

ADVANCED DOMAIN DECOMPOSITION METHOD BY LOCAL AND MIXED LAGRANGE MULTIPLIERS

JUNYOUNG KWAK¹, TAEYOUNG CHUN², HAESEONG CHO¹, SANGJOON SHIN^{1†} AND OLIVIER A. BAUCHAU²

¹DEPARTMENT OF MECHANICAL AND AEROSPACE ENGINEERING, SEOUL NATIONAL UNIVERSITY, SEOUL 151-741, SOUTH KOREA

E-mail address: k jy84@snu.ac.kr, nicejjo@snu.ac.kr, ssjoon@snu.ac.kr

²LIGNEX1, SOUTH KOREA

E-mail address: sao671@snu.ac.kr

³DEPARTMENT OF MECHANICAL ENGINEERING, UNIVERSITY OF MICHIGAN--SHANGHAI JIAO TONG UNIVERSITY JOINT INSTITUTE, CHINA

E-mail address: olivier.bauchau@sjtu.edu.cn

ABSTRACT. This paper presents development of an improved domain decomposition method for large scale structural problem that aims to provide high computational efficiency. In the previous researches, we developed the domain decomposition algorithm based on augmented Lagrangian formulation and proved numerical efficiency under both serial and parallel computing environment. In this paper, new computational analysis by the proposed domain decomposition method is performed. For this purpose, reduction in computational time achieved by the proposed algorithm is compared with that obtained by the dual-primal FETI method under serial computing condition. It is found that the proposed methods significantly accelerate the computational speed for a linear structural problem.

1. INTRODUCTION

Domain decomposition approach based on finite element analysis is an efficient algorithm which can be applied to large scale structural analysis. One of the most successful approaches for large scale structural analysis with finite element method was the finite element tearing and interconnecting (FETI) method [1-2]. The method was applied to the second- (original FETI) and the fourth- (two-level FETI) order partial differential equations [3-4]. After that, the dual-primal FETI method was proposed in addition to the original FETI [5-6]. It is a dual sub-structuring method, which introduces Lagrange multipliers to enforce the continuity constraint by following Lagrangian method. And then it solves the

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† Corresponding author.

resulting dual problem by the process of seeking a saddle-point of the relevant Lagrangian functional. Classification of the corner interface values for these FETI algorithms is shown in Fig. 1.

More recently, the authors developed a FETI-type domain decomposition method based on augmented Lagrangian formulation (ALF) [7-10]. In [9-10], proposed computational algorithm was compared with other approaches regarding convergence characteristics, computational time and memory under serial and parallel computing environment. In that research, it was shown that use of the proposed domain decomposition algorithm improved conditioning of the flexibility matrix in interface system.

In this paper, the proposed domain decomposition algorithm will be introduced first. And its optimal computational efficiency range will be studied within a serial computing environment.

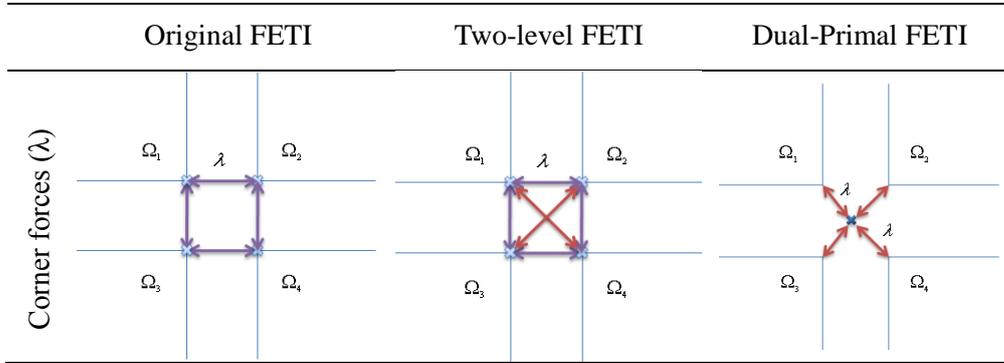


FIGURE 1. INTERFACE SYSTEM USED BY A FEW DIFFERENT FETI APPROACHES.

2. GOVERNING EQUATIONS

In the FETI algorithm, the solution procedure for the interface problem influences the efficiency of the entire computational process. An efficient iterative solution of the interface problem requires a good pre-conditioner. The original FETI method used Dirichlet pre-conditioner. A preconditioned conjugate projected gradient (PCPG) algorithm was used for the solution of the coupled system between the local rigid mode components and Lagrange multipliers. All of the original FETI, two-level FETI, and FETI-DP method used Dirichlet pre-conditioner. In contrast, the proposed FETI method solves the sub-domain problem using an augmented Lagrangian formulation, i.e., by adding penalty factors to the constraint problem.

In this section, we introduce basic equations of the proposed method which are used

throughout this paper. The detailed proposed domain decomposition algorithm was shown by the authors [7-10]. The total potential energy of the system can be expressed as

$$\Pi = A + \Phi + V^c, \quad (2.1)$$

Eq. (2.1) was the total potential energy obtained by combining the strain energy, A , the potential of the externally applied loads, Φ , and the potential of the constraints, V^c . The strain energy can be further evaluated by summing up the strain energies, A_i , of each sub-domain.

$$A = \sum_{i=1}^{N_s} A_i = \frac{1}{2} \sum_{i=1}^{N_s} \underline{u}_i^T K_i \underline{u}_i = \frac{1}{2} \underline{u}^T \text{diag}(K_\alpha) \underline{u}, \quad (2.2)$$

The total work done by the externally applied loads was

$$\Phi = \sum_{i=1}^{N_s} \Phi_i = - \sum_{i=1}^{N_s} \underline{u}_i^T \underline{Q}_i = - \underline{u}^T \underline{Q}, \quad (2.3)$$

From the augmented Lagrangian formulation, potential of the constraints was written as

$$V^c = \underline{\lambda}_G^T \text{diag}(p_{G\alpha}) \underline{C}_G + \frac{1}{2} \underline{C}_G^T \text{diag}(\bar{p}_{G\alpha}) \underline{C}_G + \sum_{i=1}^{N_s} [p_{Li} \underline{\lambda}_{Li}^T \underline{C}_{Li} + \frac{1}{2} \bar{p}_{Li} \underline{C}_{Li}^T \underline{C}_{Li}], \quad (2.4)$$

where $\underline{\lambda}_G$ and $\underline{\lambda}_L^T = \{\underline{\lambda}_{L1}^T, \underline{\lambda}_{L2}^T, \dots, \underline{\lambda}_{LN_s}^T\}$ were the global and local Lagrange multipliers used to enforce the global and local constraints, respectively. $\text{diag}(p_{G\alpha})$ was a block diagonal matrix of the scaling factors for the global Lagrange multipliers, p_{Gi} the scaling factor for sub-domain, p_{Li} that for the local Lagrange multipliers, and \bar{p}_{Gi} and \bar{p}_{Li} the penalty coefficients for the global and local constraints, respectively. The global Lagrange multipliers were the classical Lagrange multipliers that enforce the linear constraints at the nodes that belong to the adjacent sub-domains. The local Lagrange multipliers enforced the linear constraints at the boundary nodes that belong to the sub-domains connected by coarse mesh nodes. The coarse mesh nodes were a by-product of the local Lagrange multiplier technique that was proposed by Park *et al.* [11].

The resulting equations of motion were obtained by minimizing the total potential energy of the system as follows.

$$\begin{bmatrix} K^* & -\text{diag}(\bar{p}_{L\alpha} S_\alpha^T) C R^T \text{diag}(p_{G\alpha}) & \text{diag}(\bar{p}_{L\alpha} S_\alpha^T) \\ -C^T \text{diag}(\bar{p}_{L\alpha} S_\alpha) & C^T \text{diag}(\bar{p}_{L\alpha}) C & 0 & -C^T \text{diag}(p_{L\alpha}) \\ \text{diag}(p_{G\alpha}) R & 0 & 0 & 0 \\ \text{diag}(p_{L\alpha} S_\alpha) & -\text{diag}(p_{L\alpha}) C & 0 & 0 \end{bmatrix} \begin{Bmatrix} \underline{u} \\ \underline{c} \\ \underline{\lambda}_G \\ \underline{\lambda}_L \end{Bmatrix} = \begin{Bmatrix} \underline{Q} \\ \underline{0} \\ \underline{0} \\ \underline{0} \end{Bmatrix}, \quad (2.5)$$

where $K^* = \text{diag}(K_\alpha) + R^T \text{diag}(\bar{p}_{G\alpha}) R + \text{diag}(S_\alpha^T \bar{p}_{L\alpha} S_\alpha)$.

The solution in the original FETI and dual-primal FETI analysis used preconditioned

conjugate projected gradient (PCPG) algorithm with Dirichlet preconditioner. The proposed FETI-local analysis used localized Lagrange multiplier with ALF. To assess the proposed FETI-local method, all constraints were assumed to be local. The equations of motion can be easily obtained by eliminating $\underline{\lambda}_G$ and matrix R defining the global constraints.

$$\begin{bmatrix} \text{diag}(\bar{K}_\alpha) & -\text{diag}(\bar{S}_\alpha^T)C & \text{diag}(\bar{S}_\alpha^T) \\ -C^T \text{diag}(\bar{S}_\alpha)C^T \text{diag}(p_\alpha)C - C^T \text{diag}(p_\alpha) & & \\ \text{diag}(\bar{S}_\alpha) & -\text{diag}(p_\alpha)C & 0 \end{bmatrix} \begin{Bmatrix} \underline{u} \\ \underline{c} \\ \underline{\lambda}_L \end{Bmatrix} = \begin{Bmatrix} \underline{Q} \\ \underline{0} \\ \underline{0} \end{Bmatrix} \quad (2.6)$$

where $\bar{K}_i = K_i + p_i S_i^T S_i$ and $\bar{S}_i = p_i S_i$.

And the proposed FETI-mixed analysis used the mixed global and local Lagrange's multipliers, which is described in Eq. (2.5).

3. NUMERICAL RESULTS AND DISCUSSION

In this section, the algorithm presented in Section 2 is used and the resulting computational time and memory usage are examined. The structure under plane stress condition is divided into four sub-domains with different numbers of degrees of freedom [12]. The problem is divided into four sub-domains. Elastic modulus is 73GPa and Poisson's ratio is 0.3. Computations are conducted on a 64-bit Windows-operating single-processor machine. Tables 1 through 4 list the maximum displacement and computational time required by the previous and proposed FETI methods. For a fixed number of sub-domains, $N_s = 4$, increasing the problem size, the dual-primal FETI method and proposed FETI-local method show the fastest computational time.

TABLE 1. DISPLACEMENT RESULTS AND COMPUTATIONAL TIME BY CASE 1.

Case 1	Nodes	DOF's	Elastic modulus (MPa)	Poisson's ratio
	225	450	1,000	0.3
Analysis	Max disp. (m)	Max disp. (m)	Computational time	Subdomains
Original FETI	0.018	-0.004	3.37	4
FETI-DP	0.018	-0.004	0.92	4
Proposed FETI-local	0.018	-0.004	0.52	4

TABLE 2. DISPLACEMENT RESULTS AND COMPUTATIONAL TIME BY CASE 2.

Case 2	Nodes	DOF's	E(MPa)	Poisson's ratio
	441	882	1,000	0.3
Analysis	Max disp. (m)	Max disp. (m)	Computational time	Subdomains
Original FETI	0.024	-0.005	12.64	4
FETI-DP	0.024	-0.005	1.55	4
Proposed FETI-local	0.024	-0.005	0.77	4

TABLE 3. DISPLACEMENT RESULTS AND COMPUTATIONAL TIME BY CASE 3.

Case 3	Nodes	DOF's	E(MPa)	Poisson's ratio
	3,721	7,442	1,000	0.3
Analysis	Max disp. (m)	Max disp. (m)	Computational time	Subdomains
Original FETI	0.065	-0.012	3486.06	4
FETI-DP	0.065	-0.012	41.14	4
Proposed FETI-local	0.065	-0.012	24.24	4

TABLE 4. DISPLACEMENT RESULTS AND COMPUTATIONAL TIME BY CASE 4.

Case 4	Nodes	DOF's	E(MPa)	Poisson's ratio
	6,561	13,122	1,000	0.3
Analysis	Max disp. (m)	Max disp. (m)	Computational time	Subdomains
FETI-DP	0.085	-0.015	168.82	4
Proposed FETI-local	0.085	-0.015	88.61	4

Next, the dual-primal FETI, proposed FETI-local, and proposed FETI-mixed algorithms are compared. Computations are conducted on a 64-bit Windows-operating single-processor machine. Numerical analysis results are shown in Table 5. From these numerical analyses it is found that the computation time by the proposed FETI methods are shorter than that for the dual-primal FETI. However, the total memory usage of the proposed FETI-mixed method turns out to be much larger than those for the other FETI algorithms because the proposed FETI-mixed algorithm needs to solve the global interface system.

TABLE 5. COMPUTATIONAL TIME AND MEMORY USAGE.

DOF	Computation time(s)			Memory usage(MB)		
	FETI-DP	Proposed FETI-local	Proposed FETI-mixed	FETI-DP	Proposed FETI-local	Proposed FETI-mixed

98	0.378	0.241	0.181	600	588	604
162	0.377	0.242	0.176	599	588	604
242	0.382	0.241	0.19	598	589	604
338	0.389	0.245	0.188	600	588	607
450	0.401	0.254	0.206	599	589	610
882	0.629	0.31	0.266	597	591	637
1,250	0.704	0.405	0.354	597	587	652
1,682	0.902	0.553	0.518	596	588	689
2,178	1.321	0.809	0.78	596	595	745
2,738	1.973	1.283	1.263	598	594	825
3,362	2.580	1.985	2.076	601	602	935
4,418	5.665	3.327	3.484	607	611	1177
5,202	6.759	4.906	6.003	611	615	1394
7,442	18.425	10.491	11.662	627	634	2191
10,082	38.449	21.817	24.838	652	654	3504

Next numerical analysis focuses on the proposed FETI-local method, which has exhibited the shortest computational time and smallest memory usage. The proposed FETI-local and dual-primal FETI method are compared for large degree of freedom problems. It is found that when the number of the sub-domains is fixed, the proposed FETI-local algorithm is advantageous in terms of computation. The reduction in computation time is estimated by the following formula

$$\begin{aligned}
 & \text{Reduction in Computational time} & (3.1) \\
 & = \frac{(\text{Computational time by FETI DP}) - (\text{Computational time by the proposed FETI local})}{\text{Computational time by FETI DP}} \\
 & \times 100\%
 \end{aligned}$$

The computational time by the proposed FETI-local algorithm is reduced by 32-54%, when compared to that for dual-primal FETI. Figure 2 clearly illustrates the advantage of the proposed method in this example. This suggests that the computational time by the localized Lagrange multiplier with ALF has a good computational efficiency.

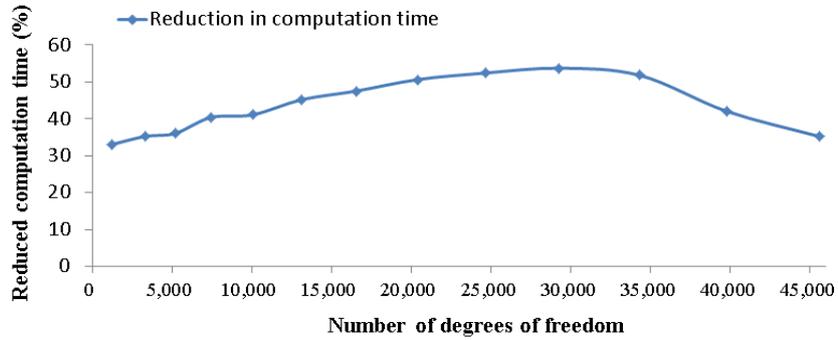
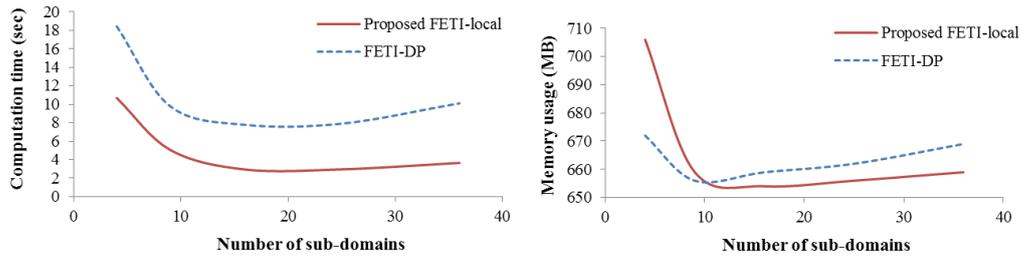
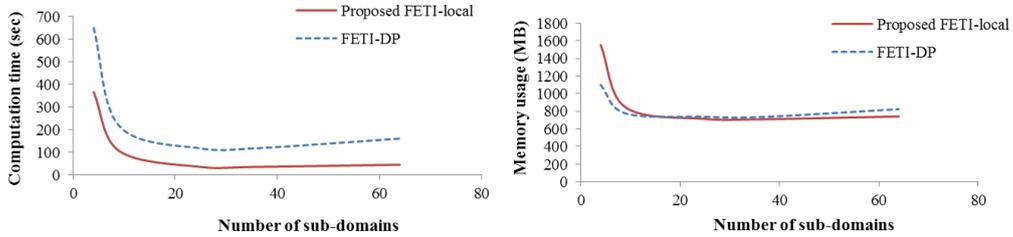


FIGURE 2. REDUCTION IN COMPUTATIONAL TIME.

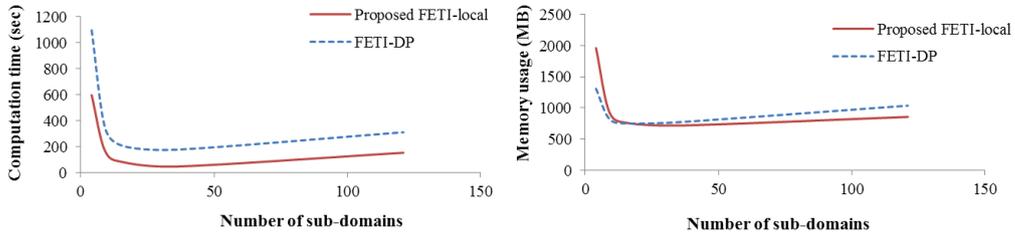
In the last numerical analysis, the number of the sub-domains is varied between 4 and 121 and then the total number of degrees of freedom will be between 7,442 and 51,842. The computational time required for the two FETI methods (The proposed FETI-local and dual-primal FETI) are summarized in Fig. 3. It is found that when the number of the sub-domains is increased up to a particular level, the computation time and memory usage will be decreased. However, when the number of the sub-domains is increased further, the computational time and memory usage will increase again. The most efficient number of the sub-domains for the proposed FETI-local is similar to that of the FETI-DP. This suggests that the proposed FETI-local method is versatile in terms of the number of elements while achieving the same level of computational efficiency.



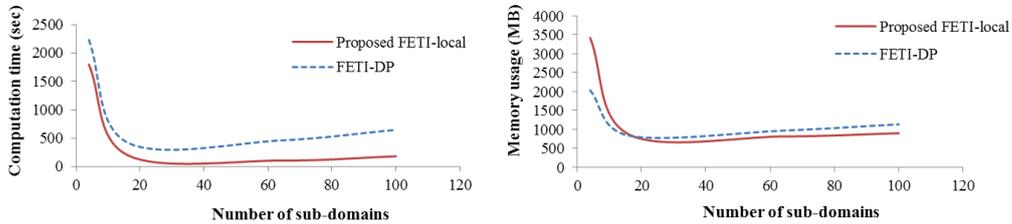
(a) Computation results (DOF: 7,442).



(b) Computation results (DOF: 29,282).



(c) Computation results (DOF: 35,378).



(d) Computation results (DOF: 51,842).

FIGURE 3. NUMERICAL RESULTS BY THE PROPOSED FETI-LOCAL METHOD FOR VARIOUS NUMBER OF THE SUB-DOMAINS.

4. CONCLUSION

A brand new finite element domain decomposition algorithm for large scale structural analysis was proposed based on local and global Lagrange multiplier with ALF in this paper. In addition to the authors' previous developments, numerical analyses using the proposed

domain decomposition method are performed. The numerical predictions obtained by the proposed algorithms are presented for a two-dimensional plane stress problem under serial computing environment. In these examples, the proposed algorithms and FETI-DP performs in a similar fashion and exhibit a better computational efficiency than the original FETI does. Even under serial computing condition, computational time for a linear structural problem is accelerated. For the fixed number of the sub-domains, the computational time by the proposed FETI method is 32-54% smaller than that for FETI-DP. Further results show that the proposed FETI method has a wide range of optimal number of sub-domains. While computational improvement is achieved by the proposed FETI method, further validation will still be required. For that purpose, the proposed procedures will be applied to the higher-order, three-dimensional structures, such as the shell problems.

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REFERENCES

- [1] C. Farhat, F.-X. Roux, *A Method of Finite Element Tearing and Interconnecting and Its Parallel Solution Algorithm*, International Journal for Numerical Methods in Engineering, **32** (1991), 1205-1227.
- [2] C. Farhat, J. Mandel, F.-X. Roux, *Optimal Convergence Properties of the FETI Domain Decomposition Method*, Computer Methods and Applied Mechanics and Engineering, **115** (1994), 367-388.
- [3] C. Farhat, P. S. Chen, J. Mandel, F.-X. Roux, *The Two-level FETI Method. Part I: an Optimal Iterative Solver for Biharmonic Systems*, Computer Methods and Applied Mechanics and Engineering, **155** (1998), 129-151.
- [4] C. Farhat, P. S. Chen, J. Mandel, F.-X. Roux, *The Two-level FETI Method. Part II: Extension to Shell Problems, Parallel Implementation and Performance Results*, Computer Methods and Applied Mechanics and Engineering, **155** (1998), 153-179.
- [5] C. Farhat, K. Pierson, M. Lesoinne, *The Second Generation FETI Methods and their Application to the Parallel Solution of Large-scale Linear and Geometrically Non-linear Structural Analysis Problems*, Computer Methods in Applied Mechanics and Engineering, **184** (2000), 333-374.
- [6] C. Farhat, M. Lesoinne, P. LeTallec, K. Pierson, D. Rixen, *FETI-DP: a Dual-primal Unified FETI method Part I: A Faster Alternative to the Two-level FETI Method*, International Journal for Numerical Methods in Engineering, **50** (2001), 1523-1544.
- [7] O.A. Bauchau, A. Epple, and C.L. Bottasso, *Scaling of constraints and augmented Lagrangian formulations in multibody dynamics simulations*, Journal of Computational and Nonlinear Dynamics, **4** (2009), 1-9.
- [8] O.A. Bauchau, *Parallel computation approaches for flexible multibody dynamics simulations*, Journal of the Franklin Institute, **347**(2010), 53-68.
- [9] J.Y. Kwak, H.S. Cho, S.J. Shin, and O.A. Bauchau, *Development of finite element domain decomposition method using local and mixed lagrange multipliers*, Journal of the Computational Structural Engineering Institute of Korea, **25** (2012), 469-476.
- [10] J.Y. Kwak, T.Y. Chun, S.J. Shin, and O.A. Bauchau, *Domain decomposition approach to flexible multibody dynamics simulation*, Computational Mechanics, **53** (2014), 147-158. doi: 10.1007/s00466-013-0898-8

- [11] K.C. Park, C.A. Felippa, and U.A. Gumaste, *A localized version of the method of Lagrange multipliers and its applications*, Computational Mechanics, **24** (2000), 476-490.
- [12] K.J. Bathe, *Finite Element Procedures*, Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1996.