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PRICING OF QUANTO OPTION UNDER THE HULL AND WHITE STOCHASTIC VOLATILITY MODEL

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Abstract. We use a power series expansion method to get an analytic approximation value for the quanto option price under the Hull and White stochastic volatility model, which turns out to be accurate enough by comparing with the simulation prices using Monte Carlo method.

1. Introduction

A quanto is a type of financial derivative whose pay-out currency differs from the natural denomination of its underlying financial variable. Its main purpose is to provide exposure to a foreign asset without taking the corresponding exchange rate risk. A quanto option is a foreign currency stock option whose payoff is converted into a domestic currency at maturity at a predetermined foreign exchange rate. The profit of quanto option is worked out a particular currency, but the pay-out of the quanto option is made by cash settlement of fixed exchange rate for another currency.

Stochastic volatility models are frequently used in pricing various kinds of European options. The most famous and popular stochastic volatility models include the Hull and White [6], the Stein and Stein [7], the Heston [5] ones. Their main purpose is to resolve a shortcoming that the Black-Scholes' constant volatility model which cannot explain long-observed features of the implied volatility surface such as volatility smiles and skews. For that reason, in valuing a quanto option it is natural to consider a stochastic volatility model.

Despite its importance, very little research has been done on pricing a quanto option using a stochastic volatility model primarily due to the sophisticated stochastic process for underlying assets and volatilities as well as the difficulty of finding analytic form of a quanto option price.

To mention some of the related previous work, Ball and Roma [4] examined alternative methods for pricing options when the underlying security volatility is stochastic.

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Alós [1] used Malliavin Calculus to construct option pricing approximation formulas under the Hull and White stochastic volatility model. Antonelli and Scarlatti [2] developed the methods of Alós to find a new approach for solving the pricing equations of European call options under stochastic volatility models by expressing the price in terms of a power series of the correlation parameter between the processes driving the dynamics of the price and of the volatility.

And then more recently, Antonelli, Ramponi and Scarlatti [3] adapted the methods of expanding and approximating the theoretical evaluation formula with respect to correlation parameters by Antonelli and Scarlatti [2] to find a new and analytic method valuating exchange options with random volatilities, which gave a strong motivation to our research.

This paper, we use the methods used in [1], [2] and [3] to find a series expansion formula to approximate the quanto option value with Hull and White stochastic volatility model. The methods used in [1] and [2] find series expansions of general option and exchange option with stochastic volatility models. We derived a series expansion pricing formula of quanto option using Hull and White stochastic volatility model with non-zero correlation. Then, we show that this is a good approximation by comparing our results with Monte Carlo simulation method.

We introduce some preliminary materials on Hull and White stochastic volatility model in Section 2. Then, in Section 3, we find an approximate value of the quanto option price by way of a PDE and a correlation expansion method. Theorem 3.3 is the main result of the paper. In Section 4, we show that simulation results using Monte Carlo method are close to our analytic approximation in both zero correlation and non-zero correlation cases.

2. Preliminaries

2.1. Hull and White stochastic volatility model

Assume that S_t is a stock price, $\sqrt{u_t}$ is a volatility of the stock price and B_t and Z_t are standard Brownian motions. In risk-neutral world, using the Hull and White stochastic volatility model, we have

$$dS_t = rS_t dt + \sqrt{u_t} S_t dB_t, \quad du_t = \mu u_t dt + \xi u_t dZ_t, \tag{1}$$

where r is a riskless interest rate and μ and ξ are constants. Put $\sigma_t = \sqrt{u_t}$ and applying Itô formula gives

$$\begin{aligned} d\sigma_t &= \frac{\partial \sigma_t}{\partial u_t} du_t + \frac{1}{2} \frac{\partial^2 \sigma_t}{\partial u_t^2} (du_t)^2 \\ &= \frac{1}{2}\left(\mu - \frac{1}{4}\xi^2\right) \sigma_t dt + \frac{1}{2}\xi \sigma_t dZ_t. \end{aligned}$$

We may rewrite the equation (1) as

$$dS_t = rS_t dt + \sigma_t S_t dB_t, \quad d\sigma_t = \tilde{\mu}\sigma_t dt + \tilde{\xi}\sigma_t dZ_t. \tag{2}$$

2.2. Quanto option with stochastic volatility

Let $[0,T]$ be a time interval, and let $(\Omega,\mathcal{F},\mathbf{P})$ be a complete probability space on which standard Brownian motions W_t, Z_t, B_t and $\tilde{B}_t$ are defined. Applying the modified Hull and White stochastic volatility model (2) to stock and foreign exchange rate dynamics. Let S_t be a stock price in foreign currency and F_t be a foreign exchange rate which is an amount of domestic currency value per one foreign currency value. For some constants μ_S, μ_F, η_1, η_2, ξ_1, ξ_2, we can obtain the following equations.

$$dS_t = \mu_S S_t dt + v_t S_t dW_t, \qquad dv_t = \eta_1 v_t dt + \xi_1 v_t dB_t, \tag{3}$$

$$dF_t = \mu_F F_t dt + \sigma_t F_t dZ_t, \qquad d\sigma_t = \eta_2 \sigma_t dt + \xi_2 \sigma_t d\tilde{B}_t. \tag{4}$$

Here, for some constant correlations ρ, ν and β, the followings hold

$$Z_t = \rho W_t + \sqrt{1-\rho^2}\bar{W}_t, \quad B_t = \nu W_t + \sqrt{1-\nu^2}\tilde{W}_t,$$
$$\tilde{B}_t = \beta\rho W_t + \beta\sqrt{1-\rho^2}\bar{W}_t + \sqrt{1-\beta^2}\hat{W}_t,$$

where W_t, $\bar{W}_t$, $\tilde{W}_t$, $\hat{W}_t$ are mutually independent standard Brownian motions. And the payoff of the quanto option at maturity time T is

$$\text{Payoff} = F_0 \max\left[S_T - K, 0\right], \tag{5}$$

where F_0 is a predetermined foreign exchange rate, S_T is a foreign stock price at maturity T and K is a strike price in foreign currency.

2.3. Qunato adjusted stock price dynamics

In risk-neutral world, the foreign exchange dynamics for domestic currency with stochastic volatility σ_t has a form as follows

$$dF_t = (r_d - r_f)F_t dt + \sigma_t F_t dZ_t,$$

where, r_d is a riskless domestic interest rate and r_f is a riskless foreign interest rate. In real world, let v_t be a volatility then by Itô lemma

$$d(S_tF_t) = S_tF_t(\mu_S + \mu_F + \rho v_t\sigma_t)dt + S_tF_t(v_t dW_t + \sigma_t dZ_t).$$

From the above equation, in risk-neutral world for domestic currency

$$d(S_tF_t) = r_d S_t F_t dt + S_tF_t(v_t dW_t + \sigma_t dZ_t). \tag{6}$$

On the other hand, in domestic currency

$$d\left(\frac{1}{F_t}\right) = \left(r_f - r_d + \sigma_t^2\right)\frac{1}{F_t}dt - \frac{\sigma_t}{F_t}dZ_t. \tag{7}$$

In risk-neutral world, using equations (6) and (7), we can find stock price dynamics in domestic currency as follows

$$\begin{aligned} dS_t &= d\left(S_tF_t \cdot \frac{1}{F_t}\right) \\ &= S_tF_t d\left(\frac{1}{F_t}\right) + \frac{1}{F_t}d(S_tF_t) + d(S_tF_t)d\left(\frac{1}{F_t}\right) \end{aligned}$$

$$= (r_f - \rho v_t \sigma_t) S_t dt + v_t S_t dW_t. \tag{8}$$

2.4. Preliminary lemmas

The well-known Duhamel's principle and the classical Feynman Kaĉ formula play important roles in the paper. Using the Feynman Kaĉ formula, we get the main PDE. After transforming inhomogeneous equations to the integral of homogeneous equations by Duhamel's principal, we can apply the Feynman Kaĉ formula again to the homogeneous equations to get the integral of expectation of random variables. Now, we introduce the Duhamel's principle and the Feynman Kaĉ formula.

Lemma 2.1 (Duhamel's principle). *Consider a inhomogeneous equation for a function*

$$c : [0, T] \times \mathbf{R}^n \to \mathbf{R}$$

with a terminal value problem

$$\begin{cases} \dfrac{\partial}{\partial t} c(t, y) + Lc(t, y) = f(t, y) \\ \qquad\qquad c(T, y) = 0, \end{cases}$$

where L is a linear differential operator. The solution of this equation is

$$c(t, y) = -\int_t^T p(t, y; s) ds,$$

where $p(t, y; s)$ is a solution of the following homogeneous equation with terminal condition

$$\begin{cases} \dfrac{\partial}{\partial t} p(t, y; s) + Lp(t, y; s) = 0 \quad \text{for} \ \ t < s, \\ \qquad\qquad p(s, y; s) = f(s, y) \quad \text{for} \ \ t = s. \end{cases}$$

Lemma 2.2 (Feynman Kaĉ formula I). *Suppose that f has a continuous derivative of order 2 and q has a continuous derivative of order 1 on $\mathbf{R}^n$. Assume that q is lower bounded. Put*

$$c(t, y) = E\left[e^{-\int_t^T q(Y_s) ds} f(Y_T) \right],$$

where Y_t is the n-dimensional Itô diffusion of the following form

$$dY_t = b(Y_t)dt + \sigma(Y_t)dB_t, \quad dB_t^i dB_t^j = \rho_{ij} dt. \tag{9}$$

Then

$$\frac{\partial c}{\partial t} + Ac = qc; \quad t > 0, \ y \in \mathbf{R}^n, \tag{10}$$

$$c(T, y) = f(y); \quad y \in \mathbf{R}^n, \tag{11}$$

where

$$Ac = \sum_i b_i \frac{\partial c}{\partial y_i} + \frac{1}{2} \sum_{i,j} \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 c}{\partial y_i \partial y_j}.$$

The following lemma is the converse version of Lemma 2.2.

Lemma 2.3 (Feynman Kaĉ formula II). *Suppose that f has a continuous derivative of order 2 and q has a continuous derivative of order 1 on $\mathbf{R}^n$. Assume that $c(t,y)$ is bounded on $[0,T]\times\mathbf{R}^n$, q is lower bounded and equations* (10) *and* (11) *hold then the solution can be written as an expectation*

$$c(t,y) = E\left[e^{-\int_t^T q(Y_s)ds} f(Y_T)\right],$$

where Y_t is the n-dimensional Itô diffusion of the form (9).

3. Quanto option value by correlation expansion

3.1. PDE for option price

Next we find a PDE for quanto option price $c(t,x,v,\sigma,\rho,\beta,\nu)$ in foreign currency with terminal condition. In general, when we value a quanto option price in domestic currency associated with the payoff equation (5), after calculating the option price in foreign currency, we multiply the predetermined foreign exchange rate F_0 to the foreign currency option price. For easy calculating, the option price $c(t,x,v,\sigma,\rho,\beta,\nu)$ is just calculated in foreign currency value. Or, we regard the predetermined foreign exchange rate as being fixed by 1. Later, we will multiply the foreign exchange rate to the option price in foreign currency in order to obtain the option price in domestic currency that we want. We get the PDE by applying the Feynman Kaĉ formula directly.

Theorem 3.1. *Let $[0,T]$ be a time interval. Under the model defined by* (3) *and* (4), *the European quanto call option price $c(t,x,v,\sigma,\rho,\beta,\nu)$ with maturity T and strike price K in foreign currency satisfies the following partial differential equation.*

$$(12)\qquad \begin{cases} \dfrac{\partial c}{\partial t} + \dfrac{1}{2}\left(v^2\dfrac{\partial^2 c}{\partial x^2} + \xi_1^2 v^2 \dfrac{\partial^2 c}{\partial v^2} + \xi_2^2\sigma^2\dfrac{\partial^2 c}{\partial \sigma^2}\right) \\ \quad + \nu\xi_1 v^2 \dfrac{\partial^2 c}{\partial x\partial v} + \beta\rho\xi_2 v\sigma\dfrac{\partial^2 c}{\partial x\partial\sigma} + \nu\beta\rho\xi_1\xi_2 v\sigma\dfrac{\partial^2 c}{\partial v\partial\sigma} \\ \quad + \left(r_f - \rho v\sigma - \dfrac{1}{2}v^2\right)\dfrac{\partial c}{\partial x} + \eta_1 v\dfrac{\partial c}{\partial v} + \eta_2\sigma\dfrac{\partial c}{\partial\sigma} = r_d c, \\ c\left(T,x,v,\sigma,\rho,\beta,\nu\right) = \max\left(e^x - K, 0\right). \end{cases}$$

Proof. We use the stock price dynamics (8) with the Hull and White stochastic volatility model and put $X_t = \ln S_t$, we can find following SDEs

$$(13)\qquad dX_t = \left(r_f - \rho v_t\sigma_t - \frac{1}{2}v_t^2\right)dt + v_t dW_t,$$

$$(14)\qquad dv_t = \eta_1 v_t dt + \xi_1 v_t dB_t,$$

$$(15)\qquad d\sigma_t = \eta_2\sigma_t dt + \xi_2\sigma_t d\tilde{B}_t.$$

The payoff of the option in foreign currency is

$$f(S_T) = \max[S_T - K, 0]$$

or

$$f\left(e^{X_T}\right) = \max\left[e^{X_T} - K, 0\right], \quad K \in \mathbf{R}.$$

In the absence of arbitrage opportunities, the option price $c(t, x, v, \sigma, \rho, \beta, \nu)$ at initial time t is

$$\begin{aligned} c(t, x, v, \sigma, \rho, \beta, \nu) &= E\left[e^{-r_d(T-t)} f\left(e^{X_T}\right)\right] \\ &= E\left[e^{-\int_t^T r_d du} f\left(e^{X_T}\right)\right]. \end{aligned}$$

Applying the Feynman Kaĉ formula (Lemma 2.2) to the above equation and using equations (13)-(15), we can find following equations directly

$$\begin{aligned} &\frac{\partial c}{\partial t} + \left(r_f - \rho v \sigma - \frac{1}{2}v^2\right)\frac{\partial c}{\partial x} + \eta_1 v \frac{\partial c}{\partial v} + \eta_2 \sigma \frac{\partial c}{\partial \sigma} \\ &+ \frac{1}{2}\left(v^2 \frac{\partial^2 c}{\partial x^2} + \xi_1^2 v^2 \frac{\partial^2 c}{\partial v^2} + \xi_2^2 \sigma^2 \frac{\partial^2 c}{\partial \sigma^2}\right) \\ &+ \nu v^2 \xi_1 \frac{\partial^2 c}{\partial x \partial v} + \beta \rho \xi_2 v \sigma \frac{\partial^2 c}{\partial x \partial \sigma} + \nu \beta \rho v \xi_1 \xi_2 \sigma \frac{\partial^2 c}{\partial v \partial \sigma} = r_d c, \\ &c(T, x, v, \sigma, \rho, \beta, \nu) = f(e^x) = \max[e^x - K, 0]. \end{aligned}$$

□

We want to calculate power series expansion formula of quanto option price function $c(t, x, v, \sigma, \rho, \beta, \nu)$. From PDE (12) we can find the PDEs which are derivatives for each correlations ρ, β, ν.

3.2. Taylor series expansion

Let $[0, T]$ be a finite time interval. For $t, s \in [0, T]$, t is initial time and less than or equal to s. From (13), integral form of X_s is

$$X_s = x + r_f(s - t) - \int_t^s \left(\rho v_u \sigma_u + \frac{1}{2}v_u^2\right) du + \int_t^s v_u dW_u.$$

Let $\mathcal{F}_s$ be a filtration which is generated by the volatilities v_u and σ_u $(t \le u \le s)$ then the distribution of X_s conditionally on $\mathcal{F}_s$ is, for a normal distribution N,

$$X_s | \mathcal{F}_s \sim N\left(x + r_f(s - t) - \int_t^s \rho v_u \sigma_u du - \frac{1}{2}\langle M \rangle_s, \langle M \rangle_s\right),$$

where M_s is denoted by the martingale $\int_t^s v_u dW_u$ and

$$\langle M \rangle_s = \int_t^s v_u^2 du$$

is its quadratic variation.

A simple modification of the Black-Scholes formula gives explicitly the value of a quanto call option. Since the underlying asset price S_t is lognormally distributed conditionally on $\mathcal{F}_t$. Let Q_{BS} be a Black-Scholes quanto option price formula in foreign currency. As we mentioned earlier, the option price that we calculate is not for domestic currency but for foreign currency. And the stock price dynamics for valuing option price is quanto adjusted. Therefore, for the strike price $K > 0$, we have

$$\begin{aligned} Q_{BS}(t,x,v,\sigma) &= E\left[e^{-r_d(T-t)}\max(e^{X_T}-K,0)\right] \\ &= e^{x+(r_f-r_d)(T-t)-\int_t^T \rho v_u\sigma_u du} N(d_1) - Ke^{-r_d(T-t)}N(d_2), \end{aligned}$$

where $N(\cdot)$ is the standard normal distribution function and

$$\begin{aligned} d_1 &= \frac{x-\ln K + r_f(T-t) - \int_t^T \rho v_u\sigma_u du + \frac{1}{2}\langle M\rangle_T}{\sqrt{\langle M\rangle_T}}, \\ d_2 &= d_1 - \sqrt{\langle M\rangle_T}. \end{aligned}$$

Moreover, if $\rho = 0$, then since $\frac{1}{K}e^{x+r_f(T-t)}N'(d_1) = N'(d_2)$,

$$\begin{aligned} &\frac{\partial Q_{BS}}{\partial x}(t,x,\Gamma) \\ &= e^{x+(r_f-r_d)(T-t)}N(d_1) \\ &\quad + e^{x+(r_f-r_d)(T-t)}N'(d_1)\frac{\partial d_1}{\partial x} - Ke^{(r_f-r_d)(T-t)}N'(d_2)\frac{\partial d_2}{\partial x} \\ &= e^{x+(r_f-r_d)(T-t)}N(d_1). \end{aligned} \tag{16}$$

We may write

$$\begin{aligned} c_0(t,x,v,\sigma) &= c(t,x,v,\sigma,0,0,0) \\ &= E\left[e^{-r_d(T-t)}\max\left(e^{X_T}-K,0\right)\right] \\ &= E\left[E\left[e^{-r_d(T-t)}\max\left(e^{X_T}-K,0\right)\Big|\mathcal{F}_T\right]\right] \\ &= E\left[e^{x+(r_f-r_d)(T-t)}N(d_1) - Ke^{-r_d(T-t)}N(d_2)\right], \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{x-\ln K + r_f(T-t) + \frac{1}{2}\langle M\rangle_T}{\sqrt{\langle M\rangle_T}}, \\ d_2 &= d_1 - \sqrt{\langle M\rangle_T}. \end{aligned}$$

On the other hand, the equation (12) can be separated by following differential operators

$$\begin{aligned} Lc = &\frac{1}{2}\left(v^2\frac{\partial^2 c}{\partial x^2} + \xi_1^2 v^2\frac{\partial^2 c}{\partial v^2} + \xi_2^2\sigma^2\frac{\partial^2 c}{\partial \sigma^2}\right) + \left(r_f - \frac{1}{2}v^2\right)\frac{\partial c}{\partial x} \\ &+ \eta_1 v\frac{\partial c}{\partial v} + \eta_2\sigma\frac{\partial c}{\partial \sigma} - r_d c, \end{aligned}$$

$$\mathcal{A}c = v\sigma\frac{\partial c}{\partial x}, \qquad \mathcal{B}c = \xi_1 v^2\frac{\partial^2 c}{\partial x\partial v},$$
$$\mathcal{A}_1 c = \xi_1\xi_2 v\sigma\frac{\partial^2 c}{\partial v\partial\sigma}, \qquad \mathcal{A}_2 c = \xi_2 v\sigma\frac{\partial^2 c}{\partial x\partial\sigma}.$$

Then we can rewrite (12) in the following way

$$\begin{cases} \dfrac{\partial c}{\partial t} + Lc - \rho\mathcal{A}c + \nu\mathcal{B}c + \nu\beta\rho\mathcal{A}_1 c + \beta\rho\mathcal{A}_2 c = 0, \\ c(T, x, v, \sigma, \rho, \nu, \beta) = \max\left(e^x - K, 0\right). \end{cases} \tag{17}$$

Consider the following first order Taylor series approximation

$$\begin{aligned} c(t, x, v, \sigma, \rho, \beta, \nu) &\approx c(t, x, v, \sigma, 0, 0, 0) + \rho\frac{\partial c}{\partial\rho}(t, x, v, \sigma, 0, 0, 0) \\ &\quad + \nu\frac{\partial c}{\partial\nu}(t, x, v, \sigma, 0, 0, 0) + \beta\frac{\partial c}{\partial\beta}(t, x, v, \sigma, 0, 0, 0). \end{aligned} \tag{18}$$

Put

$$c_0(t, x, v, \sigma) = c(t, x, v, \sigma, 0, 0, 0), \qquad c_1(t, x, v, \sigma) = \frac{\partial c}{\partial\rho}(t, x, v, \sigma, 0, 0, 0),$$
$$\phi_1(t, x, v, \sigma) = \frac{\partial c}{\partial\nu}(t, x, v, \sigma, 0, 0, 0), \quad \psi_1(t, x, v, \sigma) = \frac{\partial c}{\partial\beta}(t, x, v, \sigma, 0, 0, 0),$$

then we can rewrite approximation (18) as follows

$$\begin{aligned} c(t, x, v, \sigma, \rho, \beta, \nu) &\approx c_0(t, x, v, \sigma) + \rho c_1(t, x, v, \sigma) \\ &\quad + \nu\phi_1(t, x, v, \sigma) + \beta\psi_1(t, x, v, \sigma). \end{aligned}$$

By differentiating the equation (17) and for $(\rho, \nu, \beta) = (0, 0, 0)$, we can get the following PDE problems for $x \in \mathbf{R}, y, z > 0$ and $t \in [0, T]$. Since the terminal condition of $\psi_1(t, x, v, \sigma)$ is zero, the PDE for $\psi_1(t, x, v, \sigma)$ does not exist. Actually, the correlation between the foreign exchange rate and its volatility β does not affect to the stock price dynamics. Thus we exclude the series expansion for the correlation β. We just find the solutions for the following equations.

$$\begin{cases} \dfrac{\partial c_0}{\partial t} + Lc_0(t, x, v, \sigma) = 0, \\ \qquad c_0(T, x, v, \sigma) = \max\left[e^x - K, 0\right], \end{cases} \tag{19}$$

$$\begin{cases} \dfrac{\partial c_1}{\partial t} + Lc_1(t, x, v, \sigma) = \mathcal{A}c_0(t, x, v, \sigma), \\ \qquad c_1(T, x, v, \sigma) = 0, \end{cases} \tag{20}$$

$$\begin{cases} \dfrac{\partial \phi_1}{\partial t} + L\phi_1(t, x, v, \sigma) = -\mathcal{B}c_0(t, x, v, \sigma), \\ \qquad \phi_1(T, x, v, \sigma) = 0. \end{cases} \tag{21}$$

We can find the solution c_0 directly from Black-Scholes option pricing formula. To solve inhomogeneous equations (20) and (21) we use Duhamel's principle which is a tool for finding solutions of inhomogeneous equation. Using Duhamel's principle we can represent the inhomogeneous equation as the integral of homogeneous equations.

Proposition 3.2. *Let $c(t, x, v, \sigma, \rho, \beta, \nu)$ be a quanto option price at initial time t. Then the first order Taylor series expansion is*

$$c(t, x, v, \sigma, \rho, \beta, \nu) \approx c_0(t, x, v, \sigma) + \rho c_1(t, x, v, \sigma) + \nu\phi_1(t, x, v, \sigma), \tag{22}$$

where

$$\begin{aligned}
c_0(t, x, v, \sigma) &= E\left[e^{x+(r_f-r_d)(T-t)}N(d_1) - Ke^{-r_d(T-t)}N(d_2)\right], \\
c_1(t, x, v, \sigma) &= -e^{-r_d(T-t)}\int_t^T E\left[v_s\sigma_s e^{X_s+r_f(T-s)}N(d_1)\right]ds, \\
\phi_1(t, x, v, \sigma) &= -\frac{K\xi_1 e^{-r_d(T-t)}d_2 N'(d_2)}{\langle M\rangle_{[t,T]}}\int_t^T E\left[v_s^2\int_s^T v_u\frac{\partial v_u}{\partial v}du\right]ds.
\end{aligned}$$

Proof. We can find directly $c_0(t, x, v, \sigma)$ from $c(t, x, v, \sigma, 0, 0, 0)$. We want to find the solution of the equation (20),

$$\begin{cases}
\dfrac{\partial c_1}{\partial t} + Lc_1(t, x, v, \sigma) = \mathcal{A}c_0(t, x, v, \sigma), \\
\qquad c_1(T, x, v, \sigma) = 0.
\end{cases}$$

Since the above equation is the inhomogeneous PDE, by the Duhamel's principle (Lemma 2.1), we know that

$$c_1(t, x, v, \sigma) = -\int_t^T p_1(t, x, v, \sigma; s)ds,$$

where $t \in [0, s), x \in \mathbf{R}, v, \sigma > 0$ and $p_1(t, x, v, \sigma; s)$ is a solution of

$$\begin{cases}
\dfrac{\partial p_1}{\partial t}(t, x, v, \sigma; s) + Lp_1(t, x, v, \sigma; s) = 0, \\
\qquad p_1(s, x, v, \sigma; s) = \mathcal{A}c_0(s, x, v, \sigma).
\end{cases} \tag{23}$$

Using the Feynman Kaĉ formula, we can transform a PDE to an expectation of random variables. Applying the Feynman Kaĉ formula (Lemma 2.3) to the equation (23), we get the following expectation formula

$$p_1(t, x, v, \sigma; s) = E\left[e^{-r_d(s-t)}\mathcal{A}c_0\left(s, X_s, v_s, \sigma_s\right)\right]. \tag{24}$$

The internal expectation is actually conditioned by general filtration up to time s. From equation (16),

$$\begin{aligned}
\frac{\partial c_0}{\partial x}(t, x, v, \sigma) &= \frac{\partial}{\partial x}E\left[Q_{BS}\left(t, x + r_f(T-t), \sqrt{\langle M\rangle_T}\right)\right] \\
&= E\left[e^{x+(r_f-r_d)(T-t)}N\left(d_1\right)\right],
\end{aligned}$$

we can compute c_1 as follows

$$\begin{aligned}
&c_1(t,x,v,\sigma)\\
&= -\int_t^T E\left[e^{-r_d(s-t)}\mathcal{A}c_0\left(s,X_s,v_s,\sigma_s\right)\right]ds\\
&= -\int_t^T E\left[e^{-r_d(s-t)}v_s\sigma_s\frac{\partial c_0}{\partial x}\left(s,X_s,v_s,\sigma_s\right)\right]ds\\
&= -\int_t^T E\left[e^{-r_d(s-t)}v_s\sigma_s E\left[e^{X_s+(r_f-r_d)(T-s)}N(d_1)\right]\right]ds\\
&= -e^{-r_d(T-t)}\int_t^T E\left[v_s\sigma_s e^{X_s+r_f(T-s)}N(d_1)\right]ds.
\end{aligned}$$

Similarly applying the Duhamel's principle to equation (21)

$$\phi_1(t,x,v,s) = -\int_t^T q_1(t,x,v,\sigma;s)ds,$$

where $t\in[0,s), x\in\mathbf{R}, v,\sigma>0$ and $q_1(t,x,v,\sigma;s)$ is a solution of

$$\begin{cases}\dfrac{\partial q_1}{\partial t}(t,x,v,\sigma;s)+Lq_1(t,x,v,\sigma;s)=0,\\ \qquad\qquad q_1(s,x,v,\sigma;s)=-\mathcal{B}c_0(s,x,v,\sigma).\end{cases}$$

Then by the Feynman-Kaĉ formula (Lemma 2.3)

$$q_1(t,x,v,\sigma;s) = E\left[e^{-r_d(s-t)}(-\mathcal{B}c_0)(s,X_s,v_s,\sigma_s)\right]$$

and

$$\begin{aligned}
\phi_1(t,x,v,s) &= -\int_t^T q_1(t,x,v,\sigma;s)ds\\
&= \int_t^T E\left[e^{-r_d(s-t)}\mathcal{B}c_0(s,X_s,v_s,\sigma_s)\right]ds\\
&= \int_t^T E\left[e^{-r_d(s-t)}\xi_1 v_s^2\frac{\partial^2 c_0}{\partial x\partial v}(s,X_s,v_s,\sigma_s)\right]ds,
\end{aligned}$$

we can calculate the partial derivative of $c_0(t,x,v,\sigma)$ for v as follows

$$\begin{aligned}
&\frac{\partial}{\partial v}c_0(t,x,v,\sigma)\\
&= e^{x+(r_f-r_d)(T-t)}N'(d_1)\frac{\partial d_1}{\partial v} - Ke^{-r_d(T-t)}N'(d_2)\frac{\partial d_2}{\partial v}\\
&= N'(d_2)Ke^{-r_d(T-t)}\left(\frac{\partial}{\partial v}d_1-\frac{\partial}{\partial v}d_2\right)\\
&= Ke^{-r_d(T-t)}N'(d_2)\frac{\partial}{\partial v}\sqrt{\langle M\rangle_T}.
\end{aligned}$$

Here

$$\begin{aligned}\frac{\partial}{\partial v}\sqrt{\langle M\rangle_T} &= \frac{1}{2}\langle M\rangle_T^{-\frac{1}{2}}\frac{\partial}{\partial v}\langle M\rangle_T \\ &= \frac{1}{2\sqrt{\langle M\rangle_T}}\frac{\partial}{\partial v}\int_t^T v_u^2 du \\ &= \frac{1}{\sqrt{\langle M\rangle_T}}\int_t^T v_u\frac{\partial v_u}{\partial v}du.\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial^2}{\partial x\partial v}Q_{BS}\left(t,x,\sqrt{\langle M\rangle_T}\right) &= \frac{\partial}{\partial x}\left(Ke^{-r_d(T-t)}N'(d_2)\frac{\partial}{\partial v}\sqrt{\langle M\rangle_T}\right) \\ &= N''(d_2)\frac{\partial d_2}{\partial x}Ke^{-r_d(T-t)}\frac{\partial}{\partial v}\sqrt{\langle M\rangle_T} \\ &= \frac{-d_2N'(d_2)}{\sqrt{\langle M\rangle_T}}Ke^{-r_d(T-t)}\frac{\partial}{\partial v}\sqrt{\langle M\rangle_T} \\ &= Ke^{-r_d(T-t)}\frac{-d_2N'(d_2)}{\langle M\rangle_T}\int_t^T v_u\frac{\partial v_u}{\partial v}du.\end{aligned}$$

Therefore,

$$\begin{aligned}&\phi_1(t,x,v,\sigma) \\ &= \int_t^T e^{-r_d(s-t)}E\left[\xi_1 v_s^2 E\left[\frac{\partial^2}{\partial x\partial v}Q_{BS}\left(s,X_s,\sqrt{\langle M\rangle_{[s,T]}}\right)\right]\right] \\ &= \int_t^T e^{-r_d(s-t)}E\left[\xi_1 v_s^2 E\left[Ke^{-r_d(T-s)}\frac{-d_2N'(d_2)}{\langle M\rangle_{[s,T]}}\int_s^T v_u\frac{\partial v_u}{\partial v}du\right]\right]ds \\ &= \int_t^T Ke^{-r_d(T-t)}E\left[\xi_1 v_s^2 E\left[\frac{-d_2N'(d_2)}{\langle M\rangle_{[s,T]}}\int_s^T v_u\frac{\partial v_u}{\partial v}du\right]\right]ds.\end{aligned}$$

Let $\mathcal{F}_T$ be a filtration generated by v_u and σ_u $(t \le u \le T)$. Then

$$\begin{aligned}&\int_t^T Ke^{-r_d(T-t)}E\left[\xi_1 v_s^2 E\left[\frac{-d_2N'(d_2)}{\langle M\rangle_{[s,T]}}\int_s^T v_u\frac{\partial v_u}{\partial v}du\right]\right]ds \\ &= -\int_t^T Ke^{-r_d(T-t)}E\left[\xi_1 v_s^2\frac{d_2N'(d_2)}{\langle M\rangle_{[s,T]}}\int_s^T v_u\frac{\partial v_u}{\partial v}du\right]ds \\ &= -\int_t^T Ke^{-r_d(T-t)}E\left[\frac{\xi_1 v_s^2}{\langle M\rangle_{[s,T]}}\int_s^T v_u\frac{\partial v_u}{\partial v}du E\left[d_2N'(d_2)|\mathcal{F}_T\right]\right]ds,\end{aligned}$$

where

$$d_2 = d_2\left(s,X_s,\sqrt{\langle M\rangle_{[s,T]}}\right) = \frac{X_s - \ln K + r_f(T-s) - \frac{1}{2}\langle M\rangle_{[s,T]}}{\sqrt{\langle M\rangle_{[s,T]}}}.$$

We already know that

$$X_s|\mathcal{F}_T \sim N\left(x - \ln K + r_f(s-t) - \frac{1}{2}\langle M\rangle_{[t,s]}, \langle M\rangle_{[t,s]}\right).$$

Hence d_2 is distributed as follow

$$d_2|\mathcal{F}_T \sim N\left(\frac{x - \ln K + r_f(T-t) - \frac{1}{2}\langle M\rangle_{[t,T]}}{\langle M\rangle_{[s,T]}}, \frac{\langle M\rangle_{[t,s]}}{\langle M\rangle_{[s,T]}}\right).$$

If $z \sim N(\mu, \sigma)$, then

$$E[zN'(z)] = \frac{\mu}{\sqrt{2\pi}(1+\sigma^2)^{3/2}} e^{-\frac{\mu^2}{2(1+\sigma^2)}}. \tag{25}$$

Since

$$E\left[d_2\left(s, X_s, \sqrt{\langle M\rangle_{[s,T]}}\right)\Big|\mathcal{F}_T\right] = d_2(t,x,v,\sigma) \times \frac{\langle M\rangle_{[s,T]}}{\langle M\rangle_{[t,T]}}$$

by using the formula (25)

$$\begin{aligned}&E\left[d_2\left(s, X_s, \sqrt{\langle M\rangle_{[s,T]}}\right)N'\left(d_2\left(s, X_s, \sqrt{\langle M\rangle_{[s,T]}}\right)\right)\Big|\mathcal{F}_T\right]\\ =&\frac{\langle M\rangle_{[s,T]}}{\langle M\rangle_{[t,T]}} d_2\left(t, x, \sqrt{\langle M\rangle_{[t,T]}}\right)N'(d_2\left(t, x, \sqrt{\langle M\rangle_{[t,T]}}\right))\end{aligned}$$

we can calculate ϕ_1 as follow

$$\begin{aligned}&\phi_1(t,x,v,\sigma)\\ &= -\int_t^T \frac{Ke^{-r_d(T-t)}d_2N'(d_2)}{\langle M\rangle_{[t,T]}} E\left[\xi_1 v_s^2 \int_s^T v_u \frac{\partial v_u}{\partial v} du\right] ds\\ &= -\frac{K\xi_1 e^{-r_d(T-t)}d_2N'(d_2)}{\langle M\rangle_{[t,T]}} \int_t^T E\left[v_s^2 \int_s^T v_u \frac{\partial v_u}{\partial v} du\right] ds.\end{aligned}$$

□

Since $\langle M\rangle_T$ is a random variable sufficiently concentrated around its mean in general, one may think of approximating it by its expectation. This idea was first introduced by Alós [1]. The next theorem is the main result of the paper.

Theorem 3.3. *Replacing $\langle M\rangle_T$ by $E[\langle M\rangle_T]$ in Proposition 3.2, the approximate quanto option formula becomes*

$$c(t,x,v,\sigma,\rho,\beta,\nu) \approx \bar{c_0}(t,x,v,\sigma) + \rho\bar{c_1}(t,x,v,\sigma) + \nu\bar{\phi_1}(t,x,v,\sigma),$$

where

$$\begin{aligned}\bar{c_0}(t,x,v,\sigma) &= e^{x+(r_f-r_d)(T-t)}N(\bar{d_1}) - Ke^{-r_d(T-t)}N(\bar{d_2}),\\ \bar{c_1}(t,x,v,\sigma) &= -\frac{e^{x+(r_f-r_d)(T-t)}N(\bar{d_1})v\sigma}{\eta_1+\eta_2+\frac{1}{2}(\xi_1^2+\xi_2^2)}\left(e^{\left(\eta_1+\eta_2+\frac{1}{2}(\xi_1^2+\xi_2^2)\right)(T-t)} - 1\right),\end{aligned}$$

$$\bar{\phi_1}(t,x,v,\sigma) = -\frac{Ke^{-r_d(T-t)}\bar{d_2}N'(\bar{d_2})\xi_1 v}{2\eta_1+\xi_1^2}\left(e^{(2\eta_1+\xi_1^2)(T-t)}-1\right).$$

Proof. If we replace $\langle M\rangle_T$ by $E[\langle M\rangle_T]$, then we rewrite d_1 by $\bar{d_1}$. We have to calculate $E\left[e^{X_s+r_f(T-s)}N\left(d_1\left(s,X_s,\sqrt{\langle M\rangle_{[s,T]}}\right)\right)\right]$ in equation $c_1(t,x,v,\sigma)$, but we take an expectation of random variable X_s instead of solving the expectation as follow

$$E\left[e^{X_s+r_f(T-s)}\right] = e^{x+r_f(T-t)}.$$

Otherwise we cannot figure out the expectation problem. Thus we rewrite c_1 and ϕ_1 like follows

$$\bar{c_1}(t,x,v,\sigma) = -e^{x+(r_f-r_d)(T-t)}\int_t^T E[v_s\sigma_s]\,N(\bar{d_1}(s))ds, \tag{26}$$

(27)

$$\bar{\phi_1}(t,x,v,\sigma) = -\frac{K\xi_1 e^{-r_d(T-t)}\bar{d_2}(t)N'(\bar{d_2}(t))}{\langle\bar{M}\rangle_{[t,T]}}\int_t^T E\left[v_s^2\int_s^T v_u\frac{\partial v_u}{\partial v}du\right]ds,$$

where, for $t\le u\le T$

$$\bar{d_i}(u) = d_i\left(u, X_u, \sqrt{\langle\bar{M}\rangle_{[u,T]}}\right) \quad i=1,2,$$
$$\langle\bar{M}\rangle_{[u,T]} = E\left[\langle M\rangle_{[u,T]}\right].$$

In the equation (26), we have to calculate the integral of $N(\bar{d_1})$ but $N(\cdot)$ is an integral form itself. It is difficult to handle this formula, so we can choose an adjustment factor α which is approximate the integral of $N(\bar{d_1})$ such that

$$\int_t^T E[v_s\sigma_s]\,N(\bar{d_1}(s))ds = N(\bar{d_1}(\alpha))\int_t^T E[v_s\sigma_s]\,ds.$$

Therefore, for some $\alpha\ge 0$

$$\bar{c_1}(t,x,v,\sigma) = -e^{x+(r_f-r_d)(T-t)}N(\bar{d_1}(\alpha))\int_t^T E[v_s\sigma_s]\,ds.$$

In the risk-netral world and the uncorrelated case ($\nu=0$ and $\beta=0$) with $X_t=\ln S_t$, we have

$$dX_t = \left(r_f-\rho v_t\sigma_t-\frac{1}{2}v_t^2\right)dt+v_t dW_t,$$
$$dv_t = \eta_1 v_t dt+\xi_1 v_t dB_t,$$
$$d\sigma_t = \eta_2\sigma_t dt+\xi_2\sigma_t d\tilde{B}_t.$$

Hence, the volatilities are

$$v_u = ve^{\left(\eta_1-\frac{1}{2}\xi_1^2\right)(u-t)+\xi_1(B_u-B_t)},$$
$$\sigma_u = \sigma e^{\left(\eta_2-\frac{1}{2}\xi_2^2\right)(u-t)+\xi_2(\tilde{B}_u-\tilde{B}_t)}$$

and

$$\begin{aligned} E\left[v_u^2\right] &= E\left[v^2 e^{2\left(\eta_1 - \frac{1}{2}\xi_1^2\right)(u-t) + 2\xi_1(B_u - B_t)}\right] \\ &= v^2 e^{2\left(\eta_1 - \frac{1}{2}\xi_1^2\right)(u-t)} E\left[e^{2\xi_1(B_u - B_t)}\right] \\ &= v^2 e^{2\left(\eta_1 - \frac{1}{2}\xi_1^2\right)(u-t)} e^{2\xi_1^2(u-t)} \\ &= v^2 e^{\left(2\eta_1 + \xi_1^2\right)(u-t)}. \end{aligned}$$

This leads to the following approximations

$$\begin{aligned} E\left[\langle M\rangle_{[s,T]}\right] &= E\left[\int_s^T v_u^2 du\right] \\ &= \int_s^T E\left[v_u^2\right] du \\ &= \int_s^T v^2 e^{(2\eta_1 + \xi_1^2)(u-t)} du \\ &= v^2 \frac{e^{(2\eta_1 + \xi_1^2)(T-t)} - e^{(2\eta_1 + \xi_1^2)(s-t)}}{2\eta_1 + \xi_1^2}. \end{aligned}$$

If we apply the above equation to (26) and (27) then we can find the followings

$$\begin{aligned} \bar{c_0}(t,x,v,\sigma) &= e^{x + (r_f - r_d)(T-t)} N(\bar{d_1}) - K e^{-r_d(T-t)} N(\bar{d_2}), \\ \bar{c_1}(t,x,v,\sigma) &= -\frac{e^{x + (r_f - r_d)(T-t)} N(\bar{d_1}) v\sigma}{\eta_1 + \eta_2 + \frac{1}{2}\left(\xi_1^2 + \xi_2^2\right)} \left(e^{\left(\eta_1 + \eta_2 + \frac{1}{2}\left(\xi_1^2 + \xi_2^2\right)\right)(T-t)} - 1\right), \\ \bar{\phi_1}(t,x,v,\sigma) &= -\frac{K e^{-r_d(T-t)} \bar{d_2} N'(\bar{d_2}) \xi_1 v}{2\eta_1 + \xi_1^2} \left(e^{(2\eta_1 + \xi_1^2)(T-t)} - 1\right), \end{aligned}$$

where

$$\begin{aligned} E\left[v_s \sigma_s\right] &= E\left[v\sigma e^{\left(\eta_1 + \eta_2 - \frac{1}{2}(\xi_1^2 + \xi_2^2)\right)(s-t) + \xi_1(B_s - B_t) + \xi_2(\tilde{B}_s - \tilde{B}_t)}\right] \\ &= v\sigma e^{\left(\eta_1 + \eta_2 - \frac{1}{2}(\xi_1^2 + \xi_2^2)\right)(s-t)} e^{(\xi_1^2 + \xi_2^2)(s-t)} \\ &= v\sigma e^{\left(\eta_1 + \eta_2 + \frac{1}{2}(\xi_1^2 + \xi_2^2)\right)(s-t)}, \end{aligned}$$

$$\begin{aligned} \int_t^T E\left[v_s \sigma_s\right] ds &= \int_t^T v\sigma e^{\left(\eta_1 + \eta_2 + \frac{1}{2}(\xi_1^2 + \xi_2^2)\right)(s-t)} ds \\ &= v\sigma \frac{e^{\left(\eta_1 + \eta_2 + \frac{1}{2}\left(\xi_1^2 + \xi_2^2\right)\right)(T-t)} - 1}{\eta_1 + \eta_2 + \frac{1}{2}\left(\xi_1^2 + \xi_2^2\right)} \end{aligned}$$

and

$$\frac{\partial v_u}{\partial v} = e^{(\eta_1 - \frac{1}{2}\xi_1^2)(u-t) + \xi_1(B_u - B_t)},$$

$$\begin{aligned}
E\left[\int_s^T v_u \frac{\partial v_u}{\partial v} du\right] &= E\left[\int_s^T v e^{2(\eta_1 - \frac{1}{2}\xi_1^2)(u-t) + 2\xi_1(B_u - B_t)} du\right] \\
&= \int_s^T v e^{2(\eta_1 - \frac{1}{2}\xi_1^2)(u-t)} E\left[e^{2\xi_1(B_u - B_t)}\right] du \\
&= \int_s^T v e^{(2\eta_1 + \xi_1^2)(u-t)} du \\
&= v \frac{e^{(2\eta_1 + \xi_1^2)(T-t)} - e^{(2\eta_1 + \xi_1^2)(s-t)}}{2\eta_1 + \xi_1^2}.
\end{aligned}$$

□

4. Numerical examples

Now we can find the approximated value of the quanto option price by using the formula in Theorem 3.3. Using the main theorem, we choose the half value of the time to maturity as the adjustment factor α. We compare the result between the approximation value and the Monte Carlo simulation value. We suppose that there is a quanto European call option of S&P500 index with strike 1,200 and maturity date 13-Jun-2011. The information of the quanto option is showed in Table 1.

TABLE 1. Quanto option information sample

Information	
Name	Quanto option sample
Underlying	S&P500
Issue Volume	100
Amount per point in [Currency]	50 [USD]
Currency / quote	KRW / Unit
Option Type	European Call
Strike Value	1,200
Exchange Rate	1,100 (KRW/USD)
Maturity Date	2011-06-13

The model parameters are set to Table 2. The data are viewed at 2010-10-13. The number of the sample data which are used in making the constant volatilities and correlations is 250 from the view date. The volatilities and correlations are calculated by the moving average method which is the same as computing the standard deviation of historical market data. We use the USD riskless rate as 1 year USD LIBOR and the KRW riskless rate as 1 year KRW treasury bond rate at the view date.

Table 2. Market Data Set

Data set	
View Date	2010-10-13
S&P500	1,169.77
FX Rate (KRW/USD)	1,127
Volatility of S&P500	18.58%
Volatility of FX Rate	11.83%
Correlation between S&P500 and FX Rate	-0.2297
Correlation between S&P500 and its volatility	-0.55
Volatility of volatility of S&P500	11.72%
Volatility of volatility of FX Rate	16.8%
USD LIBOR(1Y)	0.77%
KRW Treasury Rate(1Y)	2.91%

The Monte Carlo estimates of the price were obtained by simulating 250,000 paths with 1,000 time-grid points on the interval $[0, T]$. From the Table 3 to the Table 7, the upper value is from our approximation formula, the lower value is from Monte Carlo simulation method and the right side of each entry percent point is a ratio between the approximation value and the simulation value. In the Table 3 that is the first example, we fix all the correlations zero ($\rho = \nu = 0$) and move the time to maturity and the strike value. In the Table 4, time to maturity is 1 and correlation ν is 0. We move the correlation ρ and the strike price. In the Table 5, 6, and 7, we fix the correlation between stock and stock volatility $\nu = -0.55$ and move the correlation ρ and the strike price K. The prices of our approximation method and the Monte Carlo method are calculated in domestic currency. In our approximation method, we first calculate the option price $c(t, x, v, \sigma, \rho, \beta, \nu)$ in foreign currency then we multiply the predetermined exchange rate to the price.

Table 3. Monte Carlo estimations with zero correlations

$T \setminus K$	1100		1150		1200	
0.25	478,095,931 478,511,402	(0.1%)	300,060,155 301,560,208	(0.5%)	171,427,534 172,055,249	(0.4%)
0.5	563,712,483 562,721,490	(0.2%)	400,465,433 397,836,079	(0.7%)	272,467,837 272,569,146	(0.0%)
1	693,286,693 691,414,566	(0.3%)	543,019,112 540,906,951	(0.4%)	417,477,621 413,576,358	(1%)

Table 4. $T = 1$ and correlation $\nu = 0$

$\rho \setminus K$	1100		1150		1200	
-0.4	733,707,680 731,661,695	(0.3%)	576,592,257 574,916,009	(0.3%)	443,918,065 446,471,872	(0.6%)
-0.2	713,497,186 709,296,252	(0.6%)	559,805,685 560,643,025	(0.2%)	430,697,843 427,309,852	(0.8%)
0	693,286,692 696,555,194	(0.5%)	543,019,112 542,959,975	(0.0%)	417,477,621 418,666,951	(0.3%)
0.2	673,076,198 671,322,203	(0.3%)	526,232,539 527,710,228	(0.3%)	404,257,398 402,994,168	(0.3%)
0.4	652,865,704 654,211,596	(0.2%)	509,445,966 508,860,038	(0.1%)	391,037,176 388,483,249	(0.7%)

Table 5. $T = 1$ and correlation $\nu = -0.55$

$\rho \setminus K$	1100		1150		1200	
-0.4	741,267,477 735,422,927	(0.8%)	577,758,979 576,855,830	(0.2%)	438,198,881 441,735,145	(0.8%)
-0.2	721,056,983 711,396,749	(1.3%)	560,972,406 557,444,046	(0.6%)	424,978,659 426,850,549	(0.4%)
0	700,846,489 693,801,357	(1.0%)	544,185,833 540,942,983	(0.6%)	411,758,436 412,306,861	(0.1%)
0.2	680,635,995 678,278,286	(0.4%)	527,399,260 523,628,976	(0.7%)	398,538,214 401,668,459	(0.8%)
0.4	660,425,501 662,302,097	(0.3%)	510,612,687 509,932,660	(0.1%)	385,317,992 385,985,981	(0.2%)

Table 6. $T = 0.5$ and correlation $\nu = -0.55$

$\rho \setminus K$	1100		1150		1200	
-0.4	591,062,599 585,208,735	(1.0%)	418,916,637 418,990,946	(0.0%)	280,957,939 282,192,769	(0.4%)
-0.2	580,193,189 575,986,769	(0.7%)	410,382,665 409,915,669	(0.1%)	274,963,478 276,180,713	(0.4%)
0	569,323,779 564,492,060	(0.9%)	401,848,692 399,865,410	(0.5%)	268,969,017 269,768,467	(0.3%)
0.2	558,454,369 557,417,509	(0.2%)	393,314,720 392,798,704	(0.1%)	262,974,557 264,546,984	(0.6%)
0.4	547,584,959 545,888,148	(0.3%)	384,780,747 385,181,826	(0.1%)	256,980,096 258,396,671	(0.6%)

TABLE 7. $T = 0.25$ and correlation $\nu = -0.55$

$\rho \setminus K$	1100		1150		1200	
-0.4	493,683,947 491,028,807	(0.5%)	310,057,723 309,051,984	(0.3%)	174,428,313 175,726,849	(0.7%)
-0.2	487,756,160 484,732,343	(0.6%)	305,643,449 303,809,470	(0.6%)	171,803,006 173,237,309	(0.8%)
0	481,828,372 479,168,812	(0.6%)	301,229,174 300,851,576	(0.1%)	169,177,699 168,923,701	(0.2%)
0.2	475,900,585 473,986,942	(0.4%)	296,814,899 295,859,291	(0.3%)	166,552,393 167,293,256	(0.4%)
0.4	469,972,798 471,260,262	(0.3%)	292,400,624 292,879,602	(0.2%)	163,927,086 163,996,976	(0.0%)

From the result tables, the difference ratio between our series expansion formula result and the Monte Carlo simulation result is less then 1.3%. This results show that our series expansion formula value is very close to the expected present value of the quanto option payoff. Monte Carlo simulation method is common used in pricing or calculating financial product which is not known its closed form formula. But this method is too much computation time needed for obtaining accurate pricing value. In calculating the quanto option value with the Hull and White stochastic volatility model, the correlation expansion method is a nice alternative to the Monte Carlo simulation method.

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