euler-Maruyamma (CEM) method. We denote the piecewise constant interpolant of the CEM solution by $v(x,t) = v_k(x)$ for $t \in [t_k, t_{k+1})$. We then define the piecewise linear interpolant by

$$\overline{v}(x,t) = \int_{R^\nu} K(x,y,t,0)v_0(y)dy + \int_0^T \int_{R^\nu} K(x,y,t,s)f_\lambda(y,s,v(y,s^-))dyds + \int_0^t \int_{R^\nu} K(x,y,t,s)g(y,s,v(y,s^-))dydW(s) + \int_0^t \int_{R^\nu} K(x,y,t,s)h(y,s,v(y,s^-))dyd\overline{N}(s). \quad (2.2)$$

Theorem 2.1. Suppose that $f,\ g$ and h satisfy the conditions (1.5) and (1.6), and that for some $p > 2$ there is a constant A such that

$$E \sup_{0\le t\le T} \| u(.,t) \|^p \le A,\ E \sup_{0\le t\le T} \| \overline{v}(.,t) \|^p \le A. \quad (2.3)$$

Then

$$\lim_{\Delta t\to 0} E \sup_{0\le \le T} \| \overline{v}(.,t) - u(.,t) \|^2 = 0.$$

Proof : Set

$$\tau_R = inf\{t \ge 0 : \| \overline{v}(.,t) \| \ge R\},\ \rho_R = inf\{t \ge 0 : \| u(.,t) \| \ge R\},$$

and

$$\theta_R = \tau_R \wedge \rho_R, \quad (\theta_R = \tau_R \wedge \rho_R \text{ the minimum of } \tau_R \text{ and } \rho_R).$$

Recall the following elementary inequality:

$$a^\alpha b^{1-\alpha} \le \alpha a + (1-\alpha)b,\ \forall\ a,\ b > 0, \alpha \in [0,1].$$

We thus have for any $\delta > 0$,

$$\begin{aligned} E[\sup_{0\le t\le T} \| e(.,t) \|^2] &= E[\sup_{0\le t\le T} \| e(.,t) \|^2 \, 1\{\tau_R > T,\ \rho_R > T\}] \\ &\quad + E[\sup_{0\le t\le T} \| e(.,t) \|^2 \, 1\{\tau_R \le T \ or \ \rho_R \le T\}] \\ &\le E[\sup_{0\le t\le T} \| e(.,t\wedge\theta_R) \|^2 \, 1_{\{\theta_R > T\}}] \qquad (2.4) \\ &\quad + \frac{2\delta}{p} E[\sup_{0\le t\le T} \| e(.,t) \|^p] + \frac{1-\frac{2}{p}}{\delta^{2/(p-2)}} P(\tau_R \le T \ or \ \rho_R \le T). \end{aligned}$$

Now, by using (2.3), we get,

$$P(\tau_R \le T) = E[1_{\{\tau_R \le T\}} \frac{\| \overline{v}(.,\tau_R) \|^p}{R^p}] \le \frac{1}{R^p} E[\sup_{0\le t\le T} \| \overline{v}(.,t) \|^p] \le \frac{A}{R^p}.$$

A similar result can be derived for ρ_R, so that

$$P(\tau_R \le T \ or \ \rho_R \le T) \le \frac{2A}{R^p}.$$

Using these bounds along with

$$E[\sup_{0\le t\le T} \| e(.,t) \|^p] \le 2^{p-1} E[\sup_{0\le t< T} (\| \overline{v}(.,t)) \|^p + \| u(.,t) \|^p] \le 2^p A$$

and by substituting in (2.4), we get

$$\begin{aligned} E[\sup_{0\le t\le T} \| e(.,t) \|^2] &\le E[\sup_{0\le t\le T} [\| (\overline{v}(.,t\wedge\theta_R)) - u(.,t\wedge\theta_R)] \|^2] \\ &\quad + \frac{2^{p+1}\delta A}{p} + \frac{2(p-2)A}{p\,\delta^{2/(p-2)} R^p}. \qquad (2.5) \end{aligned}$$

In order to estimating the first term on the right of (2.5) we rewrite $u(x, t\wedge\theta_R)$ as

$$\begin{aligned} u(x,t\wedge\theta_R) &= \int_{R^\nu} K(x,y,t\wedge\theta_R,0) u_0(y) dy \\ &\quad + \int_0^{t\wedge\theta_R} \int_{R^\nu} K(x,y,t\wedge\theta_R,s) f_\lambda(y,s,u(y,s^-)) dy ds \\ &\quad + \int_0^{t\wedge\theta_R} \int_{R_\nu} K(x,y,t\wedge\theta_R,s) g(y,s,u(y,s^-)) dy dW(s) \\ &\quad + \int_0^{t\wedge\theta_R} \int_{R^\nu} K(x,y,t\wedge\theta_R,s) h(y,s,u(y,s^-)) dy d\overline{N}(s). \qquad (2.6) \end{aligned}$$

From (2.2), (2.6) and by using Cauchy Schwartz inequality, we get

$$\| \overline{v}(.,t\wedge\theta_R) - u(.,t\wedge\theta_R) \|^2 \le 4\{T \int_0^{t\wedge\theta_R} \| f_\lambda(.,s,v(.,s) - f_\lambda(.,s,u(.,s)) \|^2 \, ds$$

$$+ \parallel \int_0^{t\wedge\theta_R} [g(.,s,v(.,s^-)) - g(.,s,u(.,s^-))]dW(s) \parallel^2$$

$$+ \parallel \int_0^{t\wedge\theta_R} [h(.,s,v(.,s)) - h(.,s,u(.,s))]d\overline{N}(s) \parallel^2 \}.$$

So, from (1.5) and Doob's martingale inequality , we have for any $\tau \leq T$

$$\begin{aligned}
& E[\sup_{0\leq t\leq T} \parallel \overline{v}(.,t\wedge\theta_R) - u(.,t\wedge\theta_R) \parallel^2] \\
\leq \;& 4\gamma_R(T+2)E[\int_0^{\tau\wedge\theta_R} \parallel v(.,s) - u(.,s) \parallel^2 ds] \\
\leq \;& \overline{\gamma}_R E[\int_0^{\tau\wedge\theta_R} [\parallel v(.,s) - \overline{v}(.,s) \parallel^2 + \parallel \overline{v}(.,s) - u(.,s) \parallel^2] ds] \\
\leq \;& \overline{\gamma}_R E[\int_0^{\tau\wedge\theta_R} \parallel v(.,s) - \overline{v}(.,s) \parallel^2 ds + \int_0^{\tau} \parallel \overline{v}(.,s\wedge\theta_R) - u(.,s\wedge\theta_R) \parallel^2 ds] \\
\leq \;& \overline{\gamma}_R (E[\int_0^{\tau\wedge\theta_R} \parallel v(.,s) - \overline{v}(.,s) \parallel^2 ds] \qquad (2.6) \\
& + \int_0^{\tau} E \sup_{0\leq r\leq s} \parallel \overline{v}(.,r\wedge\theta_R) - v(.,r\wedge\theta_R \parallel^2 ds).
\end{aligned}$$

Let k_c be the integer for which $c \in [t_{k_c}, t_{k_{c+1}})$, so

$$\begin{aligned}
v(x,c) - \overline{v}(x,c) &= -\int_{t_{k_c}}^{c} \int_{R^\nu} K(x,y,c,s) f_\lambda(y,s,v(y,s)) dyds \\
&\quad - \int_{t_{k_c}}^{c} \int_{R^\nu} K(x,y,c,s) g(y,s,v(y,s)) dydW(s) \\
&\quad - \int_{t_{k_c}}^{c} \int_{R^\nu} K(x,y,c,s) h(y,s,v(y,s)) dyd\overline{N}(s) \\
&= -\int_{R^\nu} K(x,y,c,0) f(y,c,v_{k_c}(y))(c - (t_{k_c}))dy \qquad (2.7) \\
&\quad - \int_{R^\nu} K(x,y,c,0) g(y,c,v_{k_c}(y))(W(c) - W(t_{k_c}))dy \\
&\quad - \int_{R^\nu} K(x,y,c,0) h(y,c,v_{k_c}(y))(\overline{N}(c) - \overline{N}(t_{k_c}))dy.
\end{aligned}$$

Hence

$$\begin{aligned}
\parallel v(.,c) - \overline{v}(.,c) \parallel^2 \leq \;& 4[\parallel f_\lambda(.,c,v_{k_c}(.) \parallel^2 (\Delta t)^2 + \parallel g(.,c,V_{k_c}(.) \parallel^2 \parallel \Delta W_{k_c}) \parallel^2 \\
& + \parallel h(.,c,v_{k_c}(.) \parallel^2 \parallel \Delta\overline{N}_{k_c}) \parallel^2] .
\end{aligned}$$

From (1.5), (1.6) and (2.3), yields

$$E\int_0^{\tau\wedge\theta_R} \parallel v(.,c)-\overline{v}(.,c)\parallel^2 ds \leq C\Delta t,$$

where C is a positive constant. By substituting in (2.7), yields

$$\begin{aligned} E[\sup_{0\leq t\leq T}\parallel \overline{v}(.,t\wedge\theta_R)-u(.,t\wedge\theta_R)\parallel^2] &\leq C\Delta t+\overline{\gamma}_R\int_0^{\tau} E\sup_{0\leq r\leq s}[\parallel \overline{v}(.,r\wedge\theta_R) \\ &\quad -u(.,r\wedge\theta_R\parallel^2]ds. \end{aligned}$$

Upon applying the Gronwall's inequality we obtain

$$E\sup_{0\leq t\leq T}\parallel \overline{v}(.,t\wedge\theta_R)-u(.,t\wedge\theta_R)\parallel^2]\leq C_1\Delta te^{T\overline{\gamma}_R},$$

for a constant $C_1=C_1(R,T,A)$. Substituting in (2.6) we obtain

$$E[\sup_{0\leq t\leq T}\parallel e(.,t)\parallel^2]\leq C_1\Delta te^{T\overline{\gamma}_R}+\frac{2^{p+1}\delta A}{p}+\frac{(p-2)2A}{p\,\delta^{2/(p-2)}R^p}.$$

Given $\epsilon>0$, we can choose $\delta>0$ such that $(2^{p+1}\delta A)/p<\frac{\epsilon}{3}$. Then choose R so that $\frac{(p-2)2A}{p\,\delta^{2/(p-2)}R^p}<\frac{\epsilon}{3}$, and then choose Δt sufficiently small such that $C_1\Delta te^{T\overline{\gamma}_R}<\frac{\epsilon}{3}$ we get,

$$E[\sup_{0\leq t\leq T}\parallel e(.,t)\parallel^2]\leq\epsilon,$$

as required.

3. CONCLUSIONS

We gave a convergence result for Euler-Maruyama method in the case where the coefficients are locally Lipschitz and moment bounds are available.

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