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ON CONVERGENCE OF THE MODIFIED GAUSS-SEIDEL ITERATIVE METHOD FOR *H*-MATRIX LINEAR SYSTEM

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Abstract. In 2009, Zheng and Miao [B. Zheng and S.-X. Miao, *Two new modified Gauss-Seidel methods for linear system with M-matrices*, J. Comput. Appl. Math. **233** (2009), 922–930] considered the modified Gauss-Seidel method for solving M-matrix linear system with the preconditioner $P_{\max}$. In this paper, we consider the modified Gauss-Seidel method for solving the linear system with the generalized preconditioner $P_{\max}(\alpha)$, and study its convergent properties when the coefficient matrix is an H-matrix. Numerical experiments are performed with different examples, and the numerical results verify our theoretical analysis.

1. Introduction

Consider the following linear system

$$Ax = b, \tag{1.1}$$

where $A = (a_{i,j})$ is an $n \times n$ nonsingular matrix, x and b are n-dimensional vectors. Without loss of generality, throughout this paper, we assume that A has the form $A = I - L - U$, where I is the identity matrix, $-L$ and $-U$ are strictly lower-triangular and strictly upper-triangular parts of A, respectively.

If A has a splitting of the form $A = M - N$, where M is nonsingular, then the splitting iterative method for solving (1.1) can be expressed as

$$x_{i+1} = M^{-1}Nx_i + M^{-1}b,\ i = 0, 1, 2, \ldots. \tag{1.2}$$

It is well known that the iterative scheme (1.2) is convergent if and only if $\rho(M^{-1}N) < 1$, where $\rho(M^{-1}N)$ denotes the spectral radius of $M^{-1}N$. The smaller is $\rho(M^{-1}N)$, the faster is the convergence. For improving the convergent rate of corresponding iterative method, preconditioning techniques are

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used [2]. Especially, we consider the following equivalent left preconditioned linear system of (1.1)

$$PAx = Pb, \tag{1.3}$$

where P, called the left preconditioner, is nonsingular. The corresponding iterative method for solving (1.3) is given by

$$x_{i+1} = M_P^{-1}N_P x_i + M_P^{-1}Pb,\ i = 0, 1, 2, \ldots, \tag{1.4}$$

based on the splitting $PA = M_P - N_P$, where M_P is nonsingular. In particular, if $M = I - L$ and M_P be the lower-triangular part of PA, then the iterative scheme (1.2) is the Gauss-Seidel method and (1.4) is the modified Gauss-Seidel (MGS) method.

Many left preconditioner P were proposed, see [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 19] and references therein. In 2002, Kotakemori et al. [9] considered the preconditioner $P_{S_{\max}} = I + S_{\max}$, where

$$(S_{\max})_{i,j} = \begin{cases} -a_{i,k_i}, & i = 1, \ldots,\ n-1,\ j > i; \\ 0, & \text{otherwise}, \end{cases}$$

with $k_i = \min\{j \mid \max_j |a_{i,j}|, i < n\}$. Here and in the sequel, $(\cdot)_{i,j}$ is used to denote the $i,\ j$-element of the corresponding matrix. The preconditioner $P_{S_{\max}}$ is constructed only by the elements from the upper triangular part of A, the preconditioning effect is not observed on the last row of matrix A. To provide the preconditioning effect on the last row for the preconditioner $P_{S_{\max}}$, Zheng and Miao [19] presented the preconditioners

$$P_{\max} = I + S_{\max} + R_{\max}, \tag{1.5}$$

where

$$(R_{\max})_{i,j} = \begin{cases} -a_{n,k_n}, & i = n,\ j = k_n, \\ 0, & \text{otherwise} \end{cases}$$

with $k_n = \min\{j \mid |a_{n,j}| = \max\{|a_{n,l}|,\ l = 1, \ldots, n-1\}\}$. It was shown in [19] that the MGS method with the preconditioner $P_{\max}$ is superior to the MGS method with the preconditioner $P_{S_{\max}}$ and the classical Gauss-Seidel method for solving the M-matrix linear system.

In this paper, we consider the generalized preconditioner

$$P_{\max}(\alpha) = I + S_{\max}(\alpha) + R_{\max}(\alpha),$$

where

$$(S_{\max}(\alpha))_{i,j} = \begin{cases} -\alpha_i a_{i,k_i}, & i = 1, \ldots, n-1,\ j > i, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$(R_{\max}(\alpha))_{i,j} = \begin{cases} -\alpha_n a_{n,k_n}, & i = n,\ j = k_n, \\ 0, & \text{otherwise} \end{cases}$$

in which α_i are positive real numbers for $i = 1, \ldots, n$. When $\alpha_i = 1$ for $i = 1, 2, \ldots, n$, the preconditioner $P_R(\alpha)$ reduces to the one considered in [19]. The basic purpose of the present paper is to prove the convergence of the MGS method with the preconditioner $P_{\max}(\alpha)$ for solving (1.1) in the case that the coefficient matrix A is an H-matrix.

The remainder of the present paper is organized as follows. Next section is the preliminaries. We study the convergence of the MGS method in Section 3. In Section 4, numerical examples for different problems are given to confirm our theoretical analysis.

2. Preliminaries

In this section, we give some of the notations, definitions and lemmas which will be used in what follows.

A vector $x = (x_1,\ x_2, \ldots,\ x_n)^T$ is called nonnegative (positive) and denoted by $x \geq 0$ $(x > 0)$, if $x_i \geq 0$ $(x_i > 0)$ for all i. Similarly, a matrix $A = (a_{i,j})$ is called nonnegative (positive) and denoted by $A \geq 0$ $(A > 0)$, if $a_{i,j} \geq 0$ $(a_{i,j} > 0)$ for all i, j. The absolute value of a matrix A is denoted by $|A| = (|a_{i,j}|)$. The comparison matrix of A is defined as $\langle A \rangle = (\tilde{a}_{i,j})$, where $\tilde{a}_{i,j}$ satisfies

$$\tilde{a}_{i,j} = \begin{cases} |a_{i,j}|, & i = j, \\ -|a_{i,j}|, & i \neq j. \end{cases}$$

Definition 2.1 ([1, 17])**.** A matrix A is called an M-matrix if $A = sI - B$, $B \geq 0$ and $s > \rho(B)$.

Definition 2.2 ([1, 17])**.** A matrix A is an H-matrix if its comparison matrix $\langle A \rangle$ is an M-matrix.

Definition 2.3 ([4])**.** The splitting $A = M - N$ is called an H-splitting if $\langle M \rangle - |N|$ is an M-matrix.

Lemma 2.1 ([4])**.** *Let $A = M - N$ be a splitting. If it is an H-splitting. Then A and M are H-matrices and $\rho(M^{-1}N) \leq \rho(\langle M \rangle^{-1}|N|) < 1$.*

Lemma 2.2 ([3])**.** *Let A have nonpositive off-diagonal entries. Then a real matrix A is an M-matrix if and only if there exists some vector $u = (u_1, u_2, \ldots, u_n)^T > 0$ such that $Au > 0$.*

3. Convergence analysis

Assume that $a_{i,k_i} \neq 0$, consider the preconditioned matrix $A_P = P_{\max}(\alpha)A$, then we have

$$\begin{aligned} A_P &= P_{\max}(\alpha)A \\ &= M_P - N_P \\ &= [I - D - L - E + R_{\max}(\alpha) - D' - E'] \\ &\quad - [U - S_{\max}(\alpha) + F + S_{\max}(\alpha)U], \end{aligned}$$

where $D,\ E$ and F are the diagonal, strictly lower triangular and strictly upper triangular parts of $S_{\max}(\alpha)L$, while $D',\ E'$ are the diagonal, strictly lower triangular parts of $R_{\max}(\alpha)(L + U)$, respectively. And now M_P and $-N_P$ are the lower-triangular and strictly upper triangular parts of $P_{\max}(\alpha)A$, hence, if M_P is nonsingular, the MGS iterative matrix is $T_P = M_P^{-1}N_P$.

Similar to Theorem 3.2 in [19], one can obtain the convergent theorem of the MGS method with the preconditioner $P_{\max}(\alpha)$ for solving M-matrices linear system. We state this as the following theorem without proof.

Theorem 3.1. *Let A be a nonsingular M-matrix. Assume that $\alpha_i \in [0,\ 1]$ for $i = 1, 2, \ldots, n$. Then $A_P = M_P - N_P$ is the convergent splitting, i.e., $\rho(T_P) < 1$.*

From Theorem 3.1, we known that the MGS method with the preconditioner $P_{\max}(\alpha)$ for solving M-matrix linear system (1.1) is convergent when the parameters α_i are located in the interval $[0,\ 1]$ for $i = 1, 2, \ldots, n$.

In what follows, the convergence of the MGS method with the preconditioner $P_{\max}(\alpha)$ for solving H-matrix linear system will be studied. For this purpose, we first give the following theorem.

Theorem 3.2. *Let A be an $n \times n$ H-matrix with unit diagonal elements, $u = (u_1,\ u_2, \ldots,\ u_n)^T$ be a positive vector such that $\langle A\rangle u > 0$. Assume that $a_{i,k_i} \neq 0,\ i = 1, 2, \ldots, n$. Then for $i = 1, 2, \ldots, n-1$*

$$\beta_i = \frac{u_i - \sum_{j=1}^{i-1} |a_{i,j}|u_j - \sum_{j=i+1, j\neq k_i}^{n} |a_{i,j}|u_j + |a_{i,k_i}|u_{k_i}}{|a_{i,k_i}| \sum_{j=1}^{n} |a_{k_i,j}|u_j}$$

and

$$\beta_n = \frac{u_n - \sum_{j=1, j\neq k_n}^{n-1} |a_{n,j}|u_j + |a_{n,k_n}|u_{k_n}}{|a_{n,k_n}| \sum_{j=1}^{n} |a_{k_n,j}|u_j}$$

are well defined and $\beta_i > 1$ $(i = 1, 2, \ldots, n)$.

Proof. As A is an H-matrix, $\langle A\rangle$ is an M-matrix. From Lemma 2.2, we known that there exists a positive vector $u = (u_1,\ u_2, \ldots,\ u_n)^T$, such that $\langle A\rangle u > 0$. It follows from the definition of $\langle A\rangle$ that

$$u_i - \sum_{j=1, j\neq i}^{n} |a_{i,j}|u_j > 0 \quad \text{for } i = 1, 2, \ldots, n-1 \tag{3.1}$$

and

$$u_n - \sum_{j=1}^{n-1} |a_{n,j}|u_j > 0. \tag{3.2}$$

For $i = 1, 2, \ldots, n-1$, one can get that

$$\begin{aligned} & u_i - \sum_{j=1}^{i-1} |a_{i,j}|u_j - \sum_{j=i+1, j\neq k_i}^{n} |a_{i,j}|u_j + |a_{i,k_i}|u_{k_i} - |a_{i,k_i}| \sum_{j=1}^{n} |a_{k_i,j}|u_j \\ = {} & u_i - \sum_{j=1, j\neq i}^{n} |a_{i,j}|u_j + |a_{i,k_i}| \left(u_{k_i} - \sum_{j=1, j\neq k_i}^{n} |a_{k_i,j}|u_j \right). \end{aligned}$$

From (3.1), we known that

$$u_i - \sum_{j=1, j\neq i}^{n} |a_{i,j}| u_j > 0 \text{ and } u_{k_i} - \sum_{j=1, j\neq k_i}^{n} |a_{k_i,j}| u_j > 0,$$

therefore,

$$u_i - \sum_{j=1}^{i-1} |a_{i,j}| u_j - \sum_{j=i+1, j\neq k_i}^{n} |a_{i,j}| u_j + |a_{i,k_i}| u_{k_i} - |a_{i,k_i}| \sum_{j=1}^{n} |a_{k_i,j}| u_j > 0,$$

which is equivalent to

$$\begin{aligned} & u_i - \sum_{j=1}^{i-1} |a_{i,j}| u_j - \sum_{j=i+1, j\neq k_i}^{n} |a_{i,j}| u_j + |a_{i,k_i}| u_{k_i} \\ & > |a_{i,k_i}| \sum_{j=1}^{n} |a_{k_i,j}| u_j \\ & > 0. \end{aligned}$$

Hence,

$$\beta_i = \frac{u_i - \sum_{j=1}^{i-1} |a_{i,j}| u_j - \sum_{j=i+1, j\neq k_i}^{n} |a_{i,j}| u_j + |a_{i,k_i}| u_{k_i}}{|a_{i,k_i}| \sum_{j=1}^{n} |a_{k_i,j}| u_j}$$

are well defined and $\beta_i > 1$ for $i = 1, 2, \ldots, n-1$.

Similarly, for $i = n$, we can deduce from (3.2) that

$$\begin{aligned} & u_n - \sum_{j=1, j\neq k_n}^{n-1} |a_{n,j}| u_j + |a_{n,k_n}| u_{k_n} \\ & > |a_{n,k_n}| \sum_{j=1}^{n} |a_{k_n,j}| u_j \\ & > 0, \end{aligned}$$

that is to say

$$\beta_n = \frac{u_n - \sum_{j=1, j\neq k_n}^{n-1} |a_{n,j}| u_j + |a_{n,k_n}| u_{k_n}}{|a_{n,k_n}| \sum_{j=1}^{n} |a_{k_n,j}| u_j}$$

is well defined and $\beta_n > 1$. The proof is completed. □

Remark 3.1. It should be remark that β_i $(i = 1, 2, \ldots, n)$ in Theorem 3.2 depends on the positive vector u. There are many such vectors u satisfying $u > 0$, how to choose applicable u is very important for practical computation. In general, we can let $u = (1, 1, \ldots, 1)^T$ when A is the strictly diagonally dominant H-matrix, while when A is not strictly diagonally dominant, it follows from [18] that the elements $m_{i,j}$ of $\langle A\rangle^{-1}$ satisfies

$$\sum_{j=1}^{n} m_{i,j} \geq 1, \ i = 1, 2, \ldots, n,$$

hence we can let $u_i = \sum_{j=1}^{n} m_{i,j}$ for $i = 1, 2, \ldots, n$ and $u = (u_1, u_2, \ldots, u_n)^T$. However, finding out β_i ($i = 1, 2, \ldots, n$) which are independent of the vector u is still an open problem and need further study.

Now we are in the position to establish the convergence of the MGS method with the preconditioner $P_{\max}(\alpha)$ for solving H-matrix linear system.

Theorem 3.3. *Let A be an H-matrix with unit diagonal elements, $P_{\max}(\alpha)A = M_P - N_P$ with $M_P = I - D - L - E + R_{\max}(\alpha) - D' - E'$ and $N_P = U - S_{\max}(\alpha) + F + S_{\max}(\alpha)U$. If for $i = 1, 2, \ldots, n$, $\alpha_i a_{i,k_i} a_{k_i,i} \neq 1$ and β_i are defined as in Theorem 3.2, then for $0 \leq \alpha_i < \beta_i$, $i = 1, 2, \ldots, n$, the splitting $P_{\max}(\alpha)A = M_P - N_P$ is an H-splitting and $\rho(T_P) < 1$.*

Proof. In order to prove that the splitting $P_{\max}(\alpha)A = M_P - N_P$ is an H-splitting, we only need to show that $\langle M_P\rangle - |N_P|$ is an M-matrix.

Let $u = (u_1, u_2, \ldots, u_n)^T$ be a given positive vector, $[(\langle M_P\rangle - |N_P|)u]_i$ be the ith element of the vector $(\langle M_P\rangle - |N_P|)u$ for $i = 1, 2, \ldots, n$. Then for $i = 1, 2, \ldots, n-1$, we known that

$$\begin{aligned}
[(\langle M_P\rangle - |N_P|)u]_i &= |1 - \alpha_i a_{i,k_i} a_{k_i,i}|u_i - \sum_{j=1, j\neq i}^{n} |a_{i,j} - \alpha_i a_{i,k_i} a_{k_i,j}|u_j \\
&\geq u_i - \alpha_i |a_{i,k_i} a_{k_i,i}|u_i - \sum_{j=1}^{i-1} |a_{i,j}|u_j \\
&\quad - \alpha_i \sum_{j=1}^{i-1} |a_{i,k_i} a_{k_i,j}|u_j - \sum_{j=i+1, j\neq k_i}^{n} |a_{i,j}|u_j \\
&\quad - \alpha_i \sum_{j=i+1, j\neq k_i}^{n} |a_{i,k_i} a_{k_i,j}|u_j - |1 - \alpha_i||a_{i,k_i}|u_{k_i},
\end{aligned} \tag{3.3}$$

and for $i = n$, it hold that

$$\begin{aligned}
[(\langle M_P\rangle - |N_P|)u]_n &= |1 - \alpha_n a_{n,k_n} a_{k_n,n}|u_n - \sum_{j=1}^{n-1} |a_{n,j} - \alpha_n a_{n,k_n} a_{k_n,j}|u_j \\
&\geq u_n - \alpha_n |a_{n,k_n} a_{k_n,n}|u_n - \sum_{j=1, j\neq k_n}^{n-1} |a_{n,j}|u_j \\
&\quad - \alpha_n \sum_{j=1, j\neq k_n}^{n-1} |a_{n,k_n} a_{k_n,j}|u_j - |1 - \alpha_n||a_{n,k_n}|u_{k_n}.
\end{aligned} \tag{3.4}$$

In what follows, for the choices of the parameters α_i $(i = 1, 2, \ldots, n)$, we consider two cases.

Case (I). $\alpha_i \in [0,\ 1],\ i = 1, 2, \ldots, n$.

For $i = 1, 2, \ldots, n-1$, it follows from (3.3) that

$$\begin{aligned}
&[(\langle M_P\rangle - |N_P|)u]_i \\
&\geq u_i - \alpha_i|a_{i,k_i}a_{k_i,i}|u_i - \sum_{j=1}^{i-1}|a_{i,j}|u_j - \alpha_i\sum_{j=1}^{i-1}|a_{i,k_i}a_{k_i,j}|u_j \\
&\quad - \sum_{j=i+1,j\neq k_i}^{n}|a_{i,j}|u_j - \alpha_i\sum_{j=i+1,j\neq k_i}^{n}|a_{i,k_i}a_{k_i,j}|u_j - (1-\alpha_i)|a_{i,k_i}|u_{k_i} \\
&= u_i - \sum_{j=1,j\neq i}^{n}|a_{i,j}|u_j + \alpha_i|a_{i,k_i}|u_{k_i} - \alpha_i|a_{i,k_i}|\sum_{j=1,j\neq k_i}^{n}|a_{k_i,j}|u_j \\
&= \left(u_i - \sum_{j=1,j\neq i}^{n}|a_{i,j}|u_j\right) + \alpha_i|a_{i,k_i}|\left(u_{k_i} - \sum_{j=1,j\neq k_i}^{n}|a_{k_i,j}|u_j\right),
\end{aligned}$$

and for $i = n$, from (3.4) we have

$$\begin{aligned}
&[(\langle M_P\rangle - |N_P|)u]_n \\
&\geq u_n - \alpha_n|a_{n,k_n}a_{k_n,n}|u_n - \sum_{j=1,j\neq k_n}^{n-1}|a_{n,j}|u_j \\
&\quad - \alpha_n\sum_{j=1,j\neq k_n}^{n-1}|a_{n,k_n}a_{k_n,j}|u_j - (1-\alpha_n)|a_{n,k_n}|u_{k_n} \\
&= \left(u_n - \sum_{j=1}^{n-1}|a_{n,j}|u_j\right) + \alpha_n|a_{n,k_n}|\left(u_{k_n} - \sum_{j=1,j\neq k_n}^{n-1}|a_{k_n,j}|u_j\right).
\end{aligned}$$

As $u_i - \sum_{j=1,j\neq i}^{n}|a_{i,j}|u_j > 0$ and $u_{k_i} - \sum_{j=1,j\neq k_i}^{n}|a_{k_i,j}|u_j > 0$ for $i = 1, 2, \ldots, n$, we get that

$$[(\langle M_P\rangle - |N_P|)u]_i > 0 \quad \text{for } i = 1, 2, \ldots, n. \tag{3.5}$$

Case (II). $\alpha_i \in (1,\ \beta_i),\ i = 1, 2, \ldots, n$.

When $i = 1, 2, \ldots, n-1$, from (3.3) and the definition of β_i, we have

$$\begin{aligned}
[(\langle M_P\rangle - |N_P|)u]_i \geq\ & u_i - \alpha_i|a_{i,k_i}a_{k_i,i}|u_i - \sum_{j=1}^{i-1}|a_{i,j}|u_j \\
& - \alpha_i\sum_{j=1}^{i-1}|a_{i,k_i}a_{k_i,j}|u_j - \sum_{j=i+1,j\neq k_i}^{n}|a_{i,j}|u_j
\end{aligned}$$

$$
\begin{aligned}
&\quad - \alpha_i \sum_{j=i+1, j\neq k_i}^{n} |a_{i,k_i} a_{k_i,j}| u_j - (\alpha_i - 1)|a_{i,k_i}| u_{k_i} \\
&= u_i - \sum_{j=1}^{i-1} |a_{i,j}| u_j - \sum_{j=i+1, j\neq k_i}^{n} |a_{i,j}| u_j \\
&\quad + |a_{i,k_i}| u_{k_i} - \alpha_i |a_{i,k_i}| \sum_{j=1}^{n} |a_{k_i,j}| u_j \\
&> 0. \qquad (3.6)
\end{aligned}
$$

For $i = n$, if follows from (3.4) and the definition of β_n that

$$
\begin{aligned}
[(\langle M_P \rangle - |N_P|)u]_n &\geq u_n - \alpha_n |a_{n,k_n} a_{k_n,n}| u_n - \sum_{j=1, j\neq k_n}^{n-1} |a_{n,j}| u_j \\
&\quad - \alpha_n \sum_{j=1, j\neq k_n}^{n-1} |a_{n,k_n} a_{k_n,j}| u_j - (\alpha_n - 1)|a_{n,k_n}| u_{k_n} \\
&= u_n - \sum_{j=1, j\neq k_n}^{n-1} |a_{n,j}| u_j + |a_{n,k_n}| u_{k_n} \\
&\quad - \alpha_n |a_{n,k_n}| \sum_{j=1}^{n} |a_{k_n,j}| u_j \\
&> 0. \qquad (3.7)
\end{aligned}
$$

Therefore, from (3.5), (3.6) and (3.7), we have

$$(\langle M_\alpha \rangle - |N_\alpha|)u > 0 \quad \text{for } 0 \leq \alpha_i < \beta_i \ (i = 1, 2, \ldots, n).$$

By Lemma 2.2, we know that $\langle M_\alpha \rangle - |N_\alpha|$ is an M-matrix for $0 \leq \alpha_i < \beta_i$ $(i = 1, 2, \ldots, n)$. From Definition 2.3, $A_\alpha = M_\alpha - N_\alpha$ is an H-splitting for $0 \leq \alpha_i < \beta_i$ $(i = 1, 2, \ldots, n)$. Hence, Lemma 2.2 yields $\rho(M_\alpha^{-1} N_\alpha) < 1$ for $0 \leq \alpha_i < \beta_i$ $(i = 1, 2, \ldots, n)$. □

Remark 3.2. From Theorem 3.3, we can see that the MGS method is convergent for all $0 \leq \alpha_i < \beta_i,\ i = 1, 2, \ldots, n-1$ with the preconditioner $P_{S_{\max}}(\alpha)$ when the coefficient matrix A of (1.1) is an H-matrix. The convergence condition when A is an H-matrix is much weaker than the ones, studied in [19] and Theorem 3.1 in this paper, when A is an M-matrix.

Remark 3.3. Comparing Theorem 3.3 with Theorem 3.1, we note that there is the larger range of the parameters α_i when A is an H-matrices. In general, the chosen range of α_i is wider than that of the parameter ω of the SOR iterative method [17].

4. Examples

In this section, we use some examples to verify our theoretical analysis in Section 3.

It is well known that the Toeplitz matrices arise in many applications, such as solutions to differential and integral equations, spline functions, and problems and methods in physics, mathematics, statistics, and signal processing [5]. Therefore, the first example, we consider the case that the coefficient matrix of (1.1) is a Toeplitz matrix.

Example 4.1. Let the coefficient matrix of (1.1) be a symmetric Toeplitz matrix as

$$A = \begin{bmatrix} a & b & c & \cdots & b \\ b & a & b & \cdots & c \\ c & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & c & b & \cdots & a \end{bmatrix}_{n\times n},$$

where $a = 1$, $b = 1/n$ and $c = 1/(n-2)$. It is clear that A is an H-matrix, and not strictly diagonally dominant. From Remark 3.1, we can let $u_i = \sum_{j=1}^{n} m_{i,j}$ for $i = 1, 2, \ldots, n$ and $u = (u_1, u_2, \ldots, u_n)^T$, here $m_{i,j}$ is the (i, j) element of $\langle A\rangle^{-1}$, then according Theorem 2 to compute β_i. After some calculations, we find that $\beta_i < n-2$ for $i = 1, 2, \ldots, n$.

The spectral radii of MGS iteration matrix with various values of α_i for $i = 1, 2, \ldots, n$ are listed in Table 1.

Table 1. The spectral radii of MGS iteration matrix for Example 4.1

	$n = 90$	$n = 180$	$n = 200$	$n = 300$
$\alpha_i = 0.8$	0.2095	0.2129	0.2132	0.2142
$\alpha_i = 1.0$	0.2078	0.2121	0.2125	0.2137
$\alpha_i = 1.3$	0.2052	0.2108	0.2113	0.2129
$\alpha_i = 2.0$	0.1993	0.2079	0.2087	0.2112
$\alpha_i = 55$	0.6175	0.3035	0.2740	0.1985

Example 4.2. When the central difference scheme on a uniform grid with $N \times N$ interior nodes ($N^2 = n$) is applied to the discretization of the two-dimension convection-diffusion equation

$$-\triangle u + \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = f$$

in the unit squire Ω with Dirichlet boundary conditions, we obtain a system of linear equations (1.1) with the coefficient matrix

$$A = I \otimes C + D \otimes I,$$

where $\otimes$ denotes the Kronecker product,

$$C = \text{tridiag}(-\frac{2+h}{8},\ 1,\ -\frac{2-h}{8}) \text{ and } D = \text{tridiag}(-\frac{1+h}{4},\ 0,\ -\frac{1-h}{8})$$

are $N \times N$ tridiagonal matrices, and the step size is $h = 1/N$.

It is clear that the matrix A is an M-matrix, see for example [19], so it is an H-matrix. Moreover, A is strictly diagonally dominant, from Remark 3.1, we can let $u = (1, 1, \ldots, 1)^T$, and then according Theorem 2 to compute β_i. After some calculations, we see that $\beta_i < 4$ for $i = 1, \ldots, n$. We list the spectral radii of MGS iteration matrix with various values of α_i for $i = 1, 2, \ldots, n$ in Table 2.

TABLE 2. The spectral radii of MGS iteration matrix for Example 4.2

	$n = 16$	$n = 81$	$n = 100$	$n = 256$
$\alpha_i = 0.8$	0.5017	0.8507	0.8754	0.9464
$\alpha_i = 1.0$	0.4579	0.8350	0.8621	0.9405
$\alpha_i = 1.5$	0.2980	0.7836	0.8190	0.9217
$\alpha_i = 2.0$	0.2844	0.7006	0.7505	0.8928
$\alpha_i = 3.8$	0.8304	0.9152	0.9158	0.9261

From Tables 1 and 2, it can be seen that the MGS method is convergent for Examples 4.1 and 4.2 when $\alpha_1 \in [0,\ \beta_i)$, i.e., $\rho(T_P) < 1$. This confirm the result of Theorem 3.3 in Section 3. In particular, if we take $\alpha_i = 1$ for $i = 1, \ldots, n-1$, then the preconditioner $P_{S_{\max}}(\alpha)$ reduces to the one considered in [19].

We also note that the MGS method with the preconditioner $P_{S_{\max}}(\alpha)$ for H-matrices linear system is convergent even if $\alpha_i > 2$. This confirms that the chosen range of α_i is wider than that of the parameter ω of the SOR iterative method [17].

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