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DUALITY THEOREM AND VECTOR SADDLE POINT THEOREM FOR ROBUST MULTIOBJECTIVE OPTIMIZATION PROBLEMS

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Abstract. In this paper, Mond-Weir type duality results for a uncertain multiobjective robust optimization problem are given under generalized invexity assumptions. Also, weak vector saddle-point theorems are obtained under convexity assumptions.

1. Introduction

Consider an uncertain multiobjective robust optimization problem:

$$\text{(MRP)} \qquad \begin{array}{ll} \text{minimize} & (f_1(x), \ldots, f_l(x)) \\ \text{subject to} & g_j(x, v_j) \leqq 0, \ \forall v_j \in \mathcal{V}_j, \ j = 1, \ldots, m, \end{array}$$

where v_i is an uncertain parameter and $v_i \in \mathcal{V}_i$ for some convex compact set $\mathcal{V}_i$ in $\mathbb{R}^q$, $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, l$ and $g_j : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $j = 1, \ldots, m$ are continuously differentiable.

When $l = 1$, (MRP) becomes an uncertain optimization problem, which has been intensively studied in ([4]-[5], [6]), associates with the uncertain program (UP) its robust counterpart [1],

$$\text{(RP)} \quad \inf_{x \in \mathbb{R}^n} \{f(x) : g_i(x, v_i) \leq 0, \ \forall v_i \in \mathcal{V}_i, \ \ i = 1, \ldots, m\},$$

where the uncertain constraints are enforced for every possible value of the parameters within their prescribed uncertainty sets $\mathcal{V}_i$, $i = 1, \ldots, m$. Recently, Jeyakumar, Li and Lee [7] established a robust duality theory for generalized convex programming problems in the face of data uncertainty. Furthermore, Kim [8] extended results of Jeyakumar, Li and Lee [7] for a uncertain multiobjective robust optimization problem. In this paper, Mond-Weir type duality results for a uncertain multiobjective robust optimization problem are given

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under generalized invexity assumptions. Also, weak vector saddle-point theorems are obtained under convexity assumptions.

Let F be the set of all the robust feasible solutions of (MRP) and $J(\bar{x}) = \{j \mid \exists v_j \in \mathcal{V}_j \text{ s.t. } g_j(\bar{x}, v_j) = 0, j = 1, \dots, m\}$.

Definition 1.1. A robust feasible solution $\bar{x}$ of (MRP) is a weakly robust efficient solution of (MRP) if there does not exist a robust feasible solution x of (MRP) such that

$$f_i(x) < f_i(\bar{x}), \quad i = 1, \dots, l.$$

Definition 1.2. (1) A vector-valued function f is said to be generalized η-quasi-invex at $(x^*, v_j) \in F \times \mathcal{V}_j$ for each $x \in \mathbb{R}^n$ there exist $\alpha_i : \mathbb{R}^n \times \mathbb{R}^n \to (0, +\infty),\ i = 1, \dots, l,\ \eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$,

$$f_i(x) \leqq f_i(x^*) \ \Rightarrow \ \alpha_i(x, x^*) \nabla f_i(x^*)^T \eta(x, x^*) \leqq 0.$$

(2) A vector-valued function f is said to be generalized η-pseudo-invex at $(x^*, v_j) \in F \times \mathcal{V}_j$ for each $x \in \mathbb{R}^n$ there exist $\alpha_i : \mathbb{R}^n \times \mathbb{R}^n \to (0, +\infty),\ i = 1, \dots, l,\ \eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$,

$$\alpha_i(x, x^*) \nabla f_i(x^*)^T \eta(x, x^*) \geqq 0 \ \Rightarrow \ f_i(x) \geqq f_i(x^*).$$

Now we define an Extended Mangasarian-Fromovitz constraint qualification for (MRP) as follows:

There exists $d \in \mathbb{R}^n$ such that for any $j \in J(\bar{x})$ and any $v_j \in \mathcal{V}_j$,

$$\nabla_1 g_j(\bar{x}, v_j)^T d < 0.$$

Now we present necessary optimality theorems for weakly robust efficient solutions for (MRP).

Theorem 1.1 ([9])**.** *Let $\bar{x} \in F$ be a weakly robust efficient solution of (MRP). Suppose that $g_j(\bar{x}, \cdot)$ are concave on $\mathcal{V}_j,\ j = 1, \dots, m$. Then there exist $\lambda_i \geqq 0,\ i = 1, \dots, l,\ \mu_j \geqq 0,\ j = 1, \dots, m$, not all zero, and $\bar{v}_j \in \mathcal{V}_j,\ j = 1, \dots, m$ such that*

$$\sum_{i=1}^{l} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j) = 0, \tag{1}$$

$$\mu_j g_j(\bar{x}, \bar{v}_j) = 0,\ j = 1, \dots, m. \tag{2}$$

Moreover, if we further assume that the Extended Mangasarian-Fromovitz constraint qualification holds, then there exist $\lambda_i \geqq 0,\ i = 1, \dots, l$, not all zero, $\mu_j \geqq 0,\ j = 1, \dots, m$, and $\bar{v}_j \in \mathcal{V}_j,\ j = 1, \dots, m$ such that (1) *and* (2) *hold.*

2. Duality results

In this section, we establish Mond-Weir type robust duality between (MRP) and (MD).

$$\text{(MD)} \qquad \text{maximize } \Big(f_1(u), \dots, f_l(u)\Big)$$

$$\text{subject to } \sum_{i=1}^{l} \lambda_i \nabla f_i(u) + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(u, v_j) = 0,$$
$$\mu_j g_j(u, v_j) \geq 0,\ j = 1, \ldots, m,$$
$$\lambda_i \geq 0,\ i = 1, \ldots, l,\ \sum_{i=1}^{l} \lambda_i = 1,$$
$$\mu_j \geq 0,\quad v_j \in \mathcal{V}_j,\ j = 1, \ldots, m.$$

Theorem 2.1 (Weak Duality)**.** *Let x be feasible for (MRP) and $(\bar{x}, \bar{v}, \lambda, \mu)$ be feasible for (MD). Suppose that $f_i(\cdot), i = 1, \ldots, l$ are generalized η-quasi-invex at $\bar{x}$ and $\mu_j g_j(\cdot, \bar{v}_j),\ j = 1, \ldots, m$ are generalized strictly η-pseudo-invex at $\bar{x}$ and $g_j(\bar{x}, \cdot)$ are concave on $\mathcal{V}_j$. Then*

$$\Big(f_1(x), \ldots, f_l(x)\Big) \not< \Big(f_1(\bar{x}), \ldots, f_l(\bar{x})\Big).$$

Proof. Let x be feasible for (MRP) and $(\bar{x}, \bar{v}, \lambda, \mu)$ be feasible for (MD). Suppose that $f_i(x) < f_i(\bar{x}),\ i = 1, \ldots, l$. Then the η-quasi-invexity of $f_i(\cdot)$ at $\bar{x}$ implies that

$$\eta(x, \bar{x})^T \nabla f_i(\bar{x}) < 0,\ i = 1, \ldots, l.$$

Since $g_j(x, \bar{v}_j) \leqq 0,\ \bar{v}_j \in \mathcal{V}_j, \mu_j \geqq 0,\ \mu_j g_j(x, \bar{v}_j) \leqq 0,\ j = 1, \ldots, m,$

$$\mu_j g_j(x, \bar{v}_j) \leqq \mu_j g_j(\bar{x}, \bar{v}_j),\ j = 1, \ldots, m.$$

Thus the strictly η-pseudo-invexity of $\mu_j g_j(\cdot, \bar{v}_j)$, $j = 1, \ldots, m$ at $\bar{x}$ implies that

$$\eta(x, \bar{x})^T \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j) < 0,\ j = 1, \ldots, m.$$

Hence $\lambda_i \geqq 0,\ i = 1, \ldots, l,\ \sum_{i=1}^{l} \lambda_i = 1,$

$$\eta(x, \bar{x})^T \Big[\sum_{i=1}^{l} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j)\Big]^T < 0.$$

This is a contradiction, since $\sum_{i=1}^{l} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j) = 0.$ □

Theorem 2.2 (Strong Duality)**.** *Let $\bar{x}$ be a weakly efficient solution of (MRP). Assume that the Extended Mangasarian-Fromovitz constraint qualification holds. Then, there exists $(\bar{v}, \bar{\lambda}, \bar{\mu})$ such that $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is feasible for (MD) and the objective values of (MRP) and (MD) are equal. If $f_i(\cdot),\ i = 1, \ldots, l$ are η-quasi-invex at $\bar{x}$, $\bar{\mu}_j g_j(\cdot, \bar{v}_j),\ j = 1, \ldots, m$ are strictly η-pseudo-invex at $\bar{x}$, and $g_j(\bar{x}, \cdot)$ are concave on $\mathcal{V}_j,\ j = 1, \ldots, m$, then $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is a weakly efficient solution of (MD).*

Proof. Since $\bar{x}$ is a weakly efficient solution of (MRP) at which the Extended Mangasarian-Fromovitz constraint qualification is satisfied, then by Theorem

1.1, there exist $\bar{\lambda}_i \geqq 0,\ i = 1, \dots, l$, not all zero, $\bar{\mu}_j \geqq 0,\ j = 1, \dots, m$, and $\bar{v}_j \in \mathcal{V}_j,\ \ j = 1, \dots, m$, such that

$$\sum_{i=1}^{l} \bar{\lambda}_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \nabla_1 g_j(\bar{x}, \bar{v}_j) = 0,$$
$$\bar{\mu}_j g_j(\bar{x}, \bar{v}_j) = 0,\ j = 1, \dots, m.$$

Thus $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is feasible for (MD) and clearly the objective values of (MRP) and (MD) are equal. If $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is weak duality holds, then there exists feasible $(\widetilde{x}, \widetilde{v}, \widetilde{\lambda}, \widetilde{\mu})$ for (MD) such that

$$\Big(f_i(\bar{x}), \dots, f_l(\bar{x})\Big) \not< \Big(f_1(\widetilde{x}), \dots, f_l(\widetilde{x})\Big).$$

Hence $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is a (MD)-feasible solution, $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is a weakly efficient solution of (MD). □

3. Weak vector saddle-point theorems

In this section, we prove weak vector saddle-point theorems for multiobjective robust optimization problem (MRP). Let

$$L(x, w, \mu) = f(x) + \mu^T g(x, w)e,$$

where $x \in \mathbb{R}^n$, $w \in \mathcal{V}$, $\mu \in \mathbb{R}^m_+$ and $e = (1, \dots, 1) \in \mathbb{R}^l$. Then, a point $(\bar{x}, \bar{w}, \bar{\mu}) \in \mathbb{R}^n \times \mathcal{V} \times \mathbb{R}^m_+$ is said to be a weak vector saddle-point if

$$L(x, \bar{w}, \bar{\mu}) \not< L(\bar{x}, \bar{w}, \bar{\mu}) \not< L(\bar{x}, w, \mu)$$

for all $x \in \mathbb{R}^n$, $w \in \mathcal{V}$, $\mu \in \mathbb{R}^m_+$.

Theorem 3.1. *Let $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{\mu})$ satisfy* (1) *and* (2). *Suppose that $f_i(\cdot),\ i = 1, \dots, l$ and $g_j(\cdot, \bar{w}_j),\ j = 1, \dots, m$ are convex and $g_j(\bar{x}, \cdot)$ are concave on $\mathcal{V}_j$. Then $(\bar{x}, \bar{w}, \bar{\mu})$ is a weak vector saddle-point of* (*MRP*).

Proof. If (1) and (2) are true. Then there exist $\bar{\lambda} \in \mathbb{R}^l_+$, $\bar{w} \in \mathcal{V}$ and $\bar{\mu} \in \mathbb{R}^m_+$ such that

$$\sum_{i=1}^{l} \bar{\lambda}_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \nabla_1 g_j(\bar{x}, \bar{w}_j) = 0 \tag{3}$$
$$\bar{\mu}_j g_j(\bar{x}, \bar{w}_j) = 0,\ j = 1, \dots, m.$$

Let $x \in \mathbb{R}^n$ be any fixed. Then $f_i(\cdot),\ i = 1, \dots, l$ and $g_j(\cdot, \bar{w}_j),\ j = 1, \dots, m$ are convex,

$$f_i(x) - f_i(\bar{x}) \geq \nabla f_i(\bar{x})^T (x - \bar{x}),$$
$$g_j(x, \bar{w}_j) - g_j(\bar{x}, \bar{w}_j) \geq \nabla_1 g_j(\bar{x}, \bar{w}_j)^T (x - \bar{x}).$$

Since $\bar{\lambda}_i \geqq 0,\ i = 1, \dots, l$, $\sum_{i=1}^{l} \lambda_i = 1$, $\bar{\mu}_j \geqq 0,\ j = 1, \dots, m$,

$$\bar{\lambda}_i \Big\{ f_i(x) - f_i(\bar{x}) \Big\} \geqq \bar{\lambda}_i \nabla f_i(\bar{x})^T (x - \bar{x}),\ i = 1, \dots, l,$$

$$\sum_{i=1}^{l}\bar\lambda_i\Big\{\bar\mu_j g_j(x,\bar w_j)-\bar\mu_j g_j(\bar x,\bar w_j)\Big\}\geqq\sum_{i=1}^{l}\bar\lambda_i\bar\mu_j\nabla_1 g_j(\bar x,\bar w_j)^T(x-\bar x),\ j=1,\dots,m.$$

Summing up all these inequalities, it follows from (3) that

$$\begin{aligned}&\sum_{i=1}^{l}\bar\lambda_i\Big\{f(x)+\sum_{j=1}^{m}\bar\mu_j g_j(x,\bar w_j)\Big\}-\sum_{i=1}^{l}\bar\lambda_i\Big\{f(\bar x)+\sum_{j=1}^{m}\bar\mu_j g_j(\bar x,\bar w_j)\Big\}\\ &\geqq\Big\{\sum_{i=1}^{l}\bar\lambda_i\nabla f_i(\bar x)+\sum_{j=1}^{m}\bar\mu_j\nabla_1 g_j(\bar x,\bar w_j)\Big\}^T(x-\bar x)\\ &=0.\end{aligned}$$

Since $\bar\lambda_i \geqq 0$, not all zero,

$$f(x)+\bar\mu^T g(x,\bar w)e \not< f(\bar x)+\bar\mu^T g(\bar x,\bar w)e \quad \text{for any } x\in\mathbb{R}^n,$$

i.e., $L(x,\bar w,\bar\mu)\not< L(\bar x,\bar w,\bar\mu)$ for any $x\in\mathbb{R}^n$.
Now, since $\bar\mu^T g(\bar x,\bar w)=0$, $\mu^T g(x,w)\leqq 0$ for any $\mu\in\mathbb{R}^m_+$, $w\in\mathcal{V}$,

$$\bar\mu^T g(\bar x,\bar w)-\mu^T g(x,\bar w)\geqq 0 \quad \text{for any } \mu\in\mathbb{R}^m_+.$$

Thus

$$f(\bar x)+\bar\mu^T g(\bar x,\bar w)e-\Big\{f(\bar x)+\mu^T g(x,w)e\Big\}\in\mathbb{R}^l_+,$$

and hence

$$L(\bar x,\bar w,\bar\mu)\not< L(\bar x,w,\mu).$$

Therefore, $(\bar x,\bar w,\bar\mu)$ is a weak vector saddle-point of (MRP). □

Corollary 3.1. *Suppose that $f_i(\cdot)$, $i=1,\dots,l$ and $g_j(\cdot,\bar w_j)$, $j=1,\dots,m$ are convex and $g_j(\bar x,\cdot)$ are concave on $\mathcal{V}_j$. If $\bar x$ is a weakly efficient solution of* (*MRP*) *at which the Extended Mangasarian-Fromovitz constraint qualification is satisfied, then there exists $(\bar v,\bar\lambda,\bar\mu)$ such that $(\bar x,\bar v,\bar\lambda,\bar\mu)$ is a weak vector saddle-point of* (*MRP*)*.*

Theorem 3.2. *If there exists $\bar\mu\in\mathbb{R}^m_+$ such that $(\bar x,\bar w,\bar\mu)$ is a weak vector saddle-point of* (*MRP*)*, then $\bar x$ is a weakly efficient solution of* (*MRP*)*.*

Proof. Let $(\bar x,\bar w,\bar\mu)$ be a weak vector saddle-point of (MRP). From the right inequality of saddle-point conditions,

$$f(\bar x)+\bar\mu^T g(\bar x,\bar w)e\not< f(\bar x)+\mu^T g(\bar x,w)e$$

for any $\mu\in\mathbb{R}^m_+$. Thus

$$\bar\mu^T g(\bar x,\bar w)e\not<\mu^T g(\bar x,w)e$$

for any $\mu\in\mathbb{R}^m_+$, $w\in\mathcal{V}$ and hence we have

$$\bar\mu^T g(\bar x,\bar w)\geqq\mu^T g(\bar x,w) \quad \text{for any } \mu\in\mathbb{R}^m_+,\ w\in\mathcal{V}. \tag{4}$$

Letting $\mu=0$ in (4), $\bar\mu^T g(\bar x,\bar w)\geqq 0$. Letting $\mu=2\bar\mu$ in (4), $w_i=\bar w_i$, $\bar\mu^T g(\bar x,\bar w)\leqq 0$. Therefore,

$$\bar\mu^T g(\bar x,\bar w)=0.$$

Now, from the left inequality of saddle-point conditions and $\bar{\mu}^T g(\bar{x}, \bar{w}) = 0$, we have, for any feasible solution x of (MRP), $f(x) \not< f(\bar{x})$. Hence $\bar{x}$ is a weakly efficient solution of (MRP). □

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